SUMS OF VALUES OF A RATIONAL FUNCTION

BJORN POONEN

Abstract. Let $K$ be a number field, and let $f \in K(x)$ be a nonconstant rational function. We study the sets

$$\left\{ \sum_{i=1}^{n} f(x_i) : x_i \in K - \{\text{poles of } f\} \right\}$$

and

$$\left\{ \sum_{i=1}^{n} f(x_i) - \sum_{i=n+1}^{2n} f(x_i) : x_i \in K - \{\text{poles of } f\} \right\}$$

for large $n$. These are rational function analogues of Waring’s Problem.

1. Introduction

Lagrange proved that every nonnegative integer is a sum of four integer squares. Waring claimed that for each $k \geq 1$, there exists $n \geq 1$ such that every nonnegative integer is a sum of $n$ nonnegative $k^{th}$ powers. Hilbert proved this, and later the circle method was developed to give a simpler approach to this and other such questions. Analogues over number fields are known. There is also the easier problem which asks for representations of an integer as

$$\sum_{i=1}^{n} x_i^k - \sum_{i=n+1}^{n+n'} x_i^k$$

when $n$ and $n'$ are large relative to $k$. See the beginning of the book [Vau97] for an introduction to some of these problems.

Each of these results for integers implies its analogue for rational numbers. This paper studies what happens when the function $f(x) = x^k$ is replaced by an arbitrary rational function $f(x)$. The problem can be generalized further by considering number fields instead of $\mathbb{Q}$, but already over $\mathbb{Q}$ the problem seems very difficult: see Section 5.

Our two main theorems give partial answers to these questions:

Date: October 24, 2004.

2000 Mathematics Subject Classification. Primary 11P05; Secondary 11D68.

Key words and phrases. Waring’s Problem.

This research was supported by NSF grant DMS-9801104, and a Packard Fellowship.

This article has been published in Acta Arith. 112.4 (2004), 333–343.
Theorem 1.1. Suppose $K$ is a finite extension of $\mathbb{Q}$. Let $f \in K(x)$ be a nonconstant rational function with all poles in $K \cup \infty$. For $n \gg 1$ it is true that for all $c \in K$, there exist $x_1, \ldots, x_{2n} \in K - \{\text{poles of } f\}$ such that
\[ \sum_{i=1}^{n} f(x_i) - 2n f(x_{n+1}) = c. \]

Theorem 1.2. Keep the hypotheses of Theorem 1.1 and assume in addition that $f$ has at most 3 poles, all of which are simple. Then for $n \gg 1$ it is true that for all $c \in K$, there exist $x_1, \ldots, x_n \in K - \{\text{poles of } f\}$ such that
\[ \sum_{i=1}^{n} f(x_i) = c. \]

Conjecture 1.3. Theorem 1.2 holds even for $f$ having more than 3 poles, provided that all the poles are simple and in $K \cup \infty$.

