VARIETIES WITHOUT EXTRA AUTOMORPHISMS III: HYPERSURFACES

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Abstract. For any field k and integers $n \geq 1$, $d \geq 3$, with (n,d) not equal to $(1,3)$ or $(2,4)$, we exhibit a smooth hypersurface X over k of degree d in $\mathbf{P}^{n+1}$ such that X has no nontrivial automorphisms over $\overline{k}$. For $(n,d) = (2,4)$, we find a smooth hypersurface X with the weaker property of having no nontrivial automorphism induced by an automorphism of the ambient $\mathbf{P}^{n+1}$.

1. Introduction

Let k be a field, and let p be its characteristic, which may be 0. Fix an algebraic closure $\overline{k}$ of k. Let X in $\mathbf{P}^{n+1}$ be a smooth hypersurface of degree d over k. Let $\overline{X} = X \times_k \overline{k}$. Let $\operatorname{Aut} \overline{X}$ be the group of automorphisms of $\overline{X}$ over $\overline{k}$. Call $\gamma \in \operatorname{Aut} \overline{X}$ *linear* with respect to the embedding $X \hookrightarrow \mathbf{P}^{n+1}$ if γ is induced by an automorphism of $\mathbf{P}^{n+1}$ over $\overline{k}$, i.e., by a linear transformation of the homogeneous coordinates. The linear automorphisms form a subgroup $\operatorname{Lin} \overline{X}$ of $\operatorname{Aut} \overline{X}$.

We will study $\operatorname{Lin} \overline{X}$ primarily. Before stating our main result, Theorem 1.6, let us briefly survey known related results. First, it is known that for most (n,d), there is no difference between $\operatorname{Aut} \overline{X}$ and $\operatorname{Lin} \overline{X}$:

Theorem 1.1. *If X is a smooth hypersurface in $\mathbf{P}^{n+1}$ of degree d, where $n \geq 1$, $d \geq 3$, and (n,d) does not equal $(1,3)$ or $(2,4)$, then $\operatorname{Aut} \overline{X} = \operatorname{Lin} \overline{X}$.*

Proof. The case $n = 1$ is Theorem 1 of [Cha78]. The case $n \geq 2$ is Theorem 2 of [MM63]. □

Remark 1.2. The exclusion of $(1,3)$ and $(2,4)$ in Theorem 1.1 is necessary. When $(n,d) = (1,3)$, a choice of flex in $X(\overline{k})$ makes $\overline{X}$ an elliptic curve, and if $P \in X(\overline{k})$ satisfies $3P \neq 0$, then translation by P is a nonlinear automorphism of $\overline{X}$. (See the proof of Theorem 1.3 below.) For $(n,d) = (2,4)$, the equality fails only for certain X; an example due to Fano and Severi is described in the proof of Theorem 4 in [MM63], for instance. What makes the proofs fail for $(n,d) = (2,4)$ is that the canonical bundle is trivial, and that $\operatorname{Pic} \overline{X}$ can be larger than $\mathbf{Z}$. In fact, the Tate conjecture predicts that the latter is automatic for X over $\overline{\mathbf{F}}_p$ with $(n,d) = (2,4)$.

Theorem 1.3. *If $n \geq 1$ and $d \geq 3$, then $\operatorname{Lin} \overline{X}$ is finite.*

Proof. See the "Historical Remarks" section at the end of [OS77]. The result has apparently been known for at least one hundred years, at least when $p = 0$. Matsumura and Monsky [MM63] give a proof in arbitrary characteristic, at least when $n \geq 2$. If $n = 1$ and $d \geq 4$, then X is a curve of genus $g = (d-1)(d-2)/2 \geq 2$, so $\operatorname{Aut} \overline{X}$ is finite [Sch38].

