PROBLEM SET 3, FOR 18.102, FALL 2007
DUE THURSDAY 27 SEPT, AT 2:30 IN CLASS (2-102).

Second version – some errors noted by Urs Niesen have been corrected!

All homework should either be submitted to me by email (it does not have to be in TeX, scanned written pages are fine) or on paper of reasonable quality, so it goes through the scanner! If you need decent paper I am happy to supply some.

The problem for this week is just to show that a continuous (real-valued) function on $[a, b]$ is Lebesgue integrable in the sense of Definition 2.3.1 (as in class on Thursday) – when extended as zero outside the interval. This is proved later in Chapter 2 but using much more machinery. Doing it directly now may help you to understand the rather strange-looking definition.

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function on a compact interval. Define

$$\tilde{f}(x) = \begin{cases} f(x) & x \in [a, b) \\ 0 & x \in \mathbb{R} \setminus [a, b) \end{cases}, \ \tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}.$$

Consider for each $n \in \mathbb{N}$ the 2^n half-open intervals

$$I_{n,j} = [a + 2^{-n}j(b - a), a + 2^{-n}(j + 1)(b - a)), \ 0 \le j \le 2^n - 1. \tag{1}$$

Define step functions

$$g_n = \begin{cases} \inf_{I_{n,j}} f & \text{on } I_{n,j} \\ 0 & \text{otherwise.} \end{cases}$$

(1) Show that $\{g_n\}$ is a non-decreasing sequence of step functions and that

$$\lim_{n\to\infty} g_n(x) = \tilde{f}(x) \ \forall \ x \in \mathbb{R}.$$

(2) Show that $\int g_n$ is a non-decreasing sequence which converges to the Riemann integral $\int_a^b f$.

(3) Show that $\int |g_n|$ converges to $\int_a^b |f|$.

(4) Show that there is a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that

$$\int_a^b |f| - \int |g_{n_j}| < 2^{-j}.$$

(5) Define $h_1 = g_{n_1}$ and $h_j = g_{n_j} - g_{n_{j-1}}$ for $j > 1$. Show that $\{h_j\}$ is a sequence of step functions such that

$$\sum_j \int |h_j| < \infty$$

$$\sum_j h_j(x) = \tilde{f}(x) \ \forall \ x \in \mathbb{R},$$

$$\sum_j \int h_j = \int_a^b f.$$

For no extra credit, but maybe to amuse yourself, show that you can modify the discussion above to find a sequence of step functions $\{f_j\}$ such that

$$\sum_j \int |f_j| < \infty$$

$$\sum_j f_j(x) = \begin{cases} f(x) & x \in [a,b], \\ 0 & x \in \mathbb{R} \setminus [a,b] \end{cases}.$$