To give a sense of the main ideas of the paper, let us sketch a proof of Theorem 1.1 in the case that $K = \mathbb{Q}$ and all poles of $f$ are simple and in $\mathbb{Q}$. We will find a “generic” solution: namely, we will $g_1, \ldots, g_{n+n'} \in \mathbb{Q}(x)$ such that
\[ \sum_{i=1}^{n} f(g_i(x)) - \sum_{i=n+1}^{n+n'} f(g_i(x)) = x. \]
Then by specializing $x$ we can represent any rational number in the desired form. (Actually, a further trick is needed to force $n = n'$ and to represent the rational numbers at which the $g_i$ have poles, but let us ignore these technicalities for now.) To find the $g_i$, we let
\[ S := \left\{ \sum_{i=1}^{n} f(g_i(x)) - \sum_{i=n+1}^{n+n'} f(g_i(x)) \mid n, n' \geq 0, g_i \in \mathbb{Q}(x) \text{ and } \deg g_i = 1 \right\} \subset \mathbb{Q}(x) \]
and let $P_1$ be the set of $\gamma \in S$ such that all poles of $\gamma$ lie in $\mathbb{Z}$ (they are automatically simple). Each $\gamma \in P_1$ has the form
\[ \gamma(x) = \sum_{i=1}^{s} \frac{a_i}{x-r_i} + b \]
where the $r_i$ are distinct integers, $a_i \in \mathbb{Q}^*$, and $b \in \mathbb{Q}$. The trick is to associate to $\gamma$ the Laurent polynomial
\[ \bar{\gamma} := \sum_{i=1}^{s} a_i T^{r_i} \in \mathbb{Q}[T, T^{-1}], \]
and let \( M := \{ \gamma : \gamma \in P_1 \}. \) Clearly \( M \) is an additive subgroup of \( \mathbb{Q}[T, T^{-1}] \); moreover, since the operations \( \gamma(x) \mapsto \gamma(x \pm 1) \) map \( P_1 \) into itself, \( M \) is a \( \mathbb{Z}[T, T^{-1}] \)-submodule, and \( \mathbb{Q} \cdot M \) is an ideal of \( \mathbb{Q}[T, T^{-1}] \). With a little work, one shows that for each \( \alpha \in \overline{\mathbb{Q}} \) there exists a Laurent polynomial in \( \mathbb{Q} \cdot M \) not vanishing at \( \alpha \), so that by the Hilbert Nullstellensatz, \( \mathbb{Q} \cdot M \) is the unit ideal. (Here we used the Nullstellensatz only for \( \mathbb{A}^1 - \{0\} \), but when we prove our theorem for number fields other than \( \mathbb{Q} \), we will apply it to \( \mathbb{A}^1 - \{0\} \).) Knowing that \( 1 \in \mathbb{Q} \cdot M \) means that some function \( ax + b \) with \( a \neq 0 \) belongs to \( S \). Substituting the inverse fractional linear transformation into \( x \) shows that \( x \) itself belongs to \( S \), completing the proof.

**Remark 1.4.** Without the assumption that the poles of \( f \) are in \( K \cup \infty \), Theorems 1.1 and 1.2 can fail. See Section 5.

**Question 1.5.** Do Theorems 1.1 and 1.2 hold for arbitrary fields \( K \)? Probably both can fail.

We now outline the structure of the paper. Section 2 uses Hensel’s Lemma to prove an analogous (but much easier) result over \( p \)-adic fields; this is not needed for the global results, but helps motivate the discussion in Section 5. Sections 3 and 4 prove Theorems 1.1 and 1.2, respectively. Section 5 raises questions about the number field case not yet addressed by our results. Finally, Section 6 discusses potential implications for diophantine definability.

### 2. Sums over \( p \)-adic fields

**Proposition 2.1.** Suppose that \( [K_v : \mathbb{Q}_p] < \infty \) for some finite prime \( p \). Let \( f \in K_v(x) \) be nonconstant. Then there exists \( c \in K_v \) and an open additive subgroup \( G \) of \( K_v \) such that for all sufficiently large \( n \),

\[
\{ f(t_1) + \cdots + f(t_n) \mid t_1, \ldots, t_n \in K_v \} = nc + G.
\]

**Remark 2.2.** The open additive subgroups of \( \mathbb{Q}_p \) are \( \mathbb{Q}_p \) and \( p^n \mathbb{Z}_p \) for \( n \in \mathbb{Z} \). For other local fields \( K_v \), there are others, such as \( \mathbb{Z}_p + p^n \mathcal{O} \), where \( \mathcal{O} \) is the ring of integers of \( K_v \).

**Proof.**

**Case 1:** \( f \) has a pole at some point \( P \in \mathbb{P}^1(K_v) \).

Expand \( f \) in a Laurent series in a uniformizer \( t \) at \( P \). Let \( \epsilon \) be the coefficient of \( t^r \), where \( r \) is the order of the pole. By scaling \( f \), we may assume that \( \epsilon = 1 \). There is a power series \( g = t + \cdots \in K_v[[t]] \) such that \( g^{-r} = f \) and \( g \) converges for sufficiently small \( t \). By Hensel’s Lemma, the set of values taken by \( g \) on any neighborhood of 0 contains a neighborhood of 0.

\(^{1}\)A. Okounkov pointed out to me that up to some normalizations, \( \gamma(T) \) is the Fourier transform of \( \gamma(x) \)!
Thus every sufficiently large $r$-th power in $K_v$ is a value of $f$. Next we must show that there exists $n$ such that any $r \in K_v$ is a sum of large $r$-th powers.

