18.102: NOTES ON HILBERT SPACE

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Abstract. Here are some notes summarizing and slightly reorganizing the material covered from Chapter 3 of Debnaith and Mikusiński

(1) Baire's theorem. Let M be a complete, non-empty, metric space. If $F_i \subset M$ are closed subsets for $i \in \mathbb{N}$ and

$$M = \bigcup_i F_i \tag{1}$$

then at least one of the F_i has non-empty interior, i.e. contains a ball of positive radius.

Proof. Assume on the contrary that each of the F_i has empty interior. Since M contains a ball of positive radius, $F_1 \neq M$. Thus there is a point $p_1 \in M \setminus F_1$ and since this set is open, for some $\epsilon_1 > 0$ $B(p_1, \epsilon_1) \cap F_1 = \emptyset$. Now, consider the set $B(p_1, \epsilon_1/3) \setminus F_2$. This must be non-empty since otherwise F_2 contains a ball, so we can choose $p_2 \in B(p_1, \epsilon_1/3)$ and $0 < \epsilon_2 < \epsilon_1/3$ such that $B(p_2, \epsilon_2) \cap F_2 = \emptyset$ and $B(p_2, \epsilon_2) \subset B(p_1, 2\epsilon_1/3)$. Proceeding step by step this gives a sequence $\{p_n\}$ in M such that

$$0 < \epsilon_n < \epsilon_{n-1}/3,\ B(p_n, \epsilon_n) \cap F_n = \emptyset, d(p_n, p_{n-1}) < \epsilon_{n-1}. \tag{2}$$

Thus the balls are nested, $B(p_n, \epsilon_n) \subset B(p_{n-1}, 2\epsilon_{n-1}/3)$ and the sequence is Cauchy, $\epsilon_n < 3^{-n+1}\epsilon_1$ and $f(p_n, p_j) < \epsilon_n$ for all $j \geq n$. By the assumed completeness of M, the sequence converges, $p_n \to p$, and the limit cannot be in any of the F_i since it lies inside the closed ball $\{q; |p - q_i| \leq 2\epsilon_i/3\}$ which does not meet F_i. This contradicts (1) so at least one of the F_i must have non-empty interior.

(2) Uniform boundedness Principle – which is another theorem. Suppose B is a Banach space, V is a normed space and $L_i : B \longrightarrow V$ is a sequence of bounded linear maps. If, for each $b \in B$, the set

$$\{L_i(b) \subset B\} \text{ is bounded (i.e. in norm) in } V \tag{3}$$

then

$$\sup_i \|L_i\| < \infty, \tag{4}$$

meaning that the sequence is uniformly bounded.

Proof. Let M be the complete metric space $\{b \in B; \|b\|_B \leq 1\}$, the closed unit ball in B. Consider the sets

$$F_n = \{b \in M; L_i(b) \leq n \ \forall\ i\}. \tag{5}$$

These are closed sets, since

$$F_n = \bigcap_i L_i^{-1}(\{|z| \leq n\} \cap M \tag{6}$$

which are all closed by the assumed continuity of the L_i. Moreover, the assumption (3) is that every point in M is in one of the F_n, so

(7) $$M = \bigcup_n F_n.$$

Baire's theorem therefore shows that at least one of the F_n has non-empty interior. This means that for some $b \in M$, and some $\epsilon > 0$,

(8) $$|b' - b| \le \epsilon \Longrightarrow |L_i(b)| \le n \ \forall\ i.$$

This set also contains an open ball in the interior of M so we can assume that (8) holds and $|b| + \epsilon < 1$. Now, the triangle inequality shows that

(9) $$|b'| < \epsilon \Longrightarrow |L_i(b')| \le |L_i(b + b')| + |L_i(b)| \le C$$

where we again use (3). Thus in fact $\|L_i\| \le C/\epsilon$.

(3) If $\{x_n\}$ is a weakly convergent sequence in a Hilbert space H, $x_n \rightharpoonup x$, meaning that for a fixed element $x \in H$ and all $v \in H$

(10) $$\langle x_n, v\rangle \longrightarrow \langle x, v\rangle$$

then $\sup_n \|x_n\| < \infty$.

Proof. We may apply the Uniform Boundedness Principle with $B = H$ the Hilbert space, $V = \mathbb{C}$ and $L_i(v) = \langle v, x_i\rangle$. These are bounded linear functionals, since

(11) $$|L_i(v)| \le \|x_i\|\|v\| \text{ by Schwarz inequality}$$

and in fact $\|L_i\| = \|x_i\|$ as follows by setting $v = x_i$. Moreover (3) holds since $\langle v, x_i\rangle \to \langle v, x\rangle$ so $\|L_i\|$ and hence $\|x_i\|$ are bounded in $\mathbb{R}$.

(4) Conversely if x_i is a bounded sequence in H and $\langle x_i, v\rangle \to \langle x, v\rangle$ for all v in a dense subset $D \subset H$ then $x_i \rightharpoonup x$.

(5) Suppose that $\|x\|, \|x_n\| \le C$ for all n and $w \in H$. Then there is a sequence $D \ni v_k \to w$. Given $\epsilon > 0$ choose k so large that

(12) $$2C\|w - v_k\| < \epsilon/2$$

then

(13) $$|\langle x_n, w\rangle - \langle x, w\rangle| \le |\langle x_n, v_k\rangle - \langle x, v_k\rangle| + |\langle x_n - x, w - v_k\rangle| \le |\langle x_n, v_k\rangle - \langle x, v_k\rangle| + \epsilon/2 < \epsilon$$

for n large. Thus $\langle x_n, w\rangle \to \langle x, w\rangle$ for all $w \in H$ so $x_n \rightharpoonup x$.

