# 18.102: NOTES ON INTEGRATION

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Abstract. Here are some notes summarizing and slightly reorganizing the material covered from Chapter 2 of Debnaith and Mikusiński

(1) It might be a good idea to review absolute convergence of series and the rearrangement of series of non-negative numbers from Rudin.

(2) Step functions. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to be a step function if there exist finitely many, finite, disjoint, semi-open intervals $[a_i, b_i)$ $i = 1, \dots, N$ such that $f$ takes a constant value $c_i$ on each $[a_i, b_i)$ and vanishes outside the union. By reordering we can assume $b_i \le a_{i+1}$ for $1 \le i < N$ (since they are disjoint). Some of the intervals may be redundant, with $c_i = 0$ – these may be dropped, and there may be cases where $b_i = a_{i+1}$ and $c_i = c_{i+1}$ in which case two intervals can be combined. Doing this a finite number of times gives the minimal presentation of a step function. The integral

$$\int f = \sum_{i=1}^{N} c_i(b_i - a_i) \tag{1}$$

is independent of the presentation – check this.

(3) Let $\mathcal{F}$ be the set of all step functions. It is a linear space (over the reals) and $f \in \mathcal{F}$ implies that $|f| \in \mathcal{F}$ and

$$|\int f| \le \int |f|. \tag{2}$$

Similarly $\max(f, g)$ and $\min(f, g)$ are in $\mathcal{F}$ if $f,\ g \in \mathcal{F}$.

(4) The covering property of intervals. If $[a_i, b_i)$ is a countable collection of intervals and $[a, b) \subset \bigcup_i [a_i, b_i)$ then

$$b - a \le \sum_i (b_i - a_i) \tag{3}$$

with equality if the $[a_i, b_i)$ are disjoint.

This is Lemma 2.2.4 in case the intervals are disjoint, the general case follows by removing from each interval the union of the preceeding ones.

(5) If $\{f_i\}$ is a sequence in $\mathcal{F}$ such that $f_i(x)$ is non-increasing and $\lim_{i\to\infty} f_i(x) = 0$ then $\lim_{i\to\infty} \int f_i = 0$. This is Theorem 2.2.6 and uses the covering lemma.

(6) If $\{f_i\}$ is a sequence in $\mathcal{F}$ such that $f_i(x)$ is non-increasing and $\lim_{i\to\infty} f_i(x) \le 0$ then $\lim_{i\to\infty} \int f_i \le 0$ (in both cases the possibility that the 'limit' is $-\infty$ is allowed). This is Corollary 2.2.7 with the signs reversed. It follows by applying the preceeding result to $\max(f_i, 0)$.

(7) We say that a sequence $\{f_i\}$ in $\mathcal{F}$ is *absolutely summable* if

$$\sum_i \int |f_i| < \infty. \tag{4}$$

(8) A subet $Z \subset \mathbb{R}$ is said to be of *measure zero* if there is an absolutely summable sequence $\{f_i\}$ in $\mathcal{F}$ such that

$$\sum_i |f_i(x)| = \infty \ \forall \ z \in Z. \tag{5}$$

(9) A countable union of sets of measure zero is of measure zero. This is somewhere later in the Chapter. Anyway, the proof is straightforward and indicative of things to come. Let $Z_i$ be the countable collection of sets of measure zero. Thus for each $i$ there is an absolutely summable sequence $\{f^i_k\}_{k=1}^{\infty}$ in $\mathcal{F}$ such that

$$\sum_k |f^i_k(x)| = \infty \ \forall \ x \in Z_i. \tag{6}$$

Now both the condition (6) and the absolute summability are conditions on the tail of the sequence. That is, if we drop a finite number of terms from the sequence they both continue to hold. The absolute summability of each sequence means that we can drop some terms and see that

$$\sum_{k=n_i}^{\infty} \int |f^i_k| < 2^{-i} \ \forall \ i. \tag{7}$$

So, let's just relabel the tails so that this is true with $n_i = 1$ for each $i$. Now, choose a sequence $\{h_n\}$ in $\mathcal{F}$ which diagonalizes all these sequences – each $f^i_k$ appears exactly once. This is absolutely summable,

$$\sum_n \int |h_n| < \infty \tag{8}$$

since it is a rearrangement of the double sequence $\sum\limits_{i,k} \int |f^i_k|$ which converges by arrangement. On the other hand

$$\sum_n |h_n(x)| = \infty \ \forall \ x \in \bigcup_i Z_i \tag{9}$$

since the subseries $\sum\limits_k |f^i_k(x)|$ diverges on $Z_i$.