We are left with the easiest case, in which $n = 1$ and $d = 3$. Without loss of generality, $k = \overline{k}$. We can make X an elliptic curve by choosing a flex P as origin. The automorphism group

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$\operatorname{Aut}(X,P)$ of the elliptic curve is finite and of order dividing 24 [Sil92, Theorem III.10.1]. Also, $\operatorname{Aut}(X,P) \subseteq \operatorname{Lin} X$, since $\mathcal{O}_X(1)$ for the embedding $X \hookrightarrow \mathbf{P}^2$ is the line sheaf $\mathcal{L}(3P)$ on X. The orbit of P under $\operatorname{Lin} X$ is contained in the set of points $P' \in X(k)$ such that $\mathcal{L}(3P') \cong \mathcal{L}(3P)$; this is the set of 3-torsion points of the elliptic curve (X,P), which is of size at most 9. Hence $(\operatorname{Lin} X : \operatorname{Aut}(X,P)) \le 9$, so $\#(\operatorname{Lin} X) \le 216$ (with equality if and only if X is supersingular and $p=2$). ☐

Remark 1.4. Suppose that $p=0$ and $d \ge 3$. In unpublished work, Bott and Tate [BT61] used homological methods to show that there exists an upper bound for $\#(\operatorname{Lin} \overline{X})$ depending only on n and d. For $n=1$ and $d \ge 4$, one can use Hurwitz's theorem that $\#(\operatorname{Aut} \overline{X}) \le 84(g-1)$ for any curve X of genus g. For $n \ge 2$, Howard and Sommese [HS81] prove that there is a constant c_n depending only on n such that $\#(\operatorname{Lin} \overline{X}) \le c_n d^n$.

Let $N = \binom{d+n+1}{d}$ be the number of monomials of degree d in variables $x_0, \dots, x_{n+1}$. Over any field k, smooth hypersurfaces of degree d in $\mathbf{P}^{n+1}$ correspond to the points of a dense open subset $\mathcal{H}_{n,d}$ of $\mathbf{P}^{N-1}$, on which the homogeneous coordinates are the coefficients of the polynomial defining the hypersurface. For $n \ge 1$, $d \ge 3$, and $(n,d) \ne (1,3)$, Katz and Sarnak [KS99, Lemma 11.8.5] show that there is an open subset $U_{n,d} \subset \mathcal{H}_{n,d}$ whose points correspond to the smooth hypersurfaces X with $\operatorname{Lin} \overline{X} = \{1\}$.

Theorem 1.5. *Suppose that $n \ge 1$, $d \ge 3$, and $(n,d) \ne (1,3)$. Then $U_{n,d}$ is nonempty. In other words, the generic hypersurface X of degree d in $\mathbf{P}^{n+1}$ has $\operatorname{Lin} \overline{X} = \{1\}$.*

Proof. Matsumura and Monsky [MM63] prove this for $n \ge 2$, $d \ge 3$, and their methods can be adapted to the case $n=1$, $d \ge 4$. A proof for $n=1$, $d \ge 4$ written out in full can be found in [Cha78] for $p=0$, and in [KS99, 10.6.18] for arbitrary p using an alternative method. ☐

Combining Theorem 1.5 with the Lang-Weil method as in Corollary 11.8.7 of [KS99], one can show that for these (n,d), there exists $N_{n,d} > 0$ such that for any field k with $\#k > N_{n,d}$ (in particular, any infinite field), there exists a smooth hypersurface X of degree d in $\mathbf{P}^{n+1}$ over k with $\operatorname{Lin} \overline{X} = \{1\}$. Our main result is that the same conclusion holds for all k:

Theorem 1.6. *For any field k and integers $n \ge 1$, $d \ge 3$ with $(n,d) \ne (1,3)$, there exists a smooth hypersurface X over k of degree d in $\mathbf{P}^{n+1}$ such that $\operatorname{Lin} \overline{X} = \{1\}$.*

Remark 1.7. The exclusion of $(1,3)$ is necessary. If $(n,d)=(1,3)$, then we may choose a flex to make X an elliptic curve, and then multiplication by -1 on the elliptic curve is a nontrivial linear automorphism.

Remark 1.8. There is a small overlap between Theorem 1.6 and the main result of [Poo00a], since a smooth hypersurface X of degree 4 in $\mathbf{P}^2$ with $\operatorname{Lin} \overline{X} = \{1\}$ is the same thing as a genus 3 curve X with $\operatorname{Aut} \overline{X} = \{1\}$.

