Basic Notions of Quantum Network Science

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July 28, 2014

A brief history of quantum computation

and

A brief introduction to Network Science

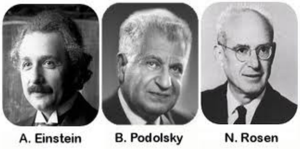
A (qu)Bit of History

Early 1900's : Birth of Quantum Mechanics

A (qu)Bit of History

Early 1900's : Birth of Quantum Mechanics

1935 : EPR paper



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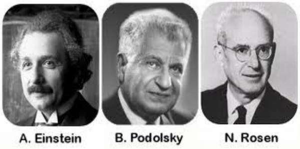
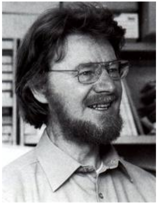


1964 : John Bell



A (qu)Bit of History

It's kind of quiet

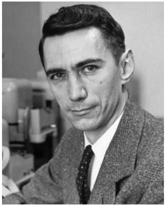
A (qu)Bit of History

It's kind of quiet

BUT

Classical Computation and Communication is Roaring

Claude Shannon 1948



Alan Turing 1936

A (qu)Bit of History

It's kind of quiet

Richard Feynman : 1982

A (qu)Bit of History

It's kind of quiet

Richard Feynman : 1982

Quantum computation remains a semi- crackpot science

A (qu)Bit of History

It's kind of quiet

Richard Feynman : 1982

Peter W. Shor: 1994

A (qu)Bit of History

It's kind of quiet

Richard Feynman : 1982

Peter W. Shor: ~~1994~~ 1997

A (qu)Bit of History

Entanglement becomes part of science

Daniel Salart et al,
Testing the speed of 'spooky action at a distance'
Nature 454, 861-864 (14 August 2008)

Why bother with building a quantum computer?

- Certainly a reason is factoring and the risk for our current security systems
- Search algorithms (Grover's) give a quadratic speed up
- Super-dense coding, teleportation, quantum Fourier transform.
- Quantum computers can help us understand quantum many-body systems
- Quantum algorithms (existing and yet to come) help accomplish tasks that are 'practically' impossible or very slow on our current computers
- etc.

Why bother with building quantum technologies?

- It is a misconception that the goal of quantum information science depends solely on one day having a QC
- Quantum encryption offers further security for communication
- There is quantum communication (things we do)
- Cryptography
- Many quantum technologies (existing and yet to come)
    - Many of these are happening and can happen well before one day having a large scale QC

Network Science: Age of Complexity

ECONOMY **Factoid:**

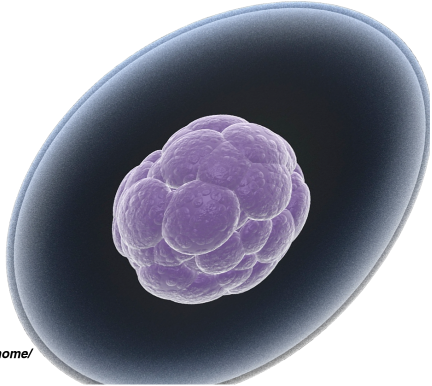
Network Science: Age of Complexity

THE HUMAN CELL | **Factoid:**

How Many Genes are in the Human Genome?

23,299

http://www.ornl.gov/sci/techresources/Human_Genome/faq/genenumber.shtml



Network Science: Age of Complexity

COMPLEX SYSTEMS

Complex

[adj., v. kuh m-pleks, kom-pleks; n. kom-pleks]
–adjective
1.
composed of many interconnected parts; compound; composite: a complex highway system.
2.
characterized by a very complicated or involved arrangement of parts, units, etc.: complex machinery.
3.
so complicated or intricate as to be hard to understand or deal with: a complex problem.

Source: Dictionary.com

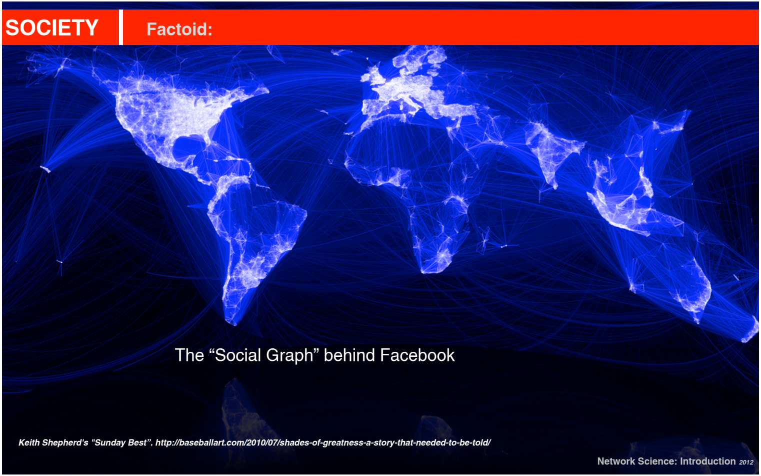
Complexity, a **scientific theory** which asserts that some systems display behavioral phenomena that are completely inexplicable by any conventional analysis of the systems' constituent parts. These phenomena, commonly referred to as emergent behaviour, seem to occur in many complex systems involving living organisms, such as a stock market or the human brain.