To accomplish this, first use Hensel’s Lemma to write $0 = \alpha_1 + \cdots + \alpha_n$ for some $n \geq 1$ and $\alpha_1, \ldots, \alpha_n \in K_v^\times$. Let $\beta_i = M\alpha_i$ for some $M \in K_v$ much larger than $r$, and use Hensel’s Lemma to replace $\beta_1$ by some $\tilde{\beta}_1$ closer to $\beta_1$ than to $0$, such that

$$\tilde{\beta}_1^r + \beta_2^r + \cdots + \beta_n^r = r.$$

Thus we may take $c = 0$ and $G = K_v$.

**Case 2:** $f$ has no poles in $\mathbb{P}^1(K_v)$.

Let $O$ be the ring of integers in $K_v$, and let $\pi$ be a uniformizer. Since $f$ is nonconstant, there exists $\alpha \in K_v$ such that $f'(\alpha) \neq 0$. By Hensel’s Lemma, $f(K_v)$ contains a neighborhood of $f(\alpha)$. By considering $f – f(\alpha)$ instead of $f$, we reduce to the case where $f(\alpha) = 0$. Now $f(K_v)$ contains an open subgroup $H := \pi^r O$ for some $r \in \mathbb{Z}$. On the other hand, since $f$ has no poles, compactness implies that $f(\mathbb{P}^1(K_v)) \subset \pi^R O$ for some $R \in \mathbb{Z}$. Let $S_n \subseteq K_v/H$ be the set of cosets that contain $f(t_1) + \cdots + f(t_n)$ for some $t_1, \ldots, t_n$. Since $0$ is a value of $f$, the $S_n$ form an increasing sequence. On the other hand, each $S_n$ is contained in the finite set $\pi^R O/\pi^r O$, so there exists $n$ such that $S_N = S_n$ for all $N \geq n$. Since $S_n$ is finite and closed under addition, it is a subgroup of $K_v/H$. Let $G$ be the union of the cosets in $S_n$. Then $G$ is an open subgroup of $K_v$, and all values of $f$ are in $G$. On the other hand, every element of $G$ is a sum of $n + 1$ values of $f$, by definition of $S_n$, since we can arrange to have $f(t_{n+1})$ equal any desired element of $H$.

**Corollary 2.3.** Under the hypotheses of Proposition 2.1, the values of

$$f(t_1) + \cdots + f(t_n) - f(t_{n+1}) - \cdots - f(t_{2n})$$

form an open subgroup of $K_v$.

Analogous results for rational functions in many variables over $p$-adic fields can be proved in the same way.

### 3. Sums and Differences over Number Fields

This section is devoted to the proof of Theorem 1.1. The first lemma of this section is a thinly disguised version of Hilbert’s Nullstellensatz, as its proof will reveal. Its relevance will become clear in the proof of Lemma 3.2.

We fix an integer $d \geq 1$ (which eventually will be taken to be $[K : \mathbb{Q}]$) and for any ring $R$, we define $R[T, T^{-1}] = R[T_1, T_1^{-1}, \ldots, T_d, T_d^{-1}]$. If $k$ is a field and $t \in (\overline{k})^d$, let $ev_t : k[T, T^{-1}] \to \overline{k}$ denote the evaluation map, which induces $ev_t : V \otimes_k k[T, T^{-1}] \to V \otimes_k \overline{k}$ for any $k$-vector space $V$. 
Lemma 3.1. Let $V$ be a finite-dimensional vector space over a field $k$. If $M$ is a $k[T, T^{-1}]$-submodule of $N := V \otimes_k k[T, T^{-1}]$ and $M \neq N$, then there exist nonzero $\lambda \in \text{Hom}_k(V \otimes_k k, k)$ and $t \in (k^*)^d$ such that $\lambda(\text{ev}_t(F)) = 0$ for all $F \in M$.

Proof. Without loss of generality, we may assume $k = k$. Let $A = k[T, T^{-1}]$, which is a noetherian ring. Then $N$ is a noetherian $A$-module, so we may assume $M$ is a maximal proper submodule of $N$. The $A$-module homomorphism $A \to N/M$ sending 1 to any $n \in N \setminus M$ must then be surjective, with kernel equal to a maximal ideal $m$. Hence $mN \subseteq M$. The ring $A$ is the ring of regular functions on the affine variety $(A^1 \setminus \{0\})^d$, so by Hilbert’s Nullstellensatz, $A/m \simeq k$ is an isomorphism induced by $\text{ev}_t$ for some point $t \in (k^*)^d$.