(10) A sequence of functions $g_n : \mathbb{R} \longrightarrow \mathbb{R}$ is said to converge *almost everywhere* to $g : \mathbb{R} \longrightarrow \mathbb{R}$ if there is a set $Z$ of measure zero such that

$$g(x) = \lim_{n\to\infty} g_n(x) \ \forall \ x \in \mathbb{R} \setminus Z. \tag{10}$$

This is written

$$g(x) = \lim_{n\to\infty} g_n(x) \text{ a.e.} \tag{11}$$

where $Z$ is not mentioned.

(11) Equality of functions a.e. has a similar meaning. Note that it is an equivalence relation with transitiviy using the fact that the union of two sets of measure zero is of measure zero.

(12) Now, the main definition, slightly changed, is that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is *integrable* if there is an absolutely summable sequence $f_i \in \mathcal{F}$ such that

$$f(x) = \sum_i f_i(x) \text{ a.e.} \tag{12}$$

In the book the set $Z$ where equality is to hold in (12) is taken to be precisely the set (if any) where $\sum_i |f_i(x)| = \infty$. This does not really matter.

(13) If $\{f_i\}$ is an absolutely summable sequence in $\mathcal{F}$ such that

$$\sum_i f_i(x) \geq 0 \text{ a.e.} \tag{13}$$

then

$$\sum_i \int f_i \geq 0. \tag{14}$$

This is really Lemma 2.3.2 but I have changed the assumptions a little bit. To reduce to the case in the book just note that (13) means that there is a set of measure zero $Z$ such that the series converges and the inequaltiy holds in $\mathbb{R} \setminus Z$. By definition of 'measure zero' there is an absolutely summable sequence $h_i$ in $\mathcal{F}$ such that $\sum_i |h_i(x)| = \infty$ for $x \in Z$. Now, interlace $h_i$ and $-h_i$ in the sequence and define

$$f'_i(x) = \begin{cases} f_k(x) & i = 3k-2 \\ h_k(x) & i = 3k-1 \\ -h_k(x) & i = 3k. \end{cases} \tag{15}$$

Then $f'_i \in \mathcal{F}$ is an absolutely summable sequence and (14) implies that

$$\sum_i f'_i(x) \geq 0 \ \forall \ x \text{ such that } \sum_i |f'_i(x)| < \infty. \tag{16}$$

Indeed, the last condition excludes all the points of $Z$. Now the proof of Lemma 2.3.2 applies and shows that

$$\sum_{i=1}^{\infty} \int f'_i \geq 0. \tag{17}$$

This sum converges to $\sum_i \int f_i$ since the finite sums to $n$ differ by at most a $\pm \int h_n$ which tends to zero with $n$ by the absolute summability.

(14) Thus the integral of an integrable function is well-defined by

$$\int f = \sum_n \int f_n \tag{18}$$

for *any* absolutely summable sequence in $\mathcal{F}$ such that (12) holds – see Corollary 2.3.3.

(15) Now Theorem 2.3.4 shows that the space of integrable functions is a vector space – if $f$ and $g$ are integrable and $\{f_i\}$ and $\{g_i\}$ are absolutely summable sequences in $\mathcal{F}$ which sum to them almost everywhere then the alternating sequence

$$h_n = \begin{cases} f_k & n = 2k-1 \\ g_k & n = 2k \end{cases} \tag{19}$$

is absolutely summable and converges to $f + g$ almost everywhere (namely off the union of the sets of measure zero where the component sequences fail to sum to $f$ and $g$).

(16) The absolute value of an integrable function is integrable. This is Theorem 2.4.1 but easier since I have shifted the work back. Namely if $\{f_i\}$ is an absolutely summable sequence in $\mathcal{F}$ converging almost everywhere to $f$ then as discussed in the proof of Theorem 2.4.1, the sequence $\{|f_i|\}$ in $\mathcal{F}$ is absolutely summable and converges to $|f|$ a.e.

(17) Theorem 2.4.3 is another property of integrable functions inherited from step functions.

(18) So we denote by $L^1(\mathbb{R})$ the linear space of integrable functions.