Source: John L. Casti, Encyclopædia Britannica

Complexity

Network Science: Introduction 2012

Network Science: Age of Complexity



Network Science: Age of Complexity

Network Science: Historical remarks

THE HISTORY OF NETWORK ANALYSIS

Graph theory: 1735, Euler

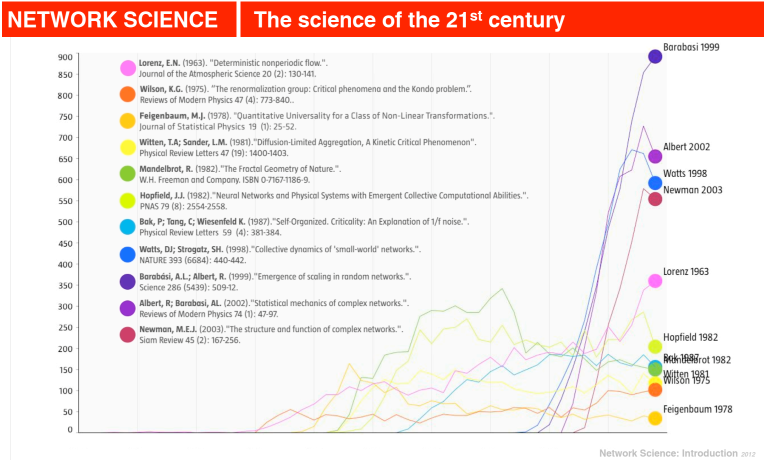
Social Network Research: 1930s, Moreno

Communication networks/internet: 1960s

Ecological Networks: May, 1979.

Network Science: Introduction *2012*

Network Science: Historical remarks



Network Science: Historical remarks

THE TOOLS OF MODERN NETWORK THEORY

- > **Graph theory**
- > **Social network theory**
- > **Statistical physics**
- > **Computer science**
- > **Biology**
- > **Statistics**

Network Science: Introduction 2012

Network Science: References

Original papers:

• 1998: Watts-Strogatz paper in the most cited **Nature** publication from 1998; highlighted by ISI as one of the ten most cited papers in physics in the decade after its publication.

•1999: Barabasi and Albert paper is the most cited **Science** paper in 1999;highlighted by ISI as one of the ten most cited papers in physics in the decade after its publication.

•2001: Pastor -Satorras and Vespignani is one of the two most cited papers among the papers published in 2001 by **Physical Review Letters**.

•2002: Girvan-Newman is the most cited paper in 2002 **Proceedings of the National Academy of Sciences**.

Network Science: Introduction 2012

Network Science: References

REVIEWS:

•The first review of network science by Albert and Barabasi, 2001) is the second most cited paper published in **Reviews of Modern Physics**, the highest impact factor physics journal, published since 1929. The most cited is ***Chandaseklar***'s 1944 review on solar processes, but it will be surpassed by the end of 2012 by Albert ***et al.***

•The SIAM review of Newman on network science is the most cited paper of any **SIAM journal**.

•BIOLOGY: "Network Biology", by Barabasi and Oltvai (2004) , is the second most cited paper in the history of **Nature Reviews Genetics**, the top review journal in genetics.

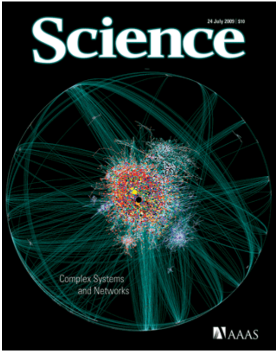
Network Science: Introduction 2012

Network Science: References

JOURNAL

•**Science**:

Special Issue for the 10 year anniversary of Barabasi&Albert 1999 paper.



Network Science: Introduction 2012

Classical and Quantum Physics (Basics)

Classical state

For n particles, the state at any given time is given by $6n$ parameters

$$(\mathbf{x_1}, \mathbf{x_2}, \cdots, \mathbf{x_n}) \quad \text{and} \quad (\mathbf{v_1}, \mathbf{v_2}, \cdots, \mathbf{v_n})$$

each $\mathbf{x_i}$ has 3 components and each $\mathbf{v_i}$ has 3 components.
The state can be seen as a point in the $6n$ dimensional phase space.

Classical evolution

Given the state at any given time t_0 **and** the laws of interaction among particles, we can in principle, integrate equations of motion (e.g. $\mathbf{F} = m\mathbf{a}$) and specify the state at another time t **deterministically.**

Quantum state: pure states

For n particles, the state at any given time is given by

$$|\psi\rangle = \sum_{i_1,\cdots,i_N=1}^{d} \psi^{i_1,i_2,\cdots,i_N} \quad |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle$$

d^N number of parameters

Each particle (e.g. spin) contributes d (e.g. 2 for electrons) dimensions multiplicatively.