Since $mN \subseteq M \subseteq N$, the image of $M$ under $\text{ev}_t : N = V \otimes_k A \to V$ is a proper subspace of $V$, so there exists a nonzero $\lambda \in \text{Hom}_k(V, k)$ such that $\lambda(\text{ev}_t(M)) = 0$, as desired. □

The main step in the proof of Theorem 1.1 is the following lemma, which obtains a representation of the rational function $x$ as a combination of values of $f$.

Lemma 3.2. Suppose $[K : \mathbb{Q}] < \infty$. Let $f \in K(x)$ be nonconstant with all poles in $K \cup \infty$. For some $n, n' \geq 1$, there exist $g_1, \ldots, g_{n+n'} \in K(x)$ of degree 1 such that

$$\sum_{i=1}^n f(g_i(x)) - \sum_{i=n+1}^{n+n'} f(g_i(x)) = x.$$

Remark 3.3. Whenever we write $f(g_i(x))$, there is also the tacit requirement that $g_i(x)$ should not be a constant equal to a pole of $f$.

Proof. Define

$$S := \left\{ \sum_{i=1}^n f(g_i(x)) - \sum_{i=n+1}^{n+n'} f(g_i(x)) \mid n, n' \geq 0, g_i \in K(x) \text{ and } \deg g_i = 1 \right\} \subset K(x).$$

We need to show that $x \in S$. Below we will frequently use without mention the easy fact that if $j \in S$, and $g \in K(x)$ is of degree 1, then $j \circ g \in S$.

For $j \in K(x)$, let $m(j)$ denote the maximum order of all poles of $j$. Since $S$ contains nonconstant rational functions, we may choose a nonconstant $j \in S$ minimizing $m := m(j)$.

Case 1. $j$ has a unique pole of order $m$.

If $m = 1$, then $\deg j = 1$, so $x = j \circ g \in S$, where $g$ is the inverse function of $j$. If $m > 1$, then by replacing $j$ with $j \circ g$ for some $g$ of degree 1,
we may assume that the pole is at $\infty$. Then $j(x + 1) - j(x) \in S$, but $0 < m(j(x + 1) - j(x)) = m - 1 < m$, contradicting the definition of $j$.

Case 2. $j$ has more than one pole of order $m$.

Let $d = [K : \mathbb{Q}]$. Let $\alpha_1, \ldots, \alpha_d$ be a $\mathbb{Z}$-basis for the ring of integers $\mathcal{O}_K$ of $K$. Let $P_m$ be the set of $\gamma \in S$ such that $m(\gamma) \leq m$, and such that all poles of $\gamma$ of order $m$ are in $\mathcal{O}_K$. By replacing the given $j$ with $j \circ g$ for some $g$ of degree 1, we may assume first that $j$ has no pole at $\infty$, and then that $j \in P_m$.

Given any $\gamma \in P_m$, write $\gamma$ as

\begin{equation}
\gamma(x) = \sum_{i=1}^{s} \frac{a_i}{(x - r_i)^m} + \text{(terms with lower order poles)}
\end{equation}

where the $r_i$ are distinct elements of $\mathcal{O}_K$ and $a_i \in K^*$, and define the $-$ operation by

$$\bar{\gamma} := \sum_{i=1}^{s} a_i T^{k_i} \in K[T, T^{-1}],$$

where each vector of exponents $k_i = (k_{i1}, \ldots, k_{id}) \in \mathbb{Z}^d$ is such that $r_i = k_{i1}\alpha_1 + \cdots + k_{id}\alpha_d$. Since $P_m$ is an additive group, so is $M := \{ \bar{\gamma} \mid \gamma \in P_m \}$. If $1 \leq i \leq d$ and $k \in \mathbb{Z}$, and $\tau(x)$ is the polynomial $x - k\alpha_i$, then $\bar{\gamma} \circ \tau = T^{k}\bar{\tau}$.