Our proof of Theorem 1.6 does not use Theorem 1.5, so it gives a new proof of Theorem 1.5. We can also combine Theorems 1.1 and 1.6 to obtain the following:

Corollary 1.9. *For any field k and integers $n \ge 1$, $d \ge 3$ with (n,d) not equal to $(1,3)$ or $(2,4)$, there exists a smooth hypersurface X over k of degree d in $\mathbf{P}^{n+1}$ such that $\operatorname{Aut} \overline{X} = \{1\}$.*

Remark 1.10. Remark 1.7 shows that the exclusion of $(1,3)$ in Corollary 1.9 is necessary. But it may be that Corollary 1.9 holds for $(n,d)=(2,4)$.

Section 2 gives the definition of X for Theorem 1.6, which will depend on n, d, and p. Section 3 proves that X is smooth. Most of the rest of the paper is devoted to proving that $\operatorname{Lin} \overline{X}$ is trivial in the various cases. Finally, in Section 11, we mention a few consequences for the *automorphism group scheme* $\mathbf{Aut}\, \overline{X}$.

2. Construction of X

The hypersurface X in Theorem 1.6 will be the subvariety of $\mathbf{P}^{n+1}$ defined by a homogeneous polynomial $f(x_0, x_1, \ldots, x_{n+1})$. In order to help us control the automorphisms, we will choose an f that "endows the variables with an ordering." As a first attempt, we could try

$$x_0 x_1^{d-1} + x_1 x_2^{d-1} + \cdots + x_n x_{n+1}^{d-1},$$

but this fails for two reasons: first, the resulting hypersurface is singular at $(1:0:0:\cdots:0)$; and second, it has nontrivial automorphisms if $d-1$ is not a power of p, since one can multiply x_{n+1} by a nontrivial $(d-1)$-th root of unity. In fact, if we choose any form with $n+1$ or fewer monomials, there will be a nontrivial diagonal action of $\mathbf{G}_m$ on X in which $\lambda \in \overline{k}^*$ acts as

$$(x_0 : x_1 : \cdots : x_{n+1}) \mapsto (\lambda^{a_0} x_0 : \lambda^{a_1} x_1 : \cdots : \lambda^{a_{n+1}} x_{n+1}),$$

for some integers a_i not all equal.

These problems can be fixed for most triples (n, d, p) by adding a few terms to the "ends" of f. In particular, we will show that adding cx_0^d and x_{n+1}^d to f will work when $d \not\equiv 0, 1 \pmod{p}$, if we choose $c \in k \setminus \{0, c_{\text{bad}}\}$, where

$$c_{\text{bad}} := -d^{(1-d)^{n+1}-1}(1-d)^{\frac{(1-d)^{n+2}-(1-d)}{d}} \in k^*,$$

except that we must also avoid $c = 2^4 3^{-9}$ if $(n, d) = (2, 3)$. The hypersurface in Case I becomes singular for $c = 0$ or $c = c_{\text{bad}}$. If $(n, d, c) = (2, 3, 2^4 3^{-9})$, then the resulting cubic surface in any characteristic not 2 or 3 has a nontrivial linear automorphism given by

$$\begin{bmatrix} 324 & 6561 & 1458 & 4374 \\ 16 & 324 & -72 & -216 \\ 0 & 0 & 648 & 0 \\ 48 & -972 & -216 & 0 \end{bmatrix} \in PGL_4(k).$$

When $d \equiv 0 \pmod{p}$, we need to add a term to rule out automorphisms mapping $x_0 \mapsto x_0 + \lambda x_1$ and fixing all other x_i. (Actually such automorphisms create a problem only when d is a *power* of p.) When $d \equiv 1 \pmod{p}$, we add a few terms in order that some of the second partial derivatives of f be nonvanishing, because our method for controlling the automorphisms relies on the fact that most, but not all, of the second partial derivatives of f vanish.

The definition of f in all cases is given in Table 1. The congruence conditions on d defining the cases are congruences modulo p. Note that in Cases I and II, we have $p \neq 2$, and if $(n, d) = (2, 3)$ in Case I, then $p \neq 3$ also, so there is always at least one choice for $c \in k$. The reader who prefers to have c prescribed explicitly may take $c = 2c_{\text{bad}}$ in Case I and $c = 2(-2)^{d-2}$ in Case II.

3. Smoothness of X

This section proves that X is smooth in each case. This is not especially difficult. The hard part was *finding* the f for which this would be easy, and for which our methods for controlling the automorphisms would apply.