Entanglement

If a state $|\psi\rangle$ is entangled

$$|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$$

Generally a state of a composite system becomes entangled because of interaction of the constituents.

Quantifying Entanglement

There are many quantifications. A particularly important one is

Theorem

(Schmidt Decomposition, aka SVD) ***Suppose*** $|\psi_{AB}\rangle$ ***is the pure state of a composite system, AB. Then there exists orthonormal states*** $|\Phi_A^\alpha\rangle$ ***for*** A ***and orthonormal states*** $|\theta_B^\alpha\rangle$ ***of system*** B ***s. t.***

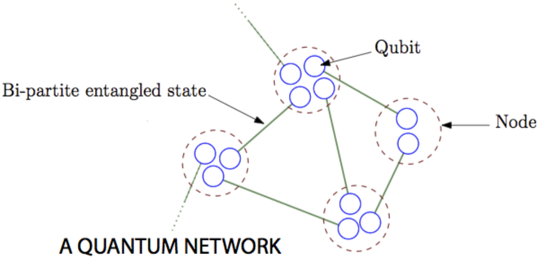
$$\psi\rangle = \sum_\alpha \lambda_\alpha |\Phi_A^\alpha\rangle \otimes |\theta_B^\alpha\rangle$$

where λ_α ***are non-negative real numbers satisfying*** $\sum_\alpha \lambda_\alpha^2 = 1$ ***known as Schmidt coefficients.***

The number of non-zero Schmidt numbers is called the **Schmidt rank** , χ, of the state (a quantification of entanglement) .

Quantum Networks (Basics)

Goal of quantum communication



- Allow quantum communication between distant points on a grid (i.e., a graph or a lattice). For example a quantum internet.
- Specifically, create an entanglement between distant nodes on a grid: can do
  - Super dense coding, Teleportation etc.

No Cloning theorem

A quantum state cannot be copied. Namely this is NOT allowed

$$|\psi\rangle|0\rangle \rightarrow |\psi\rangle|\psi\rangle$$

Classically of course one can do this and take a majority vote to protect agains errors.

Teleportation

Idea: Move quantum states around in the absence of any communication (even a quantum channel connecting sender and receiver) .

1. Alice and Bob met long ago and generated an EPR pair. Then, each holding one of the qubits, they went far away (Bob is hiding) .
2. Alice wants to send a qubit $|\psi\rangle$ to Bob if she chooses to. Alice doesn't know the state of the qubit and she can only classically communicate with Bob (e.g., phone call)
    1. To measure the qubit first, would destroy its current state.
    2. To classically describe the state of $\psi\rangle$ takes infinite bits of information.

Teleportation: Overview of the method (N&C book)

1. Alice interacts $|\psi\rangle$ with her half of the EPR pairs and measures it.
    1. She can get 00, 01, 10, 11
2. She then calls Bob and tells him what he got
3. Depending on what Alice says, Bob measures his half of the EPR pair
4. By doing so can recover $|\psi\rangle$.

Teleportation

Looking at

$$
\begin{aligned}
|\psi_2\rangle &= \frac{1}{2}\{|00\rangle(\alpha|0\rangle+\beta|1\rangle)+|01\rangle(\alpha|1\rangle+\beta|0\rangle) \\
&+ |10\rangle(\alpha|0\rangle-\beta|1\rangle)+|11\rangle(\alpha|1\rangle-\beta|0\rangle)\}
\end{aligned}
$$

Quantum computation 27

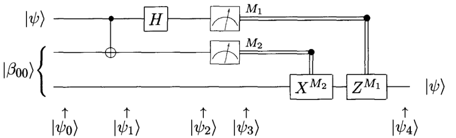


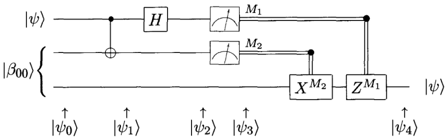
Figure 1.13. Quantum circuit for teleporting a qubit. The two top lines represent Alice's system, while the bottom line is Bob's system. The meters represent measurement, and the double lines coming out of them carry classical bits (recall that single lines denote qubits).

Teleportation

Alice measurement $\implies$ Bob's qubit state .

$$
\begin{aligned}
00 &\to |\psi_3(00)\rangle \to |\psi_{\text{bob}}\rangle = \alpha|0\rangle + \beta|1\rangle \\
01 &\to |\psi_3(01)\rangle \to |\psi_{\text{bob}}\rangle = \alpha|1\rangle + \beta|0\rangle \\
10 &\to |\psi_3(10)\rangle \to |\psi_{\text{bob}}\rangle = \alpha|0\rangle - \beta|1\rangle \\
11 &\to |\psi_3(11)\rangle \to |\psi_{\text{bob}}\rangle = \alpha|1\rangle - \beta|0\rangle
\end{aligned}
$$

Quantum computation 27



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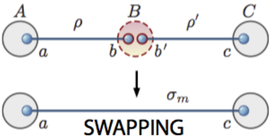
Entanglement Swapping

Suppose Alice has an EPR pair with Bob. Moreover, Bob has an EPR pair with Carol. So Alice has one qubit, Bob 2 and Carol one



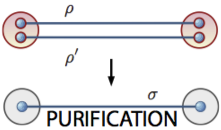
Bob can treat his first EPR (shared with Alice) as the state to be teleported to Carol. He measures it as before, calls Carol and Carol does his measurement. The final state that Carol has is half of the EPR pair with Alice.

