**Problem 1: Nonlinear diffusion (20 Points)**

Consider a concentration field $c(t, x)$ defined on $x \geq 0$ and governed by the nonlinear diffusion equation

$$\frac{\partial c}{\partial t} = D \partial_x \left[ c^p (\partial_x c) \right]$$  \hspace{1cm} (1)

where $p$ and $D$ are strictly positive constants. Show that the self-similar solution of the form

$$c(t, x) = \frac{M^{2/(2+p)}}{(D^t)^{1/(2+p)}} F(\xi), \quad \xi = \frac{x}{(M^p D^t)^{1/(2+p)}}$$  \hspace{1cm} (2)

which satisfies

$$\int_0^\infty c(t, x) \, dx = M, \quad \partial_x c(t, 0) = 0, \quad c(t, x \to \infty) = 0$$

for some constant $M > 0$, is given by

$$F(\xi) = \left[ A - \frac{p \xi^2}{2(2 + p)} \right]^{1/p}, \quad 0 < \xi < \left[ \frac{2(2 + p) A}{p} \right]^{1/2}$$  \hspace{1cm} (3)

and $F \equiv 0$ otherwise. For the case $p = 1$, prove that $A = (3/8)^{1/3}$.

**Problem 2: Phytoplankton-zooplankton (20 Points)**

A space-dependent phytoplankton-zooplankton model can be reduced to the following equations

$$\begin{align*}
\partial_t u &= \partial_{xx} u + u + u^2 - \gamma uv \\
\partial_t v &= d \partial_{xx} v + \beta uv - v^2
\end{align*}$$  \hspace{1cm} (4)

where $u(t, x)$ and $v(t, x)$ are the plankton concentrations, respectively, and $\beta, \gamma, d$ are positive parameters. Find the regions in the $(\beta, \gamma)$-plane (a) in which there is a stable, homogeneous state $(u_0, v_0)$ such that neither $u_0$ nor $v_0$ is zero and (b) in which that state may be unstable to a Turing instability. In case (b), for what values of $d$ will the instability occur, and what is the critical wavenumber for the onset of the instability?
**Problem 3: A spinning cylinder (30 points)**

In this question, you will use a complex potential to analyze a spinning cylinder of radius \(a\) in a uniform flow of speed \(U\) (and density \(\rho\)). (This is described in section 4.5 of Acheson, although the notation is slightly different.)

(i) Verify that the complex potential

\[ w(z) = U \left( z + \frac{a^2}{z} \right) + i \frac{\Gamma}{2\pi} \ln \frac{z}{a} \quad (6) \]

correctly describes this situation: Check that (1) the boundary of the cylinder \(z = ae^{i\theta}\) is a streamline in this flow, and (2) far away from the cylinder, the flow is the uniform flow given by \(u = (U, 0)\).

(ii) Use Bernoulli’s theorem to calculate the pressure \(p\) at any location \(\theta\) on the cylinder and the total lift and drag. (Use \(p_0\) to refer to the pressure far away from the cylinder.)

(iii) Find a formula for the location \(z\) of the stagnation points and draw a sketch of the cylinder, the stagnation points, and the streamlines for the three cases \(\Gamma/4\pi aU < 1\), \(\Gamma/4\pi aU = 1\), and \(\Gamma/4\pi aU > 1\). (Assume \(\Gamma > 0\). For \(\Gamma < 0\) the problem is identical but with the cylinder moving downward instead.)

**Problem 4: Bending of a thin elastic sheet under gravity (30 Points)**

In the linearized limit of small deformations, the elastic bending energy of a two dimensional sheet with shape \(y(x)\) is

\[ U[y(x)] = \frac{1}{2} \left( \frac{Yh^3}{12(1 - \sigma^2)} \right) \int_0^l \left( \frac{d^2y}{dx^2} \right)^2 dx, \quad (7) \]

where \(l\) is the projected length of the beam, \(h\) is the thickness of the sheet and \(Y\) is its Young’s modulus. The term \(B = \frac{Yh^3}{12(1 - \sigma^2)}\) is often referred to as the bending modulus; it measures how stiff the sheet is under bending deformations.