Case I: $d \not\equiv 0, 1 \pmod{p}$

	Case	f	Conditions
I	$d \not\equiv 0, 1$	$cx_0^d + \left(\sum_{i=0}^{n} x_i x_{i+1}^{d-1}\right) + x_{n+1}^d$	$c \neq 0, c_{\text{bad}}$ $(n, d, c) \neq (2, 3, 2^4 3^{-9})$
II	$d \equiv 0$ $p \neq 2$	$cx_0^d + x_0^2 x_1^{d-2} + \left(\sum_{i=0}^{n} x_i x_{i+1}^{d-1}\right) + x_{n+1}^d$	$c \neq 0, (-2)^{d-2}$
III	$d \equiv 0$ $p = 2$	$x_0^{d-1} x_1 + c(x_1^d + x_2^d) + \left(\sum_{i=0}^{n} x_i x_{i+1}^{d-1}\right) + x_{n+1}^d$	$c = \begin{cases} 0, & \text{if } n = 1 \\ 1, & \text{if } n \geq 2 \end{cases}$
IV	$d \equiv 1$ $p \neq 2$	$x_0^d + \left(\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} x_{2i}^2 x_{2i+1}^{d-2}\right) + \left(\sum_{i=0}^{n} x_i x_{i+1}^{d-1}\right) + x_{n+1} x_0^{d-1}$	
V	$d \equiv 1$ $p = 2$ $n = 1$	$x_0 x_1^{d-2} x_2 + x_0 x_1^{d-1} + x_1 x_2^{d-1} + x_2 x_0^{d-1} + x_1^2 x_2^{d-2}$	
VI	$d \equiv 1$ $p = 2$ $n > 1$	$x_1^d + \left(\sum_{i=0}^{\lfloor \frac{n-1}{3} \rfloor} x_{3i} x_{3i+1}^{d-2} x_{3i+2}\right) + \left(\sum_{i=1}^{n} x_i x_{i+1}^{d-1}\right) + x_{n+1} x_0^{d-1}$	

TABLE 1. Definition of $f(x_0, x_1, \ldots, x_{n+1})$.

Suppose P is a singular point. At P the derivative $\partial f / \partial x_i$ must vanish for each i:

$$\begin{aligned} 0 &= cdx_0^{d-1} + x_1^{d-1} \\ 0 &= (d-1)x_0 x_1^{d-2} + x_2^{d-1} \\ 0 &= (d-1)x_1 x_2^{d-2} + x_3^{d-1} \\ &\vdots \\ 0 &= (d-1)x_{n-1} x_n^{d-2} + x_{n+1}^{d-1} \\ 0 &= (d-1)x_n x_{n+1}^{d-2} + dx_{n+1}^{d-1}. \end{aligned} \tag{1}$$

Note that if $0 \leq i \leq n$, and $x_i = 0$ at P, then $x_{i+1} = 0$ by the equation

$$0 = (d-1)x_{i-1} x_i^{d-2} + x_{i+1}^{d-1}$$

(or the first equation, if $i = 0$), so that by induction $x_j = 0$ for all $j \geq i$. On the other hand, if $2 \leq i \leq n+1$, and $x_i = 0$, we find from

$$0 = (d-1)x_{i-2} x_{i-1}^{d-2} + x_i^{d-1}$$

that either $x_{i-1} = 0$ or $x_{i-2} = 0$, and the latter also implies $x_{i-1} = 0$ by what we just proved, so that $x_{i-1} = 0$ in any case. Also, if $x_1 = 0$, then $x_0 = 0$ by the first equation in (1).

Thus if any x_i is zero at P, all are zero at P. Hence if there is a singular point P, all its projective coordinates are nonzero. Without loss of generality, assume $x_{n+1} = 1$. Then from the last equation in (1) we find

$$x_n = d(1-d)^{-1}.$$

Substituting into the penultimate equation in (1), we find

$$x_{n-1} = d^{2-d}(1-d)^{d-3}.$$

Working our way up the list of equations, using all of them up to but not including the first, we prove by induction on i that

$$x_{n+1-i} = d^{\frac{1-(1-d)^i}{d}}(1-d)^{\frac{1-(i+1)d-(1-d)^{i+1}}{d^2}}$$

for $i = 1, 2, \ldots, n+1$. The values of x_0 and x_1 so computed contradict the first equation in (1), provided that $c \neq c_{\text{bad}}$.