Thus we arrive at the following key observation:

$M$ is a $\mathbb{Z}[T, T^{-1}]$-submodule of $K[T, T^{-1}]$.

If $\mathbb{Q} \cdot M = K[T, T^{-1}]$, then there exists $\gamma \in P_m$ such that $\bar{\gamma} \in \mathbb{Q}^* \subset K[T, T^{-1}]$. Then $\gamma$ has a single pole (at 0) of order $m$, and we have reduced to Case 1.

Otherwise, if $\mathbb{Q} \cdot M \neq K[T, T^{-1}]$, then by Lemma 3.1 applied with $V = K$, $k = \mathbb{Q}$, and $\mathbb{Q} \cdot M$ as $M$, there exist a nonzero $\lambda \in \text{Hom}_{\mathbb{Q}^*}(K \otimes \overline{\mathbb{Q}}, \mathbb{Q})$ and $\mathbf{t} \in (\overline{\mathbb{Q}}^*)^d$ such that $\lambda(\text{ev}_t(\bar{\gamma})) = 0$ for all $\bar{\gamma} \in M$. Pick a finite extension $L$ of $\mathbb{Q}$ over which $\lambda$ and $\mathbf{t}$ are defined; i.e., so that $\lambda$ maps $K \otimes L$ into $L$, and $\mathbf{t} \in (L^*)^d$. Replacing $\lambda$ by an integer multiple, we may assume that $\lambda$ maps $\mathcal{O}_K \otimes \mathcal{O}_L$ into $\mathcal{O}_L$. Define $a_i, r_i \in K$ so that (1) holds with $\gamma$ replaced by our given $j$. For any prime $p$ of $\mathbb{Q}$, let $\mathcal{O}_{K,p}$ (resp. $\mathcal{O}_{L,p}$) denote the subring of $K$ (resp. $L$) of elements that are integral at all the primes above $p$. By the Chebotarev Density Theorem, there exists a prime $p$ of $\mathbb{Q}$ such that

1. $p$ splits completely in $K$ and in $L$,
2. for any prime $\mathfrak{p}$ of $L$ above $p$, the $(\mathcal{O}_L/\mathfrak{p})$-linear functional
   $$\lambda_{\mathfrak{p}} : \mathcal{O}_{K,p}/(p) \otimes (\mathcal{O}_L/\mathfrak{p}) \to \mathcal{O}_L/\mathfrak{p} \simeq \mathbb{F}_p$$
   induced by $\lambda$ is nonzero,
3. $\mathbf{t} \in (\mathcal{O}_{L,p}^*)^d$
(4) \(a_i \in \mathcal{O}_{K,p}^* \) and \(r_i - r_k \in \mathcal{O}_{K,p}^* \) for all \(1 \leq i < k \leq s\).

(The conditions after the first one exclude only finitely many \(p\).) Fix \(p\) as in condition 2.

Replacing \(j(x)\) by \(j(x + c)\) for some \(c \in \mathcal{O}_K\), we may assume that \(r_1 = p\). Then the other \(r_i\) are prime to \(p\), because of condition 4. Let \(R = r_1r_2 \ldots r_s \neq 0\). Then \(\eta(x) := p^m j(R/x) \in S\) has poles at \(R/r_i\) for \(1 \leq i \leq d\), so \(\eta \in P_m\). The coefficient \(b_i\) of \((x - R/r_i)^{-m}\) in the partial fraction decomposition of \(\eta(x)\) equals the value of
\[
p^m \left( x - \frac{R}{r_i} \right)^m \frac{a_i}{(x - r_i)^m}
\]
at \(x = R/r_i\) (which makes sense after terms are cancelled), so
\[
b_i = \left( -\frac{p}{r_i} \right)^m \left( \frac{R}{r_i} \right)^m a_i.
\]

Since the \(r_i\) are in \(\mathcal{O}_{K,p}^*\) except for \(r_1 = p\), and since \(a_i \in \mathcal{O}_{K,p}^*\), each \(b_i\) lies in \(\mathcal{O}_{K,p}^*\); in fact, \(b_1 \in \mathcal{O}_{K,p}^*\) and \(b_i \in p^m \mathcal{O}_{K,p}\) for \(2 \leq i \leq s\). Let \(\mu(x) = \eta(x + R/r_1) \in P_m\), to move the pole at \(R/r_1\) to 0. Then
\[
\bar{\mu} \equiv b_1 \pmod{p \mathcal{O}_{K,p}[T, T^{-1}]).}
\]

