Compactly of integration.

**Theorem 1.** The integration operator on \([a, b]\):

\[
(1) \quad L^2([a, b]) \ni u \mapsto -\int_a^x u(s) \, ds \in L^2([a, b])
\]

is a compact operator.

**Proof.** We need to show that \(A\) maps bounded sets into precompact sets, and to do this it suffices to show that for any sequence \(\{u_n\}\) with \(\|u_n\| \leq 1\) in \(L^2([a, b])\), \(v_n = Au_n\) has a convergent subsequence in \(L^2([a, b])\). In fact we may assume that \(u_n \rightharpoonup u\) is weakly convergent and then we will show that \(v_n \rightarrow v = Au\) is strongly convergent. So, observe first that \(v_n \in C([a, b])\).

\[
(2) \quad |A(w)(x) - A(w)(y)| = |\int_x^y w(s) \, ds| \leq |x - y|^{1/2} \|w\|_{L^2}.
\]

This shows that the sequence \(v_n = Au_n\) is (uniformly) equicontinuous (each \(v_n\) is uniformly continuous since it is continuous on a compact set). Moreover

\[
(3) \quad v_n(x) = \int_a^b \chi_{[a, x]} u_n(s) \, ds \rightarrow v(x) \text{ for each } x \in [a, b]
\]

since \(\chi_{[a, x]} \in L^2([a, b])\) and \(u_n \rightharpoonup u\). In fact it follows that \(v_n \rightarrow v\) uniformly on \([a, b]\) by combining (3) and (2) since given \(\epsilon > 0\) we may choose \(n\) so large that

\[
(4) \quad |v_n(x) - v(x)| \leq \epsilon/3 \forall x = a + m(b - a)/N, \ m \in \{0, \ldots, N\}, \ N < \epsilon^2/4.
\]

Then

\[
(5) \quad |v(x) - v_n(x)| \leq |v(x) - v(x')| + |v(x') - v_n(x')| + |v_n(x') - v_n(x)| < \epsilon
\]

where \(x'\) is one of the points in (4).

Now \(v_n \rightarrow v\) uniformly implies \(v_n \rightarrow v\) in \(L^2([a, b])\), proving the compactness of \(A\).

Let me first apply this to a ‘trivial’ problem. Suppose we are interested in the Dirichlet problem

\[
(6) \quad \frac{d^2 u(x)}{dx^2} + V(x) u(x) = f(x) \text{ on } [0, 1], \ u(0) = u(1) = 0.
\]

By the ‘trivial case’ I mean \(V \equiv 0\). Then of course we can solve (6) by integration, assuming for instance that \(f\) is continuous (but this works for \(f \in L^2([0, 1])\)). Thus,
integrating twice gives a solution of the differential equation

\[ \int_0^x \int_0^t f(s)dsdt. \]

From the discussion above we see immediately that

\[ Bf(0) = 0 \]

but on the other hand

\[ Bf(1) = \int_0^1 \int_0^t f(s)dsdt = \int_0^1 (1 - s)f(s)ds = B_1f \]

need not vanish. However there are solutions to the homogeneous equation, (6) with \( f \equiv 0 \), namely any linear function. Of course we don’t want to mess up the fact that \( Bf(0) = 0 \) so we should only add \( cx \) to this. Choosing \( c \) correctly, namely

\[ Af(x) = Bf(x) - (B_1f)x \]

ensures that \( u(x) = Af(x) \) satisfies (6) (at least if \( f \) is continuous) including the boundary conditions.

**Proposition 1.** The operator \( A \) in (9) is a compact and self-adjoint as an operator on \( L^2([0,1]) \).

**Proof.** Since \( x \) is a fixed function and \( B_1f \) is a constant, the extra term in (9) is finite rank. Thus the compactness of \( A \) follows from that of \( B \) – in fact \( B \) is the composite of two compact operators.

To see the self-adjointness of \( A \) we just need to compute its adjoint! First we can change the order of integration to write

\[ \int_0^x (x - s)f(s)ds. \]

Then, computing the adjoint of \( B \), by again changing the order of integration,

\[ \langle Bf, \phi \rangle = \int_0^1 \int_0^t (x - s)f(s)ds\phi(s)(x)dx = \int_0^1 \int_0^1 (x - s)f(s)\phi(s)dxds \]

\[ = \int_0^1 f(s)(B^*\phi)(s)dxds = \langle f, B^*\phi \rangle, \]

where

\[ (B^*\phi)(x) = \int_0^1 (s - x)\phi(s)ds = \int_0^1 (s - x)\phi(s)ds + \int_0^x (x - s)\phi(s)ds. \]

Similarly,

\[ \langle (B_1f)x, \phi \rangle = \int_0^1 \int_0^1 (1 - s)f(s)dx\phi(s)(x)dx \]

\[ = \int_0^1 f(x)(1 - x) \int_0^1 s\phi(s)dsdx = \langle f, (B_1\phi)x \rangle + \int_0^1 f(x)\int_0^1 (s - x)\phi(s)dsdx. \]

Thus

\[ \langle A^*x, \phi \rangle = \langle B - (B_1f)x \rangle^* = A. \]

Thus \( A \) is indeed a compact self-adjoint operator. Let’s compute its eigenvalues, using the uniqueness of solutions to ODEs (should I prove this?)
First the non-zero eigenvalues. If \( Au = \lambda u \) with \( \lambda \neq 0 \) then we can write
\[
u = \lambda^{-1} Au.
\]
Since \( Au \) is \( C^1 \) and vanishes at \( x = 0 \) and \( x = 1 \), so is and does \( u \). Moreover if \( u \) is \( C^1 \) then \( Au \) is \( C^2 \) and hence so is \( u \). In fact this argument shows that \( u \) is \( C^\infty \) but we do not need to check that, since by differentiating we see that
\[
\frac{d^2 u}{dx^2} = \lambda^{-1} u, \quad u(0) = u(1) = 0.
\]
So, now we can use the uniqueness of solutions to this ODE. There is a unique solution of the Cauchy problem, with \( u(0) = 0 \) and \( u'(0) = 1 \). In fact this function is \( \sin(x/\sqrt{-\lambda}) \). So \( \lambda = -T^2 < 0 \) necessarily and then for \( \sin(T) = 0 \) we must have \( T = \pi n, \ n \in \mathbb{N} \). Thus the general solution to \( Au - \lambda u \) with \( \lambda \neq 0 \) is
\[
u(x) = A \sin(\pi nx), \quad \lambda = \frac{-1}{\pi^2 n^2}, \ n \in \mathbb{N}.
\]
Now, we have to think about the null space of \( A, Au = 0 \). Differentiation once shows that
\[
\int_0^x u(s)ds = 0 \text{ a.e.}
\]
We already know that this implies that \( u = 0 \) in \( L^2([0,1]) \). Thus the null space is trivial.

Now, applying the spectral theorem, we conclude that
\[
\phi_n(x) = \sin(\pi nx), \ n \in \mathbb{N} \text{ is a complete orthonormal basis of } L^2([0,1]).
\]