Entanglement Distillation

Distillation: Convert a large number of known pure states $|\psi\rangle$ to as many $|\beta_{00}\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ as possible using LOCC, requiring not to succeed exactly, but with high fidelity. Then using the Bell pairs, they can quantum communicate. A way of quantifying how much entanglement $\psi\rangle$ possesses.

$$m \text{ copies of } |\psi\rangle \to n \text{ copies of } |\beta_{00}\rangle \quad \text{high fidelty}$$

$$n/m: \quad \text{distillable entanglementof } |\psi\rangle$$



Entanglement Dilution

Dilution (the opposite) : Use a large number of Bell states to produce as many copies of a desired $|\psi\rangle$ as possible using LOCC with high fidelity

$$n \text{ copies of } |\beta_{00}\rangle \to m \text{ copies of } |\psi\rangle \quad \text{high fidelty}$$

$$n/m: \quad \text{entanglement of formation of } |\psi\rangle$$

By no means obvious but for a pure state $\psi\rangle$ both processes give the same n/m.

Entanglement decay with distance

How does entanglement diminish with distance for whatever reason? e.g., LOCC not being perfect, the channel being noisy, etc.

Answer: exp(physical distance).

Current technologies:

- 150 km between stations
- 250 km in the lab

Protocols for quantum networks

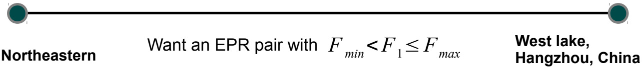
1. "Quantum Repeaters for Communication", H.-J. Briegel, W. Dür, J. I. Cirac, P. Zoller,
   PRL 81, 26, (1998)
2. "Entanglement Percolation in Quantum Networks", Antonio Acin, J. Ignacio Cirac, and Maciej Lewenstein,
   Nature Physics 3, 256 - 259 (2007)
3. "Distribution of entanglement in large-scale quantum networks" S. Persequers, G. Lapeyre, D. Calvalcanti, M Lewenstein.
   Reports on Progress in Physics 76 (2013).
4. "Cascading dynamics in complex quantum networks" Liang Huang and Ying-Cheng Lai
   Chaos 21, 025107 (2011).
5. "Quantum random networks" S Perseguers, M Lewenstein, A Acín, JI Cirac
   Nature Physics, 6(7), 539-543 (2010).

Quantum Repeaters: Summary

Allow communication between distant nodes (inspired by classical repeaters):

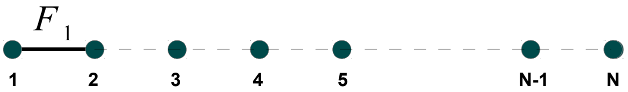
1. Method for creating entanglement between distant nodes using intermediate "connecting points" and a nested purification
2. entanglement purification with imperfect means
3. Protocol for which the time needed for entanglement creation scales polynomially whereas the required resources per connection point grow only logarithmically with distance.