Case II: $d \equiv 0 \pmod{p}$, $p \neq 2$

This time, the vanishing of the derivatives gives rise to the system

$$\begin{aligned} 0 &= 2x_0x_1^{d-2} + x_1^{d-1} \\ 0 &= -x_0x_1^{d-2} - 2x_0^2x_1^{d-3} + x_2^{d-1} \\ 0 &= -x_1x_2^{d-2} + x_3^{d-1} \\ &\vdots \\ 0 &= -x_{n-1}x_n^{d-2} + x_{n+1}^{d-1} \\ 0 &= -x_nx_{n+1}^{d-2}. \end{aligned} \tag{2}$$

As in Case I, if $2 \le i \le n$ and $x_i = 0$, then $x_{i+1} = 0$ by the equation

$$0 = -x_{i-1}x_i^{d-2} + x_{i+1}^{d-1}.$$

On the other hand, if $4 \le i \le n+1$ and $x_i = 0$, then from

$$0 = -x_{i-2}x_{i-1}^{d-2} + x_i^{d-1}$$

we obtain $x_{i-1} = 0$ or $x_{i-2} = 0$, and the latter also implies $x_{i-1} = 0$ by what we just proved, so that $x_{i-1} = 0$ in any case.

From the last equation in (2), we obtain $x_n = 0$ or $x_{n+1} = 0$, so we immediately deduce $x_i = 0$ for $3 \le i \le n+1$. If $x_1 = 0$, then we obtain $x_0 = 0$ from the original equation $f = 0$, and $x_2 = 0$ from the second equation in (2), which is a contradiction, as desired. Thus we may assume $x_1 = 1$, and then the first and third equations in (2) yield $x_0 = -1/2$ and $x_2 = 0$. For $(-\frac{1}{2} : 1 : 0 : 0 : \cdots : 0)$ to be a point on X we must have

$$c\left(-\frac{1}{2}\right)^d + \frac{1}{4} - \frac{1}{2} = 0,$$

so we obtain the desired contradiction, provided that $c \neq (-2)^{d-2}$.

Case III: $d \equiv 0 \pmod{p}$, $p = 2$

The vanishing of the derivatives gives rise to the system

$$\begin{aligned} 0 &= x_0^{d-2}x_1 + x_1^{d-1} \\ 0 &= x_0^{d-1} + x_0x_1^{d-2} + x_2^{d-1} \\ 0 &= x_1x_2^{d-2} + x_3^{d-1} \\ &\vdots \\ 0 &= x_{n-1}x_n^{d-2} + x_{n+1}^{d-1} \\ 0 &= x_nx_{n+1}^{d-2}. \end{aligned} \tag{3}$$

We deduce as in Case II that $x_3 = x_4 = \cdots = x_{n+1} = 0$. If $x_1 = 0$, then we obtain $x_2 = 0$ from the original equation $f = 0$, and $x_0 = 0$ from the second equation in (3), so all x_i are zero, a contradiction. Thus we may assume $x_1 = 1$, and the third and first equations in (3) yield $x_2 = 0$ and $x_0^{d-2} = 1$. The original equation $f = 0$ becomes

$$x_0 + 1(1 + 0) + x_0 + 0 + 0 + \cdots + 0 = 0,$$

a contradiction in characteristic 2.

Case IV: $d \equiv 1 \pmod{p}$, $p \neq 2$

The vanishing of the derivatives gives rise to the system

$$\begin{aligned}
0 &= x_1^{d-1} + 2x_0 x_1^{d-2} + x_0^{d-1} \\
0 &= x_2^{d-1} - x_0^2 x_1^{d-3} \\
0 &= x_3^{d-1} + 2x_2 x_3^{d-2} \\
0 &= x_4^{d-1} - x_2^2 x_3^{d-3} \\
&\vdots \\
0 &= x_{n+1}^{d-1} \qquad (-x_{n-1}^2 x_n^{d-3} \quad \text{if } n \text{ is odd}) \\
0 &= x_0^{d-1}.
\end{aligned} \tag{4}$$

The last equation implies $x_0 = 0$. The first then implies $x_1 = 0$, and going down the list of equations we show by induction that $x_i = 0$ for all i. (Note that the conditions defining this case imply $d \geq 4$, so the exponent $d - 3$ and anything larger will be positive.)