Since \(p\) splits completely in \(k\),
\[
\mathcal{O}_{K,p}/(p) \simeq \mathbb{F}_p \times \cdots \times \mathbb{F}_p,
\]
and since \(b_1 \in \mathcal{O}_{K,p}^*\), \(b_1\) reduces mod \(p\) to a vector of elements of \(\mathbb{F}_p^*\) on the right. Since \(\lambda_p\) is nonzero, one of the factors on the right (tensored with \(\mathcal{O}_L/p\)), say the \(i\)-th, is not killed by \(\lambda_p\). Choose \(c \in \mathcal{O}_K\) whose image in \(\mathcal{O}_{K,p}/(p) \simeq \mathbb{F}_p \times \cdots \times \mathbb{F}_p\) is zero in all coordinates except the \(i\)-th, and let \(\theta(x) = \mu(x/c)\). A short calculation shows that \(\theta \in P_m\) and
\[
\bar{\theta} \equiv c^m b_1 \pmod{p \mathcal{O}_{K,p}[T, T^{-1}]).}
\]

Now
\[
\text{ev}_t(\bar{\theta}) \equiv c^m b_1 \otimes 1 \pmod{p(\mathcal{O}_{K,p} \otimes \mathcal{O}_L))}.
\]

By choice of \(c\), the right hand side is not killed by \(\lambda_p\), so \(\lambda(\text{ev}_t(\bar{\theta}))\) cannot possibly be zero. This contradicts the construction of \(\lambda\) and \(t\). \(\square\)

**Theorem 3.4.** Let \(K\) be a finite extension of \(\mathbb{Q}\). Let \(f \in K(x)\) be a non-constant rational function all of whose poles are in \(K \cup \infty\). If \(n \geq 1\) is sufficiently large, then for any \(h \in K(x)\), there exist \(g_1, \ldots, g_{2n} \in K(x)\) such that
\[
\sum_{i=1}^{n} f(g_i(x)) - \sum_{i=n+1}^{2n} f(g_i(x)) = h(x).
\]
Proof. Find a representation of $x$ as in Lemma 3.2, using $n$ plus terms and $n'$ minus terms. Write $h = h_1 - h_2$ where $h_1, h_2 \in K(x)$ are nonconstant. Substitute $h_1$ for $x$ in the identity giving $x$, then substitute $h_2$ for $x$ in the same identity, and subtract the two equations to obtain a representation of $h$ using $n + n'$ plus terms and $n + n'$ minus terms. We can add pairs of cancelling terms to obtain representations with more than $n + n'$ terms of each sign. □

To prove Theorem 1.1, apply Theorem 3.4 with $h(x)$ as the constant $c \in K$, and substitute an element of $K$ for $x$: all but finitely many elements of $K$ will yield a representation of the required form.

4. Sums over number fields

Fix a number field $K$ for this section. If $f, h \in K(x)$, we write $h \preceq f$ to mean that for some $n \geq 1$, there exist $g_1, \ldots, g_n \in K(x)$ of degree 1 such that $\sum_{i=1}^{n} f(g_i(x)) = h(x)$. The set of $h$ such that $h \preceq f$ is closed under addition, and closed under $h \mapsto h \circ j$ for any $j \in K(x)$ of degree 1, so it follows that $\preceq$ is transitive.

Lemma 4.1. Suppose $f$ is a nonconstant function in $K(x)$. Suppose that the poles of $f$ are simple and in $K \cup \infty$. If there is a constant function $c \in K$ such that $c \preceq f$, then $x \preceq f$.

