PROBLEM SET 4, FOR 18.102, FALL 2007
DUE THURSDAY 4 OCT, AT 2:30 IN CLASS (2-102).

All homework should either be submitted to me by email (it does not have to be in TeX, scanned written pages are fine) or on paper of reasonable quality, so it goes through the scanner! If you need decent paper I am happy to supply some.

(1) Let f_n be a sequence of step functions such that $\sum\limits_n \int |f_n| < \infty$. Show that the set $Z = \{x \in \mathbb{R}; \sum\limits_n |f_n(x)| = \infty\}$ can, for any $\epsilon > 0$, be covered by a countable collection of semi-open intervals $[a_i, b_i)$ of total length $< \epsilon$.

Hint: Fix a large constant A and look at the set where $\sum\limits_{k\le n} |f_k(x)| > A$ for some n. Show that this is a countable union as desired and that the total length tends to 0 as $A \to \infty$.

Remark: This is the usual definition of 'Lebesgue measure zero' for a subset of $\mathbb{R}$.

(2) Prove the following converse of this. Namely if $Z \subset \mathbb{R}$ is a set which, for each $\epsilon > 0$, is covered by a countable union of semi-open intervals of total length $< \epsilon$ then there exists a sequence $f_n \ge 0$ of step functions such that $\sum\limits_n \int f_n < \infty$ but $\sum\limits_n f_n(x) = \infty$ on Z (and possibly a larger set).

(3) Write down in your own words a proof the following result from Chapter 1:- A normed space is complete if and only if every absolutely convergent series is convergent, where absolute convergence of a series with terms $\{u_n\}$ means that $\sum\limits_n \|u_n\| < \infty$.

Remark: This is used in the proof of completeness of $\mathcal{L}^1(\mathbb{R})$.