Quantum Repeaters: Naive way



Want an EPR pair with $F_{min} < F_1 \leq F_{max}$

Quantum Repeaters: Naive way

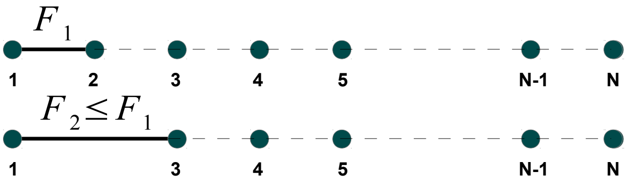


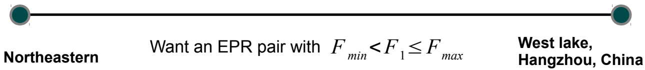
Northeastern

Want an EPR pair with $F_{min} < F_1 \leq F_{max}$

West lake, Hangzhou, China

Quantum Repeaters: Naive way





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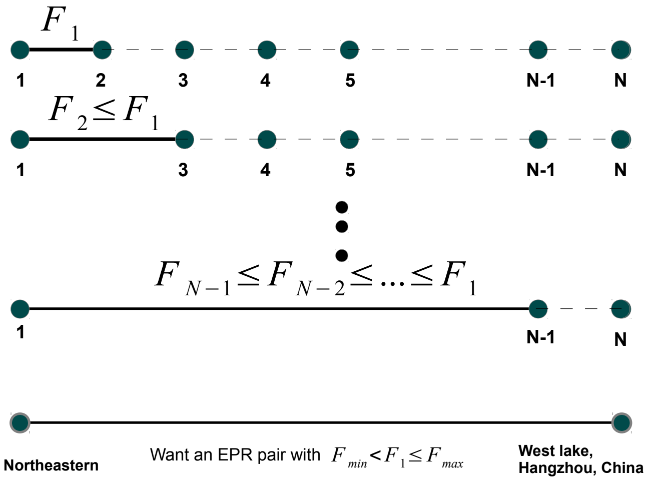
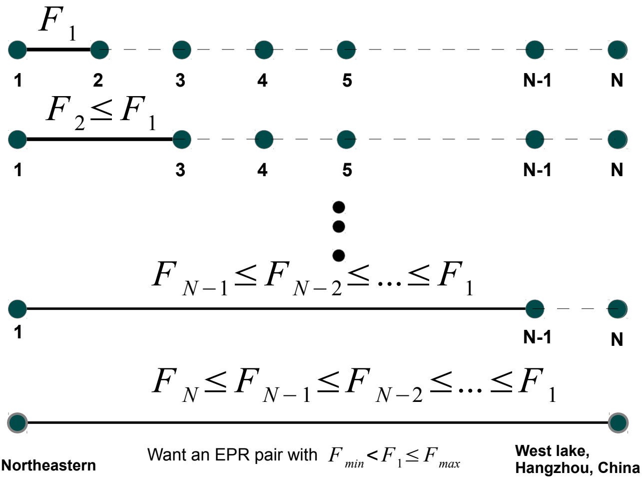


Quantum Repeaters: Naive way



Quantum Repeaters: Naive way

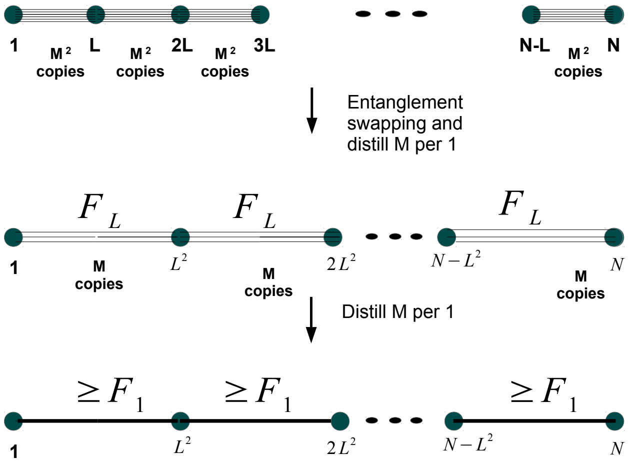
Why isn't it good enough

- Every step of entanglement swapping decreases the fidelity
- F_N decreases with distance N exponentially.
- For sufficiently large N, we are bound to get $F_N < F_{min}$
- One needs to use $\exp(N)$ pairs to hope to succeed.

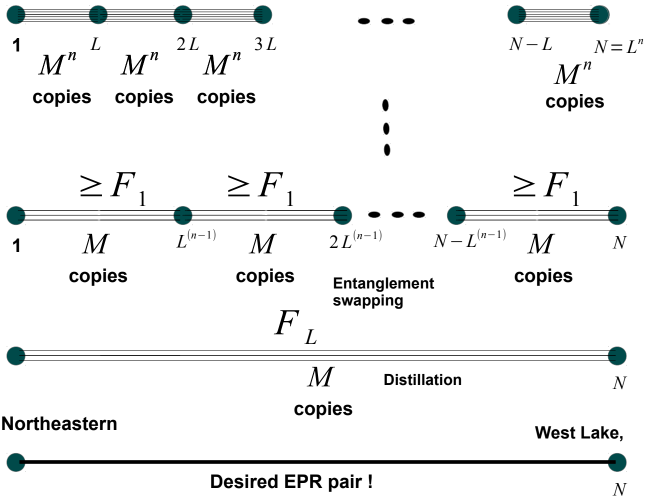
Quantum Repeaters: Actual

Suppose we have $N = L^n$ stations.

Quantum Repeaters



Quantum Repeaters

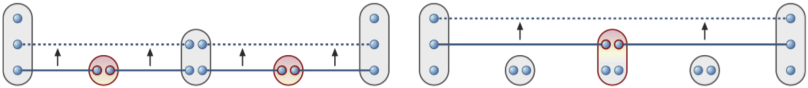


Quantum Repeaters

In the actual quantum repeater scheme we have $N = L^n$ and need R pairs

$$\begin{aligned} R &= (LM)^n = NM^n \\ &= N^{\log_L M + 1} \end{aligned}$$

We used $c^d = b^{d \log_b c}$.