Proof. We are given an identity $\sum_{i=1}^{n} f(g_i(x)) = c$. Let $h(x) = f(g_1(x))$, which is a nonconstant function with poles in $K \cup \infty$ such that $h \preceq f$ and $c - h \preceq f$. Applying Lemma 3.2 to $h$ yields an identity

$$\sum_{i=1}^{n} h(j_i(x)) - \sum_{i=n+1}^{n+n'} h(j_i(x)) = x$$

for some $j_i \in K(x)$ of degree 1. Then

$$\sum_{i=1}^{n} h(j_i(x)) + \sum_{i=n+1}^{n+n'} (c - h(j_i(x))) = x + n'c$$

and each summand on the left is $\preceq f$, so $x + n'c \preceq f$. Substituting $x - n'c$ for $x$ shows that $x \preceq f$. □

Lemma 4.2. If $f \in K(x)$ is nonconstant with $\leq 3$ poles, all simple and in $K \cup \infty$, then there is a constant function $c \in K$ such that $c \preceq f$.

Proof. First suppose that $f$ has $\leq 2$ poles. Composing with a degree 1 function, we may assume without loss of generality that the poles are contained in $\{0, \infty\}$, so

$$f(x) = ax + \frac{b}{x} + r$$

...
for some \( a, b, r \in K \). Then \( 2r = f(x) + f(-x) \preceq f \), and \( 2r \) is constant.

If \( f \) has 3 poles, then we may assume they are 0, 1, and \( \infty \). Then \( 2r = f(x) + f(-x) \) has 2 poles (at 1 and \(-1\)), and \( f(x) + f(-x) \preceq f \), so apply the previous paragraph and use transitivity of \( \preceq \).

**Proof of Theorem 1.2.** Applying Lemmas 4.1 and 4.2, we see that \( x \preceq f \).
Thus \( \sum_{i=1}^{m} f(g_i(x)) = x \) for some \( g_i \in K(x) \) of degree 1. Then \( \sum_{i=1}^{m} f(g_i(x)) + \sum_{i=1}^{m} f(g_i(c-x)) = c \). Substitute an element of \( K \) for \( x \): all but finitely many choices lead to a representation of \( c \) as \( \sum_{i=1}^{2m} f(x_i) \) with \( x_i \in K \).

To obtain a representation with \( n \) terms for \( n > 2m \), choose \( x_{2m+1}, \ldots, x_n \in K - \{ \text{poles of } f \} \) arbitrarily, let \( c' = c - \sum_{i=2m+1}^{n} f(x_i) \), and use the previous paragraph to find \( x_1, \ldots, x_{2m} \) such that \( \sum_{i=1}^{2m} f(x_i) = c' \).

**5. LOCAL-GLOBAL QUESTIONS**

Throughout this section \( K \) denotes a number field, and \( f \in K(x) \) is a nonconstant rational function.

Theorem 1.2 cannot be generalized to all nonconstant \( f \) with poles in \( K \cup \infty \), since there can be local obstructions at the real places. For instance, if \( K = \mathbb{Q} \) and \( f(x) = x^2 \), then the equation is not solvable when \( c < 0 \).

**Question 5.1.** Is it possible that Theorem 1.2 can be extended to the case where \( f \) has all poles in \( K \cup \infty \) (not necessarily simple), and the highest order pole is of odd order?

Without the assumption that the poles of \( f \) are in \( K \cup \infty \), even Theorem 1.1 can fail. For example, suppose that \( K = \mathbb{Q} \) and \( f(x) = 2/(x^2 - 2) \). Local considerations show that

\[
f(t) \in R := \left\{ \frac{r}{s} \in \mathbb{Q} \mid r, s \in \mathbb{Z}, \text{ and } s \text{ is a product of primes of the form } 8k \pm 1 \right\}
\]

for any \( t \in \mathbb{Q} \). If \( c \not\in R \), then for any \( n \),

\[
\sum_{i=1}^{n} f(x_i) - \sum_{i=n+1}^{2n} f(x_i) = c
\]

has no solution over \( K \).

**Remark 5.2.** Nevertheless, there are some rational functions having some poles outside \( K \cup \infty \) for which the conclusions of Theorems 1.1 and 3.4 still hold. For instance, if \( K = \mathbb{Q} \) again, and

\[
f(x) = \frac{x}{2} + \frac{1}{x^2 - 2},
\]
then although \( f \) has poles outside \( \mathbb{Q} \cup \infty \), the combination \( f(x) - f(-x) \) yields \( z \), from which any other \( h \in \mathbb{Q}(x) \) can be obtained. (See the proof of Theorem 3.4 for this last step.)