(6)

Theorem 1 (Open mapping theorem). *Let $T : B_1 \longrightarrow B_2$ be a bounded linear map between Banach spaces and suppose that T is surjective, then $T(O) \subset B_2$ is open for each open set $O \subset B_1$.*

Proof. Consider the open ball around the origin in B_1 of unit radius $B(0, 1)$. By the assumed surjectivity

(14) $$\bigcup_{k\in\mathbb{N}} T(B(0, k)) = B_2$$

since every point in B_1 is in the ball of radius k for k large enough. By Baire's theorem it follows that one of the $T(B(0, k))$ must have closure with non-empty interior in B_1, that is for some $p \in B_2$, some $\epsilon > 0$ and some k,

(15) $$B(p, \epsilon) \subset \overline{T(B(0, k)}.$$

This means that the set $T(B(0,k))\cap B(p,\epsilon)$ is dense in $B(p,\epsilon)$. In particular it is non-empty, so there is some $q\in B(0,k)$ such that $T(q)\in B(p,\epsilon)$. If $\delta>0$ is small enough then $B(T(q),\delta)\subset B(p,\epsilon)$ so $T(B(0,k))\cap B(T(q),\delta)$ is also dense in $B(T(q),\delta)$. So, reverting to the earlier notation, we can assume that $p=T(q)$, the centre of the ball, is in $T(B(0,k))$. The triangle inequality shows that $B(q,2k)\supset B(0,k)$. It follows that

$$B(0,\epsilon)\subset\overline{T(B(0,2k))}. \tag{16}$$

Indeed, if $f\in B(0,\epsilon)$ then $T(q)+f=\lim_n T(q_n)$, $q_n\in B(T(q),\epsilon)$. Thus $f=\lim_n T(q_n-q)$ and $q_n-q\in B(0,2k)$.

Now we use the linearity of T to make (16) look more uniform. Given $f\in B(0,\epsilon)$ and $\eta>0$, (16) asserts the existence of $u\in B(0,2k)$ such that $\|Tu-f\|<\eta$. Suppose that $f\in B_2$ and $f\neq 0$. Then $f'=\epsilon f/2\|f\|\in B(0,\epsilon)$. So there exists $u'\in B(0,2k)$ with $\|Tu'-f'\|<\epsilon\eta/2\|f\|$. This means that $u=2\|f\|u'/\epsilon$ shows that

$$\exists\ u\in B_1,\ \|u\|\le C\|f\|,\ \|Tu-f\|<\eta,\ C=2k/\epsilon. \tag{17}$$

So this is the uniformity – this is possible for every $f\in B_2$ and every $\eta>0$ with C independent of η and f – including $f=0$ for which we can take $u=0$ for any η.

So, now fix $f\in B(0,1)\subset B_2$ and choose a sequence $u_n\in B_1$ using (17). First apply (17) to f with $\eta=1/2$ and let u_1 be the result. Now, suppose by induction we have obtained u_k for $k\le n$ which satisfy

$$\|u_k\|\le 2^{-k}C,\ \|T(\sum_{k=1}^{n}u_k)-f\|\le 2^{-n}. \tag{18}$$

Then apply (17) again but to $f_{n+1}=f-T(\sum_{k=1}^{n}u_k)$ with $\eta=2^{-n-1}$ and let u_{n+1} be the result. So $\|u_{n+1}\|\le 2^{-n}C$ and

$$\|Tu_n-(f-T(\sum_{k=1}^{n-1}u_k))\|\le 2^{-n-1}. \tag{19}$$

This is just the inductive hypothesis (18) for $n+1$ so in fact we have (18) for every n. It follows that

$$v_n=\sum_{k=1}^{n}u_k\Longrightarrow\|v_n-v_m\|<2^{-|n-m|}C,\ \|v_n\|<C \tag{20}$$

so v_n is Cauchy in B_1, hence convergent by the assumed completeness of B_1. Thus $v_n\to V$ in B_1. However the second part of (18) shows that $Tv_n\to f$ so by the continuity of T, $Tv=f$. Moreover, $v\in B(0,3C)$ so we conclude that

$$T(B(0,3C))\supset B(0,1) \tag{21}$$

without taking the closure.

Now, from (21) it follows for any $\epsilon>0$, $T(0,3C\epsilon)\supset B(0,\epsilon)$. If O is open and $q\in T(O)$, so $q=T(p)$ for $p\in O$ then, for some $\epsilon>0$, $B(p,3C\epsilon)\subset O$ so

$$T(O)\supset T(B(p,3C\epsilon))\supset B(q,\epsilon) \tag{22}$$

and it follows that $T(O)$ is open. □

(7)

Corollary 1. *If* $T : B_1 \longrightarrow B_2$ *is a continuous linear map between Banach spaces which is 1-1 and onto then* $T^{-1} : B_2 \longrightarrow B_1$ *is continuous.*

Proof. For safety of notation, let $S : B_2 \longrightarrow B_1$ be the inverse of T. If $O \subset B_1$ is open then $S^{-1}(O) = T(O) \subset B_2$ is open by the Open Mapping Theorem, so S is continuous. □

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