QUANTUM REPEATERS

Reference

Overview: uses network science/ and statistical mechanics for Quantum information theory

"Entanglement Percolation in Quantum Networks",
Antonio Acin, J. Ignacio Cirac, and Maciej Lewenstein,
Nature Physics 3, 256 - 259 (2007)

Goal

Design of an efficient quantum network depends on:

- Way the nodes are connected (underlying graph)
- Entanglement between the nodes

Goal: Let the state of a general network be ρ and let A and B be two distant nodes:

- Find measurements and LOCC to put A and B in a maximally entangled 'singlet' with maximum probability.
  - Denote this probability by Singlet Conversion Probability (SCP)

Networks considered here

- Nodes are spatially distributed in a regular way
- Each pair of nodes is connected by a pure state $\varphi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$

$$\varphi\rangle = \sum_{i=1}^{d} \sqrt{\lambda_i} |ii\rangle$$

Call them Pair Entangled Pure Networks

What they do:

- Introduce various protocols related to classical percolation
- Determine optimal protocols for $1D$ systems (a little counter intuitive)
- Introduce various protocols in $2D$, where dramatic improvement over classical percolation is obtained
  - Can obtain perfect quantum communication
  - Classical percolation implies an exponential decay of success rate with the number of intermediate nodes

Method

Consider a natural measurement that converts all pairs of nodes into singlets with optimal probability p^{ok}.
SCP for $\varphi\rangle$ is equal to $p^{ok} = \min(1, 2(1-\lambda_1))$.
[Vidal, PRL 83, 1046-1049 (1999)]

- a perfect quantum channel is established with prob p^{ok}, otherwise no entanglement

"This is analogous to Bond Percolation" : one distributes "entanglement" between nodes in a probabilistic manner.
Denote this measurement strategy :

"Classical Entanglement Percolation" (CEP)

Classical Entanglement Percolation (CEP)

In "Bond Percolation", there exists a threshold p_{th} such that as $n \to \infty$, then there is an infinite connected cluster with positive measure probability.

- If $p > p_{th}$, let $\theta(p)$ be the prob. that one node is in the cluster then for A and B to be in the cluster, the prob. is $\theta^2(p)$.
- If $p < p_{th}$, then this prob decays exponentially with the number of separating N nodes.

Classical Entanglement Percolation (CEP)

Threshold probability determines whether CEP is possible or not.

- $1D$, CEP is possible iff $p^{ok} = 1$
- $2D$, CEP is possible (derived from percolation arguments) if $p^{ok} = 2(1 - \lambda_1) = \frac{1}{2}$.

CEP shows that distribution of entanglement through quantum networks defines a critical phenomenon.

For instance, for $2D$ lattices, $\exists$ a continuum of λ_2 such that CEP is possible.

Is CEP optimal for any geometry and number of nodes: 1D

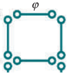
Is CEP optimal? If not, is it at least asymptotically optimal?
In $1D$, and finite N, CEP is not optimal, but asymptotically it is:

- Use a single repeater you get a channel working with p^{ok} (via entanglement swapping)
- Use percolation (CEP), then SCP is simply $\left(p^{ok}\right)^2$.

However, asymptotically the concurrence (another measure of entanglement) decreases exponentially with the number of repeaters unless every link is already maximally entangled. Exp decay of SCP follows.

Is CEP optimal for any geometry and number of nodes: 2D

CEP is not optimal:



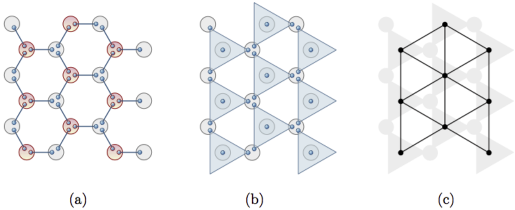
Take a square lattice and demand an entanglement for diagonal nodes. The SCP obtained from CEP is $1-\left(1-\left(p^{ok}\right)^2\right)^2$.
By concatenating the optimal measurement strategy for the one-repeater configuration, the SCP is $1-\left(1-p^{ok}\right)^2$.
HOWEVER,
You can do even better. Can design strategies where a singlet can be established whenever $\frac{1}{2} \leq \lambda_1 \leq 0.6498$.
$\therefore$ There are network geometries where, although the connections are not maximally entangled, the entanglement is still sufficient to establish a perfect quantum channel.

Beyond classical percolation: Example showing CEP not optimal

Consider a honeycomb lattice. If you choose a conversion probability smaller than the percolation threshold for honeycomb lattice, then

$$\lambda_1 = \sqrt{1 + \sin\frac{\pi}{18}} \approx 0.82$$

which is useless for CEP (Requirement: $\frac{1}{2} \leq \lambda_1 \leq 0.6498$).



BOND-TO-SITE PERCOLATION
HONEYCOMB TO TRIANGULAR LATTICE

However by entanglement swapping can turn the honeycomb to a triangular lattice. SCP for the new bond is

$$2\lambda_2 = 2\left(1 - \sqrt{1 + \sin\frac{\pi}{18}}\right) \approx 0.358 > 2\sin\left(\frac{\pi}{18}\right)$$

the nodes can now apply CEP to the new (triangular) lattice and succeed.

Comments

Although previous strategies used percolation concepts, this is not necessarily the case for optimal (unknown) strategy.