For large deformations, the corresponding bending energy (per unit width) is

\[ U[y(s)] = \frac{1}{2} B \int_0^L \left( \frac{d\theta}{ds} \right)^2 ds, \quad (8) \]

where \(L\) is the total length of the sheet, \(\theta\) is the angle that the sheet makes with the horizontal and \(s\) is the arclength along the neutral surface of the sheet. Eqn. (8) can be read in words as: “The bending energy of a thin sheet per unit width equals one half of the bending modulus times the curvature \((d\theta/ds)\) squared, integrated along its total length.”

Consider the following configuration. A thin sheet of thickness \(h\), width \(b\) and total length \(L\) is held vertically at \(s = 0\) and is free to bend under gravity otherwise. The boundary conditions are that: 1) the sheet is vertical at the clamp \(\theta(s = 0) = \pi/2\) and 2) there is no curvature at the free end \((d\theta/ds)(s = L) = 0\). Assume that the sheet has a linear density (mass per unit length) given by \(\rho_l = \rho bh\), where \(\rho\) is the volumetric density. A schematic diagram of this configuration is given in Fig. 1a).
Figure 1: a) Schematic diagram of a thin sheet clamped vertically and bending under gravity. The sheet has dimensions: length $L$, thickness $h$ and width/span $b$. Note that $y_L$ is the height of the free end of the sheet at $s = L$. b) Plot of the dimensionless parameters $\Delta$ and $y_L/L$ for equilibrium shapes $\theta_E$ that satisfy Eqn. (11). c) Table for some values of $\Delta$ v.s. $y_L/L$.

(i) Write the total energy functional, $\mathcal{E}[\theta(s)]$, for this thin strip bent under gravity. Make sure that your expression for the energy only depends on $s$, $\theta(s)$ and $d\theta/ds$. (*hint: the fact that $dy = \sin \theta ds$ i.e. $y(s) = \int_0^{s} \sin[\theta(s)]ds$ will help*).

(ii) Assuming small functional perturbations, $\delta \theta(s)$, on the equilibrium shape of the beam $\theta_E(s)$, use calculus of variations ($\delta \mathcal{E} \sim \mathcal{E}[\theta_E(s) + \delta \theta(s)] - \mathcal{E}[\theta_E(s)] \to 0$) to show that the equilibrium shape satisfies the following differential equation

$$B_b \frac{d^2 \theta_E}{ds^2} = \rho_l g (L - s) \cos \theta_E$$

(9)

(iii) Non-dimensionalize Eqn. (9) using $L$ and show that

$$\frac{d^2 \theta_E}{ds^2} = \Delta (1 - \bar{s}) \cos \theta_E,$$

(10)

where $\Delta = (L/L_c)^3$ and $L_c$ is often called the elasto-gravity lengthscale. What is $L_c$ in terms of the physical quantities in the problem? What is its physical significance of $L_c$?

(iv) Determine $L_c$ from scaling arguments and ensure that you get the same result as in (iii).

(v) Show that the differential equation in (10) is identical to

$$\frac{1}{2} \left( \frac{d\theta_E}{ds} \right)^2 = \Delta \left[ (1 - \bar{s}) \sin \theta_E + (\bar{y} - \bar{y}_L) \right],$$

(11)
where $\bar{y} = \frac{y}{L}$, $\bar{y}_L = \frac{y_L}{L}$ and $y_L$ is the height of the free end with respect to the horizontal (see the diagram in Fig. 1a). (hint: it will be easier to go backwards from Eqn. (11) to Eqn. (10).)

Note: the differential equation that describes the equilibrium shapes of the bent sheet under gravity – Eqn. (11) – is nonlinear and therefore one has to solve it numerically to find $\theta_E(s)$ (that yields the equilibrium shapes). In case you’re interested, the results for $\Delta$ as a function of $y_L/L$ are plotted in Fig. 1b) (some points of this graph are given in the inset table).