(19) Lemma 2.5.2 now says that if $f \in L^1(\mathbb{R})$ and $\epsilon > 0$ is given then there exists a sequence $f_n \in \mathcal{F}$ such that

$$\sum_n \int |f_n| \le \int |f| + \epsilon \text{ and } f(x) = \sum_n f_n(x) \text{a.e.} \tag{20}$$

(20) Look at 'Definition' 2.5.1. Instead we say that a sequence $\{f_n\}$ in $L^1(\mathbb{R})$ is absolutely summable if

$$\sum_n \int |f_n| < \infty. \tag{21}$$

(21) We can insert a result that for any such absolutely summable sequence in $L^1(\mathbb{R})$ the sum

$$\begin{gathered} \sum_n |f_n(x)| < \infty \text{ a.e. and } f(x) = \sum_n f_n(x) \text{ a.e.} \\ \Longrightarrow \sum_{k \le n} f_k \to f \in L^1(\mathbb{R}), \ \int f = \sum_n \int f_n. \end{gathered} \tag{22}$$

This is really Theorem 2.5.3.

(22) Having defined sets of measure zero, we say that a function is *null* if it vanishes outside a set of measure zero.

(23) The null functions form a linear space $\mathcal{N}$. This is clear since the support of a sum is contained in the union of the supports.

(24) A function $f \in \mathcal{N}$ if and only if it is integrable and $\int |f| = 0$. I may have messed up one half of this in lectures. If $f \in \mathcal{N}$ then the sequence of functions all zero converges to $f$ almost everywhere and to $|f|$ almost everywhere as well, hence it is integrable with $\int |f| = 0$. Conversely, if $f$ is integrable with $\int |f| = 0$ then the sequence with all entries $|f|$ is absolutely summable and converges to 0 off the support of $f$ but diverges on the support of $f$. Thus the support of $f$ must be a set of measure zero.

(25) This allows us to define what the book sometimes refers to as $\mathcal{L}^1(\mathbb{R})$. Namely this is the set of equivalence classes of integrable functions under equality almost everywhere. As a vector space it is the quotient $L^1(\mathbb{R})/\mathcal{N}$. At least for a little while I will denote the elements $[u]$ where $u \in L^1(\mathbb{R})$ and $[u] = u + \mathcal{N}$ is its equivalence class. Usually the distinction between $L^1(\mathbb{R})$ and $\mathcal{L}^1(\mathbb{R})$ is ignored.

(26) The space $\mathcal{L}^1(\mathbb{R})$ is a normed space with the norm

$$\|[u]\|_{L^1} = \int |u|. \tag{23}$$

First check that the norm is well defined. If $g \in \mathcal{N}$ then $|f+g| = |f| + p$ where $p \in \mathcal{N}$ since it has the same support as $g$. Thus

$$\int |f| = \int |f+g| \ \forall \ f \in L^1(\mathbb{R}),\ g \in \mathcal{N}. \tag{24}$$

The norm is therefore well-defined and non-negative. Moreover we showed above that $\|f\| = 0$ for $f \in L^1(\mathbb{R})$ implies $f \in \mathcal{N}$ which means $[f] = 0$ in $\mathcal{L}^1(\mathbb{R})$. We also know that $\|\lambda[f]\|_{L^1} = |\lambda|\|[f]\|_{L^1}$ and that the triangle inequality holds, since $|f(x)+g(x)| \le |f(x)| + |g(x)|$ at each point.

(27) Of course the point of the whole discussion is that $\mathcal{L}^1(\mathbb{R})$ is a Banach space. Let me rephrase the argument in the book. If $\{[f_n]\}$ is a Cauchy sequence in $\mathcal{L}^1(\mathbb{R})$ then it suffices to show that it has a convergent subsequence. We choose a subsequence of the representatives by requiring $\|f_{n_{j+1}} - f_{n_j}\|_{L^1} \le 2^{-j}$. For convenience, renumber these as $f_j$. Then the telescoping sequence $g_1 = f_1$, $g_j = f_j - f_{j-1}$ for $j > 1$ is absolutely summable,

$$\|g_j\| \le 2^{-j+1},\ j > 1 \Longrightarrow \sum_j \|g_j\|_{L^1} < \infty. \tag{25}$$

From the discussion above, for a sequence $\sum\limits_{j=1}^{n} g_j(x) = f_n(x)$ converges almost everywhere. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be equal to the limit when it exist and be zero otherwise. Then $f \in L^1(\mathbb{R})$ and $f_n \to f$ in $L^1(\mathbb{R})$. Thus $[f_n] \to [f]$ in $\mathcal{L}^1(\mathbb{R})$ which is therefore complete.