The results do not exclude the possibility of finding new strategies where very little entanglement is sufficient to allow perfect communication.

So the entanglement phase transition, can be called entanglement percolation. The critical parameter is the minimal amount of entanglement needed for perfect communication with polynomially decaying probability.

Open: what are more optimal strategies?

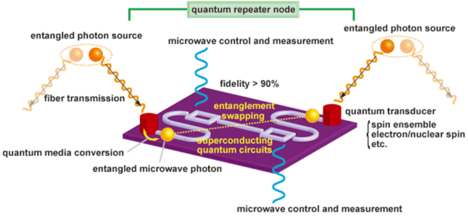
Cascading Dynamics in Complex Quantum Networks

Chaos 21, 025107 (2011)
Huang & Lai

Quantum Repeater

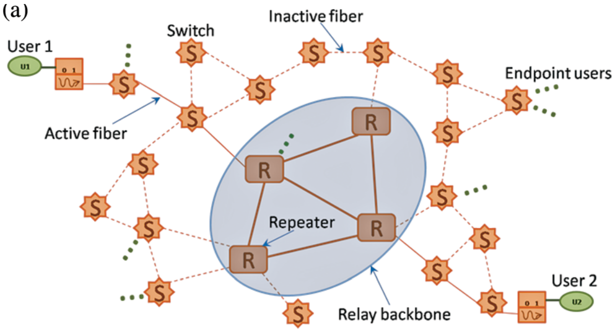
The distribution of quantum states over long distances is essential for future applications such as **quantum key distribution** (QKD) and quantum networks. The direct distribution of quantum states is limited by unavoidable transmission losses of the channel used to transmit these quantum states.

A promising alternative for long distance quantum states distribution is the use of **quantum repeaters**. A *quantum repeater* contains a light-matter interface, a quantum memory (e.g., in the form of an ensemble of atoms), and elementary quantum gates. Upon receiving a photon, a *quantum repeater* stores it in the memory and produces a new photon with the same quantum characters. *Quantum repeater* can also introduce desirable delays. A network of quantum repeaters can then transport an entangled photon carrying, e.g., a quantum key, to any part of the network.



http://www.uqcc.org/research/repeater/subjectC2013_01.html

Proposed hybrid QKD network



The proposed hybrid quantum network transports entangled photons. There are two components in such a network: **quantum repeaters** for long-distance information transport and local networks of fibers and **switches**. Each switch or repeater can have many endpoint users and those attached to repeaters have high priority in terms of security.

Photon loss as a soft cascading process

Motivation: Before constructing realistic complex quantum networks, we need to understand the **photon loss** phenomenon (triggered by external perturbations and internal traffic fluctuations). Photon loss can be modeled as a **soft cascading process**:

1. A node (either repeater or switch) usually has a **finite bandwidth B**. If a node receives too many photons at a given time, it can be jammed. For a switch-type of node, some photons will be lost.

2. While some proper routing protocol can divert some information-carrying photons to other nodes, more nodes will get jammed if they are operating near the limits of their respective bandwidths. The jamming can thus spread on the quantum network in a manner of a **cascading process**.

3. In contrast to a hard cascading process where failed nodes are disabled and are completely cut off from the network, here it is a **soft type cascading process** leading to only loss of photons, because jammed nodes are still expected to work within their respective bandwidths.

Key Result: Surprisingly, the degree of photon loss can reach maximum when the network is neither sparsely nor densely connected.

Implication: This result can be used to guide network design to mitigate photon loss and enhance the communication efficiency of quantum networks.

Photon loss as a soft cascading process

1. Assume that the photons travel along their respective **shortest paths**, i.e., **paths having minimum sum of node and link weights**. (For simplicity, we assume link weights are all one.)

2. **Node weights are dynamical** and can be adjusted to mitigate the photon flow. The initial node weights are set to be zero.

3. During each time unit τ, we assume that there are β photons to be communicated between each pair of nodes. In the steady state, within a time unit τ, the number of photons arriving at node i is (β b_i), where b_i is the number of shortest paths passing through this node, i.e., its **betweenness centrality**.

4. To ensure normal operation, we have τ $B_i > \beta$ b_i, where B_i is the bandwidth of node i. In general, we can set $B_i = C$ b_i, where C is the bandwidth parameter. Here, we have neglected the detailed processes for the repeaters and switches to transmit a photon, but assumed that each node, disregarding its type, can be built with sufficient bandwidth as required by the topological connection. As β approaches C τ, jamming can occur due to random traffic fluctuations.

5. When jamming occurs, we increase the weight of this node by one to mitigate the load running through it. This can lead to load increase and consequently jamming at nodes that are originally not jammed, i.e., soft cascading.

