SOLUTIONS TO PROBLEM SET 1, FOR 18.102, FALL 2007
WAS DUE THURSDAY 13 SEPT, AT 2:30 IN CLASS (2-102).

General comments:- I marked this rather harshly, but in fact anyone who handed in a script got full marks. Mainly I took off marks for a lack of clarity – and in some cases for using obviously more sophisticated results to prove elementary ones! If you have questions about what I wrote, please ask. The main point to remember is that you need to have an argument that is convincing to me (or to your dedicated grader).

RBM.

(1) (No.3) Prove that a subspace of a vector space is a vector space itself. [The point of course is to write this out carefully but succinctly].

My solution. By assumption V is a vector space and $S \subset V$ is a subset such that $u, v \in S$ implies $u + v \in S$ and $\alpha \in \mathbb{F}$ (the field, either reals or complexes) then $\alpha u \in S$. To see that S is a vector space we need to check the axioms. Commutativity and associativity follow since they hold in V and all the operations lead to elements of S so they hold in S. Existence of a zero element and additive inverse for each element follows similarly, since $-u = (-1)u$ and $u + (-u) = 0$ in V implies that these hold in S. □

Comments: Pretty silly question really, I just wanted to make sure that you realized there is something to check although not much.

(2) (No.7) Show that any vector of $\mathbb{R}^3$ is a linear combination of (the) vectors $(1, 0, 0)$, $(1, 1, 0)$ and $(1, 1, 1)$. [Once again, try to keep it brief and as clear as you can]

My solution. Best to do it explicitly and write

$$(a, b, c) = (a - b)(1, 0, 0) + (b - c)(1, 1, 0) + c(1, 1, 1).$$

□

(3) (No.8) Prove that every quadruple of (i.e. set of four) vectors in $\mathbb{R}^3$ is linearly dependent. [Meaning show there is a non-trivial linear relation between them.]

My solution. Given 4 elements u_i of $\mathbb{R}^3$ if any three of them are linearly depedent we are finished, so we can assume without loss of generality that the first 3 are linearly independent; we will prove that they must be a basis (rather than just quote it). Let e_i be the standard basis $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ then u_1 is a linear combination of these with at least one coefficient non-zero (otherwise it would be zero and there would be a linear relation amongst the u_i.) Thus there is a basis of u_1 and two of the e_i. Then u_2 is a linear combination of these, so the absence of a linear relation between u_1 and u_2 means that u_1, u_2 and one of the e_i form a basis. Applying the same reasoning again shows that u_1, u_2 and u_3 are a basis and hence u_4 is linearly dependent on them. □

Comment: Of course you can kill this by quoting some more general version of the result, I just wanted you to do it by hand – and of course it would be better to do the general case, that in $\mathbb{R}^n$ and $n+1$ vectors are linearly dependent.

(4) (No.9) Prove that the functions $f_n(x) = x^n$, $n \in \{0, 1, \dots\}$ are linearly independent.

My solution. It should really say in the space of functions on $\mathbb{R}$. So we must show that any finite dependence relation is trivial, that is if

$$\sum_{i=0}^{N} c_i x^i = 0 \ \forall \ x \in \mathbb{R}$$

then $c_i = 0$ for all $i = 0, \dots N$. Without loss of generality we can assume that $c_N \neq 0$. Then

$$c_N = \lim_{x \to \infty} x^{-N} \sum_{i=0}^{N} c_i x^i = 0$$

is a contradiction. You could also differentiate N times and evaluate at the origin. □

Comment: Note that you cannot write down an *infinite* dependence relation without assuming some convergence condition, but this is not what the condition of independence requires – just the absence of any finite dependenc relation.

(5) (No.11) Prove that each of the spaces $\mathcal{C}(\mathbb{R}^n)$, $\mathcal{C}^k(\mathbb{R}^n)$, $\mathcal{C}^\infty(\mathbb{R}^n)$ is infinite dimensional. [Try the case n=1 first, the other cases follow from this; also think about which of the three cases is the hardest and how it is related to the others, before you start!]

My solution. On $\mathbb{R}^n$ the polynomials in the first variable x_1 are certainly elements of $\mathcal{C}^\infty(\mathbb{R}^n)$ and by the discussion above this must be infinite dimensional. Since

$$\mathcal{C}^\infty(\mathbb{R}^n) \subset \mathcal{C}^k(\mathbb{R}^n) \subset \mathcal{C}^0(\mathbb{R}^n)$$

these must all be infinite dimensional. □