(28) Monotone convergence. If $\{f_n\}$ is a sequence of integrable functions which is monotone almost everywhere and $|\int f_n| \le M$ is uniformly bounded then there is an integrable function $f$ such that $f_n \to f$, a.e $|\int f| \le M$ and $\int |f - f_n| \to 0$. For definiteness sake assume that $f_n(x)$ is non-decreasing for $x \in \mathbb{R} \setminus Z$ where $Z$ has measure zero – otherwise work with $\{-f_n\}$. Thus $f_k - f_{k-1} \ge 0$ for all $k > 1$ and hence

$$\int |f_1| + \sum_{k=2}^{n} \int |f_k - f_{k-1}| \le 3M. \tag{26}$$

Let $n \to \infty$ and we see that $g_1 = f_1$, $g_k = f_k - f_{k-1}$ for $k \ge 2$ form an absolutely summable series. The sum of this exists almost everywhere, take $f$ to be equal to this (and zero elsewhere). Then $f_n \to f$ in $L^1(\mathbb{R})$ and $|\int f| \le M$.

(29) Fatou's Lemma. Let $f_k$ be a sequence of non-negative functions in $L^1(\mathbb{R})$ then, provided $\int f_n < M$, $g = \liminf f_k \in L^1(\mathbb{R})$ and $\int g \le \liminf \int f_n$. First consider $g_{m,k}(x) = \min\{f_m(x), \dots, f_{m+k}(x)\}$. For fixed $m$ this is a non-increasing sequence of functions with integral bounded by $\inf f_m \le M$. So by Monotone convergence it converges to $g_m = \inf_{k \ge m} f_k$ which is integrable. Now, $g_m$ is non-decreasing in $m$ and with integral bounded by $M$. Thus it converges a.e. to $g \in L^1(\mathbb{R})$ and $\int g_m \le \inf_{k \ge m} \int f_k$ for all $m$ implies that $\int g = \lim \int g_m \le \liminf \int f_n$. [I think I rejected the wise suggestion from the audience that the condition $\int f_n < M$ was necessary – of course it is, it is just that in the usual approach one concludes that $g$ is measureable and that its integral satisfies the inequality, but it can be infinite. We do not allow this here.]

(30) Lebesgue dominated convergence. If a sequence $f_n \in L^1(\mathbb{R})$ converges a.e to $f$ and there exists $h \in L^1(\mathbb{R})$ such that $|f_n| \le h$ then $f \in L^1(\mathbb{R})$ and $f_n \to f$ in $L^1(\mathbb{R})$. By changing all the $f_n$'s to be zero on the set of measure zero we ensure convergence everywhere and the conclusion is the same. So consider $h - f_n$. This is a sequence of non-negative functions to which Fatou's lemma applies, since $\int (h-f_n) \le 2\int h$. In fact we know that $h(x) - f_n(x) \to h - f(x)$ for all $x \in \mathbb{R}$ so $\liminf_n (h - f_n(x)) = h - f_n(x)$ so by the monotone convergence theorem

$$\int (h-f) = \int \liminf (h-f_n) \le \liminf \int (h-f_n) = \int h - \limsup \int f_n. \tag{27}$$

Then apply the same argument to $h + f_n$ which shows that

$$\int (h+f) = \int \liminf (h+f_n) \le \liminf \int (h+f_n) \le \int h + \liminf \int f_n. \tag{28}$$

Subtracting off the $\int h$'s gives

$$\limsup \int f_n \le \int f \le \liminf f_n \Longrightarrow \int f = \lim \int f_n. \tag{29}$$

This implies that $\liminf \int f_n = \limsup \int f_n$ since the first is always smaller than or equal to the second. For a real sequence this equality is equivalent to convergence so

$$\int f = \lim \int f_n. \tag{30}$$

The same argument can now be applied to $|f - f_n| \le 2h$ which is convergent a.e. with limit 0. Thus $f_n \to f$ in $L^1(\mathbb{R})$.

(31) Let me give an example of the usefulness of this theorem. Suppose $f \in L^1(\mathbb{R})$. For each $N \in \mathbb{N}$ consider the function

$$f_N(x) = \begin{cases} 0 & x \in \mathbb{R} \setminus [-N, N] \\ N & x \in [-N, N],\ f(x) > N \\ -N & x \in [-N, N],\ f(x) < -N \\ f(x) & \text{otherwise.} \end{cases} \tag{31}$$

Thus, $f_N$ is $f$ with both its support truncated and its values truncated (you could do this independently if you want). Clearly $f_N(x) \to f(x)$ for all $x$ – in fact $f_N(x) = f(x)$ for large enough $N$ depending on $x$. Moreover, $|f_N(x)| \le |f(x)$. Thus Lebesgue's theorem of dominated convergence applies and shows that $f_n \to f$ in $L^1(\mathbb{R})$. So, every $L^1$ function is the limit in $L^1$ of a sequence of bounded integrable functions.