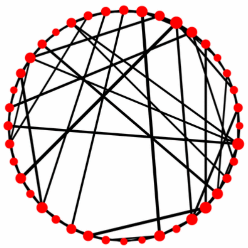
Quantify Photon loss

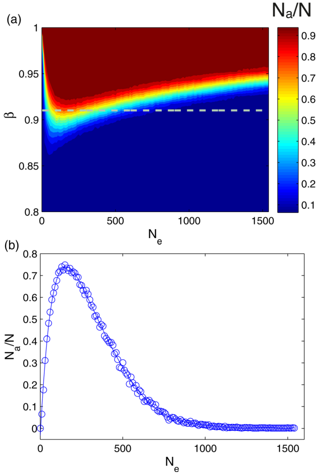
1. Apply a perturbation to the node possessing the largest bandwidth and increase its weight by a unit to trigger a soft cascading process.

2. In the steady state, the number N_a of nodes that have been jammed in the process, i.e., nodes with weight larger than one, characterizes the degree of photon loss.

3. One can study the number of affected nodes N_a as a function of network parameters.

Photon loss in small-world networks

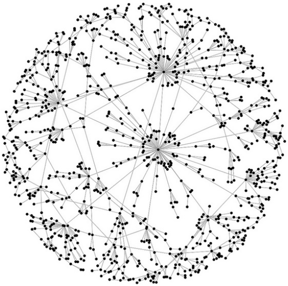


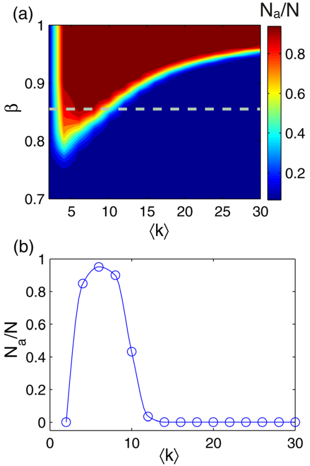


The regular backbone network is assumed to have a ring-like structure of N nodes, where each node is connected to m nearest-neighbor nodes, and the number of random shortcuts is N_e.

Photon loss reaches maximum when the network is neither sparse nor dense.

Photon loss in scale-free networks

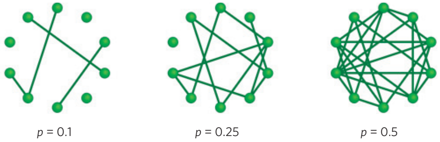




Photon loss reaches maximum when the network is neither sparse nor dense.

First standard networks

Suppose you have $G(V, E)$, V is the set of N nodes and E is the set of L edges.
Each pair of nodes get connected with probability p.
Certain features (e.g. subgraphs) appear sharply at p_c, where generically F is observed.



Features appearing: Still Classical

Suppose $p_c(N) = cN^{-n/\ell}$ for F with n nodes and ℓ edges and c independent of N. Then, classically

Table 1 | Thresholds for the appearance of classical subgraphs.

z	$-\infty$	-2	$-\frac{3}{2}$	$-\frac{4}{3}$	-1	$-\frac{2}{3}$
F						

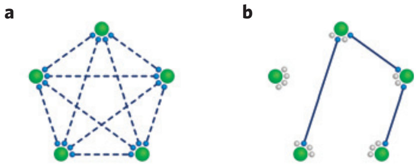
Some critical probabilities, according to equation (1), at which a subgraph F appears in random graphs of N nodes connected with probability $p \sim N^z$: cycles and trees of all orders appear at $z = -1$, whereas complete subgraphs (of order four or more) appear at a higher connection probability[30].

Quantum Random Network

Extend p to density matrix $\rho = |\varphi\rangle\langle\varphi|$

$$\varphi\rangle = \sqrt{1-p/2}|00\rangle + \sqrt{p/2}|11\rangle$$

$p = 1$ Bell pair
$p = 0$ separable state



conversion to a Bell state

Suppose every pair succeeds in tuning p such that the state is converted to $\frac{1}{\sqrt{2}}(00\rangle + 11\rangle)$. Therefore the task of determining what type of maximally entangled states remaining after these conversions is mapped exactly to the classical problem with F's emerging as previously:

Table 1 | Thresholds for the appearance of classical subgraphs.

z	$-\infty$	-2	$-\frac{3}{2}$	$-\frac{4}{3}$	-1	$-\frac{2}{3}$
F					 	

Some critical probabilities, according to equation (1), at which a subgraph F appears in random graphs of N nodes connected with probability $p \sim N^z$: cycles and trees of all orders appear at $z = -1$, whereas complete subgraphs (of order four or more) appear at a higher connection probability[30].

Main Results of this work: Collapse of the critical exponents

They have a protocol that utilizes joint measurement that gives: In the limit that $N \to \infty$ and $p \sim N^{-2}$ one is able to obtain, with probability approaching unity, any finite subgraph (multipartite entangled). Moreover, some particular states of interest in QIT (e.g., W states, Dicke states) also arise at $z = -2$.
$z = -2$ is optimal as with $z < -2$ the state is close to a product state

$$\left|\langle G_{N,p}|0\cdots 0\right|^2 = \left(1 - \frac{p}{2}\right)^{\frac{N(N-1)}{2}} \approx \exp\left(\frac{-N^{z+2}}{4}\right) \to 1 \quad .$$