18.102: NOTES ON DIRICHLET PROBLEM ON AN INTERVAL

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Abstract. Here are my notes for the lecture on November, about the Dirichlet problem for d^2/dx^2 on an interval.

Compactness of integration.

Theorem 1. *The integration operator on* $[a, b]$:

$$L^2([a,b]) \ni u \longmapsto Au(x) = \int_a^x u(s)ds \in L^2([a,b]) \tag{1}$$

is a compact operator.

Proof. We need to show that A maps bounded sets into precompact sets, and to do this it suffices to show that for any sequence $\{u_n\}$ with $\|u_n\| \le 1$ in $L^2([a,b])$, $v_n = Au_n$ has a convergent subsequence in $L^2([a,b])$. In fact we may assume that $u_n \rightharpoonup u$ is weakly convergent and then we will show that $v_n \to v = Au$ is strongly convergent. So, observe first that $v_n \in \mathcal{C}([a,b])$. In fact

$$|Aw(x) - Aw(y)| = |\int_x^y w(s)ds| \le |x-y|^{\frac{1}{2}} \|w\|_{L^2}. \tag{2}$$

This shows that the sequence $v_n = Au_n$ is (uniformly) equicontinuous (each v_n is uniformly continuous since it is continuous on a compact set). Morever

$$v_n(x) = \int_a^b \chi_{[a,x]} u_n(s)ds \to v(x) \text{ for each } x \in [a,b] \tag{3}$$

since $\chi_{[a,x]} \in L^2([a,b])$ and $u_n \rightharpoonup u$. In fact it follows that $v_n \to v$ uniformly on $[a,b]$ by combining (3) and (2) since given $\epsilon > 0$ we may choose n so large that

$$|v_n(x) - v(x)| \le \epsilon/3 \ \forall \ x = a + m(b-a)/N, \ m \in \{0, \dots, N\}, \ N < \epsilon^2/4. \tag{4}$$

Then

$$|v(x) - v_n(x)| \le |v(x) - v(x')| + |v(x') - v_n(x'| + |v_n(x') - v_n(x)| < \epsilon \tag{5}$$

where x' is one of the points in (4).

Now $v_n \to v$ uniformly imples $v_n \to v$ in $L^2([a,b])$, proving the compactness of A. □

Let me first apply this to a 'trivial' problem. Suppose we are interested in the Dirichlet problem

$$\frac{d^2u(x)}{dx^2} + V(x)u(x) = f(x) \text{ on } [0,1], \ u(0) = u(1) = 0. \tag{6}$$

By the 'trivial case' I mean $V \equiv 0$. Then of course we can solve (6) by integration, assuming for instance that f is continuous (but this works for $f \in L^2([0,1])$. Thus, integrating twice gives a solution of the differential equation

$$Bf(x)=\int_0^x\int_0^t f(s)dsdt. \tag{7}$$

From the discussion above we see immediately that $B : L^2([0,1]) \longrightarrow L^2([0,1])$ is compact. Clearly $Bf(0)=0$ but on the other hand

$$Bf(1)=\int_0^1\int_0^t f(s)dsdt=\int_0^1(1-s)f(s)ds=B_1f \tag{8}$$

need not vanish. However there are solutions to the homogeneous equation, (6) with $f\equiv 0$, namely any linear function. Of course we don't want to mess up the fact that $Bf(0)=0$ so we should only add cx to this. Choosing c correctly, namely

$$Af(x)=Bf(x)-(B_1f)x \tag{9}$$

ensures that $u(x)=Af(x)$ satisfies (6) (at least if f is continuous) including the boundary conditions.

Proposition 1. *The operator A in* (9) *is a compact and self-adjoint as an operator on* $L^2([0,1])$.

Proof. Since x is a fixed function and B_1f is a constant, the extra term in (9) is finite rank. Thus the compactness of A follows from that of B – in fact B is the composite of two compact operators.

To see the self-adjointness of A we just need to compute its adjoint! First we can change the order of integration to write

$$Bf(x)=\int_0^x(x-s)f(s)ds. \tag{10}$$

Then, computing the adjoint of B, by again changing the order of integration,

$$\begin{aligned}\langle Bf,\phi\rangle&=\int_0^1\int_0^x(x-s)f(s)ds\overline{\phi(x)}dx=\int_0^1\int_s^1(x-s)f(s)\overline{\phi(x)}dxds\\&=\int_0^1 f(s)\overline{(B^*\phi)(s)}dxds=\langle f,B^*\phi\rangle,\ \text{where}\\(B^*\phi)(x)&=\int_x^1(s-x)\phi(s)ds=\int_0^1(s-x)\phi(s)ds+\int_0^x(x-s)\phi(s)ds.\end{aligned} \tag{11}$$

□

Similarly,

$$\begin{gathered}\langle (B_1f)x,\phi\rangle=\int_0^1\int_0^1(1-s)f(s)dsx\overline{\phi(x)}dx\\=\int_0^1 f(x)\overline{(1-x)\int_0^1 s\phi(s)ds}dx=\langle f,(B_1\phi)x\rangle+\int_0^1 f(x)\overline{\int_0^1(s-x)\phi(s)ds}dx.\end{gathered} \tag{12}$$

Thus

$$A^*=(B-(B_1f)x)^*=A. \tag{13}$$

Thus A is indeed a compact self-adjoint operator. Let's compute its eigenvalues, using the uniqueness of solutions to ODEs (should I prove this?)

First the non-zero eigenvalues. If $Au = \lambda u$ with $\lambda \neq 0$ then we can write $u = \lambda^{-1}Au$. Since Au is $\mathcal{C}^1$ and vanishes at $x = 0$ and $x = 1$, so is and does u. Moverover if u is $\mathcal{C}^1$ then Au is $\mathcal{C}^2$ and hence so is u. In fact this argument shows that u is $\mathcal{C}^\infty$ but we do not need to check that, since by differentiating we see that

$$\frac{d^2u}{dx^2} = \lambda^{-1}u,\ u(0) = u(1) = 0. \tag{14}$$

So, now we can use the uniqueness of solutions to this ODE. There is a unique solution of the Cauchy problem, with $u(0) = 0$ and $u'(0) = 1$. In fact this function is $\sin(x/\sqrt{-\lambda})$. So $\lambda = -T^2 < 0$ necessarily and then for $\sin(T) = 0$ we must have $T = \pi n$, $n \in \mathbb{N}$. Thus the general solution to $Au - \lambda u$ with $\lambda \neq 0$ is

$$u(x) = A\sin(\pi n x),\ \lambda = \frac{-1}{\pi^2 n^2},\ n \in \mathbb{N}. \tag{15}$$

Now, we have to think about the null space of A, $Au = 0$. Differentiation once shows that

$$\int_0^x u(s)ds = 0 \text{ a.e.} \tag{16}$$

We already know that this implies that $u = 0$ in $L^2([0,1])$. Thus the null space is trivial.

Now, applying the spectral theorem, we conclude that

$$\phi_n(x) = \sin(\pi n x),\ n \in \mathbb{N} \text{ is a complete orthonormal basis of } L^2([0,1]). \tag{17}$$

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