Case V: $d \equiv 1 \pmod{p}$, $p = 2$, $n = 1$

The vanishing of the derivatives gives rise to the system

$$\begin{aligned}
0 &= x_1^{d-2} x_2 + x_1^{d-1} \\
0 &= x_0 x_1^{d-3} x_2 + x_2^{d-1} \\
0 &= x_0 x_1^{d-2} + x_0^{d-1} + x_1^2 x_2^{d-3}.
\end{aligned} \tag{5}$$

Working from the bottom up, we find

$$x_1 = 0 \implies x_0 = 0 \implies x_2 = 0 \implies x_1 = 0.$$

Thus if any x_i is zero, all the x_i are zero, a contradiction. Hence all the x_i are nonzero. Then the first equation in (5) implies $x_1 = x_2$. Substituting $x_2 = x_1$ in the second equation yields $0 = x_0 x_1^{d-2} + x_1^{d-1}$, so $x_0 = x_1 = x_2$. Substituting these into the third equation, we find $x_0^{d-1} = 0$, so $x_0 = 0$, a contradiction.

Case VI: $d \equiv 1 \pmod{p}$, $p = 2$, $n > 1$

The vanishing of the derivatives gives rise to the system

$$\begin{aligned}
0 &= x_1^{d-2}x_2\\
0 &= x_2^{d-1} + x_0x_1^{d-3}x_2 + x_1^{d-1}\\
0 &= x_3^{d-1} + x_0x_1^{d-2}\\
0 &= x_4^{d-1} + x_4^{d-2}x_5\\
0 &= x_5^{d-1} + x_3x_4^{d-3}x_5 \qquad (6)\\
0 &= x_6^{d-1} + x_3x_4^{d-2}\\
&\vdots\\
0 &= x_0^{d-1} \qquad (+x_{n-1}x_n^{d-2} \quad \text{if } n \equiv 1 \pmod 3)
\end{aligned}$$

(To see the pattern, pretend that the exceptional term x_1^{d-1} in the second equation were actually in the first, and group the equations in threes.) Let $m = 3\lfloor \frac{n+2}{3} \rfloor$. There are zero, one, or two equations past the first m equations, and these final ones are those that are missing a "second term," i.e., that are simply of the form $0 = x_i^{d-1}$ for some i.

By the first equation, either x_1 or x_2 is zero. If $x_2 = 0$, then $x_1 = 0$ by the second equation, so $x_1 = 0$ in any case. The third equation then yields $x_3 = 0$. For $i = 3, 6, \ldots, m-3$, the $(i+2)$-th, $(i+1)$-th, and $(i+3)$-th equations show that

$$x_i = 0 \implies x_{i+2} = 0 \implies x_{i+1} = 0 \implies x_{i+3} = 0,$$

where we should interpret x_{n+2} as x_0 if necessary. Thus we deduce $x_i = 0$ for $3 \le i \le m$.

If $m = n$, we have $x_{m+1} = 0$ and $x_0 = 0$ automatically from the last two equations in (6). If $m = n+1$, we have $x_0 = 0$ automatically from the last equation. If $m = n+2$, we have already shown $x_0 = x_{n+2} = 0$. Thus in every case we have $x_i = 0$ for all i except possibly $i = 2$. Finally, we obtain $x_2 = 0$ from the second equation in (6).

4. Controlling the automorphisms: the idea

The remainder of the paper is devoted to proving that $\operatorname{Lin} \overline{X}$ is trivial in each case. In this section, we explain the main tool to be used, and introduce some notation.

Suppose we are in Case I. Then $\partial f/\partial x_0$ is killed by $\partial/\partial x_i$ for all $i \ge 2$. If we have a linear automorphism of X given by the matrix $L = (\ell_{ij}) \in \mathrm{GL}_{n+2}(\overline{k})$, and if we set $y_i = \sum_{j=0}^{n+1} \ell_{ij}x_j$, then

$$f(x_0, x_1, \ldots, x_{n+1}) = \alpha f(y_0, y_1, \ldots, y_{n+1}) \qquad (7)$$

for some nonzero scalar $\alpha \in \overline{k}^*$, and

$$\frac{\partial}{\partial x_0} f(x_0, x_1, \ldots, x_{n+1}) = \alpha \sum_{i=0}^{n+1} \ell_{i0} \frac{\partial f(y_0, y_1, \ldots, y_{n+1})}{\partial y_i}$$

is killed by at least an $(n-2)$-dimensional subspace of the span of the operators $\partial/\partial x_j$, which is also the span of the $\partial/\partial y_i$. Such considerations will severely constrain the possibilities for the entries of the matrix L.