(32) Local integrability. We say a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is locally integrable if the function $f\chi_{[-N,N]}$ is integrable for each $N$.

(33) The space $L^2(\mathbb{R})$ consists of those locally integrable functions such that $|f|^2$ is integrable (same for general $1 < p < \infty$). Then $\mathcal{L}^2(\mathbb{R}) = L^2(\mathbb{R})/\mathcal{N}$ is the quotient with identification a.e. Note that if $g \in \mathcal{N}$ then $|f+g|^2 = |f|^2 + p$ where $p \in \mathcal{N}$ as for $\mathcal{L}^1(\mathbb{R})$. We can therefore set

$$\|[f]\|_{L^2} = \left(\int |f|^2\right)^{\frac{1}{2}}. \tag{32}$$

By the Cauchy-Schwarz inequality

(33) $$\int |f\chi_{[-N,N]}| \le (2N)^{\frac{1}{2}} \|f\|_{L^2}$$

so the $L^1$ norm of $f\chi_{[-N,N]}$ is controlled by the $L^2$ norm.

(34) The same argument as above works for $L^2(\mathbb{R})$ to shows that bounded $L^2$ functions with compact support are dense in $L^2(\mathbb{R})$. Simply take the same cutoff as in (31). Then $|f - f_n|^2 \le |f|^2$ converges to zero so $f_n \to f$ in $L^2(\mathbb{R})$.

(35) The main point of this whole departure into integration theory is that $\mathcal{L}^2(\mathbb{R})$ is a (real) Hilbert space. This requires

(36) Cauchy-Schwarz – if $f, g \in L^2(\mathbb{R})$ then $fg \in L^1(\mathbb{R})$ (Why? A good way is to use the cutoff (31), observe that $(fg)_n \in L^1$, and then use Cauchy-Schwarz to show that $(fg)_n$ is Cauchy in $L^1$ and converges to $fg$, which is therefore in $L^1$) and

(34) $$|\int fg| \le \|f\|_{L^2}\|g\|_{L^2}.$$

This is proved in the book (and means that $L^2(\mathbb{R})$ is a real pre-Hilbert space.

(37) Completeness of $\mathcal{L}^2(\mathbb{R})$ – much like completeness of $\mathcal{L}^1(\mathbb{R})$. If $\{[f_n]\}$ is Cauchy in $\mathcal{L}^2(\mathbb{R})$ then we can extract a subsequence which is relabelled $f_j$ such that $\|f_{j+1} - f_j\|_{L^2} \le 2^{-j}$ for all $j \ge 1$. Now the inequality (33) shows that $f_j\chi_{[-N,N]}$ is absolutely summable in $L^1(\mathbb{R})$ for every $N$. Thus $f_j(x)$ converges a.e. on $[-N, N]$ for every $N$ which means (countable union of measure zero being measure zero) that it converges a.e. on $\mathbb{R}$. So any limit a.e., $f$, is locally integrable. We can either apply Lebesgue dominated convergence or observe that $g_1 = |f_1|^2$, $g_j = |f_j|^2 - |f_{j-1}|^2$ is absolutely summable (meaning in $L^1(\mathbb{R})$ since using Cauchy-Scwarz

(35)
$$\|g_j\|_{L^1} = \int ||f_j|^2 - |f_{j-1}|^2| \le \int |f_j - f_{j-1}||f_j + f_{j-1}| \le \|f_f - f_{j-1}\|_{L^2}(\|f_j\|_{L^2} + \|f_{j-1}|_{L^2}| \le C2^{-j+1}.$$

Thus it sums a.e. to an $L^1$ function, but it sums a.e. to $|f|^2$. Thus $f \in L^2(\mathbb{R})$. For $k > j$ the difference $|f_j - f_k|^2$ is in $L^1$ with $\int |f_j - f_k|^2 < C2^{-j}$ (from Cauchy-Schwarz) and converges a.e. to $|f_j - f|^2$ so by Fatou's Lemma this is in $L^1$ and has

(36) $$\|f_j - f\|_{L^2} \to 0,\ j \to \infty$$

so $f_j \to f$ in $L^2$.

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