Local obstructions explain the failure of Theorem 1.1 to generalize to functions such as \( f(x) = 2/(x^2 - 2) \). It is natural to ask whether these are the only obstructions to representability of a rational numbers as a sum and difference of a fixed number of values of \( f \): More precisely, one might ask the following:

**Question 5.3.** For \( n \gg 1 \) is it true that for each \( c \in K \), the equation

\[
(2) \quad \sum_{i=1}^{n} f(x_i) - \sum_{i=n+1}^{2n} f(x_i) = c
\]

has a solution over \( K \) if and only if it has a solution over all completions? Equivalently, if \( X_{n,c} \) is the affine variety over \( K \) defined by (2) and the inequalities saying that each \( x_i \) does not equal a pole of \( f \), is it true for \( n \gg 1 \) that for all \( c \in K \), the variety \( X_{n,c} \) satisfies the Hasse principle?

The analogous question with sums only has a negative answer. For example, if \( K = \mathbb{Q} \) and \( f(x) = (x^2 - 2)^2 \) then methods similar to those used in the proof of Proposition 2.1 show that for \( n \geq 5 \),

\[
f(x_1) + \cdots + f(x_n) = 0
\]

has a solution over every completion of \( \mathbb{Q} \), while considering the equation over \( \mathbb{R} \) shows that it has no solution over \( \mathbb{Q} \). One could however, ask the following:

**Question 5.4.** Is it true for \( n \gg 1 \) that for all \( c \in K \), if

\[
(3) \quad \sum_{i=1}^{n} f(x_i) = c
\]

has a solution over every completion of \( K \), and for each real completion \( K_v \) the equation \( \sum_{i=1}^{n} f(x_i) = c' \) is solvable over \( K_v \) for all \( c' \) in a neighborhood of \( c \), then (3) has a solution over \( K \)?

6. Undecidability

A subset \( A \subseteq \mathbb{Q} \) is called *diophantine* over \( \mathbb{Q} \) if there is a polynomial \( g(t, x_1, \ldots, x_n) \) such that

\[
A = \{ a \in \mathbb{Q} : \exists x_1, \ldots, x_n \in \mathbb{Q} \text{ with } g(a, x_1, \ldots, x_n) = 0 \}.
\]

If \( \mathbb{Z} \) were diophantine over \( \mathbb{Q} \), then the (known) undecidability of Hilbert’s Tenth Problem over \( \mathbb{Z} \) would imply the undecidability of Hilbert’s Tenth Problem over \( \mathbb{Q} \), that is, that there is no general algorithm for deciding
whether a variety over \( \mathbb{Q} \) has a rational point. See the book [DLPVG00] for a discussion of this and related questions.

Given that it is unknown whether \( \mathbb{Z} \) is diophantine over \( \mathbb{Q} \), it is natural to ask whether other subrings between \( \mathbb{Z} \) and \( \mathbb{Q} \) can be proved to be diophantine over \( \mathbb{Q} \). If \( S \) is the complement of a finite subset in the set of all primes, then the semilocal ring \( \mathbb{Z}[S^{-1}] \) is known to be diophantine over \( \mathbb{Q} \): this follows from [KR92]. Currently there are no other subsets \( S \) for which \( \mathbb{Z}[S^{-1}] \) has been proved diophantine over \( \mathbb{Q} \).

If Question 5.3 has a positive answer for the example \( K = \mathbb{Q} \) and \( f(x) = 2/(x^2 - 2) \), then it would follow that the ring \( R = \mathbb{Z}[S^{-1}] \) is diophantine over \( \mathbb{Q} \), where \( S \) is the set of primes of the form \( 8k \pm 1 \). If Question 5.3 has a positive answer in general, then there would exist subsets \( S \) of arbitrarily small positive natural density such that \( \mathbb{Z}[S^{-1}] \) is diophantine over \( \mathbb{Q} \). One cannot hope to obtain \( \mathbb{Z} \) as a finite intersection of subrings arising in this way, however, since if \( L \) is the number field generated by the poles of the corresponding rational functions \( f \), then all the primes splitting completely in \( L \) will remain invertible in the intersection, and these form a set of primes of positive density, by the Chebotarev Density Theorem.

**References**


Department of Mathematics, University of California, Berkeley, CA 94720-3840, USA

E-mail address: poonen@math.berkeley.edu