In general, let A denote the Hessian matrix of f, with entries $a_{ij} := \frac{\partial^2 f}{\partial x_i \partial x_j}$. For (column) vectors $v = (v_0, v_1, \ldots, v_{n+1})$ and $w = (w_0, w_1, \ldots, w_{n+1})$ in $\overline{k}^{n+2}$, we define a symmetric $\overline{k}$-linear pairing

$$\langle v, w \rangle := v^{\mathrm{t}} A w = \sum_{i=0}^{n+1} \sum_{j=0}^{n+1} v_i w_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

taking values in $\overline{k}[x_0, x_1, \ldots, x_{n+1}]$. If $L = (\ell_{ij}) \in \mathrm{GL}_{n+2}(\overline{k})$ gives an automorphism of X, then $\langle v, w\rangle$ is a scalar multiple of the result of replacing each x_i by $\sum_{j=0}^{n+1} \ell_{ij} x_j$ in $\langle Lv, Lw\rangle$. In particular, and this is mainly what we will use,

$$\langle Lv, Lw\rangle = 0 \iff \langle v, w\rangle = 0.$$

For a vector $v \in \overline{k}^{n+2}$, define a subspace

$$v^\perp := \{\, w \in \overline{k}^{n+2} : \langle v, w\rangle = 0 \,\}.$$

For any subspace V, let $\operatorname{codim} V$ denote the codimension of V as a subspace of $\overline{k}^{n+2}$. In the subsequent sections we will repeatedly use the following (trivial) observation.

Lemma 4.1. *The number* $\operatorname{codim} v^\perp$ *equals the dimension of the* $\overline{k}$*-vector space spanned by the (polynomial) entries of the column vector* Av*.*

Proof. Both numbers equal the dimension of the image of $(Av)^{\mathrm{t}}$, considered as a linear function on $\overline{k}^{n+2}$. □

For a subspace $V \subseteq \overline{k}^{n+2}$, define a subspace

$$V^\perp := \{\, w \in \overline{k}^{n+2} : \langle v, w\rangle = 0 \text{ for all } v \in V \,\}.$$

If L gives an automorphism of X, then for all vectors v and subspaces V,

$$(Lv)^\perp = L\left(v^\perp\right) \qquad \text{and} \qquad (LV)^\perp = L\left(V^\perp\right), \tag{8}$$

so in particular

$$\operatorname{codim}(Lv)^\perp = \operatorname{codim} v^\perp \qquad \text{and} \qquad \operatorname{codim}(LV)^\perp = \dim V^\perp. \tag{9}$$

We let $\{e_0, e_1, \ldots, e_{n+1}\}$ denote the standard basis for $\overline{k}^{n+2}$.

For $0 \le m \le n+1$, define subspaces

$$S_m := \sum_{i=0}^{m} \overline{k} \cdot e_i, \qquad T_m := \sum_{i=m}^{n+1} \overline{k} \cdot e_i.$$

Also set $S_m = 0$ if $m < 0$, and $T_m = 0$ if $m > n+1$.

Once we have taken full advantage of the fact that L respects the pairing, we can usually complete the proof that L is a scalar multiple of the identity simply by equating various coefficients in (7).

5. Controlling the automorphisms: Case I

In this case we have

$$A = \begin{bmatrix} h_0 & g_1 & 0 & 0 & \cdots & 0 & 0 \\ g_1 & h_1 & g_2 & 0 & \cdots & 0 & 0 \\ 0 & g_2 & h_2 & g_3 & \cdots & 0 & 0 \\ 0 & 0 & g_3 & h_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & h_n & g_{n+1} \\ 0 & 0 & 0 & 0 & \cdots & g_{n+1} & h_{n+1} \end{bmatrix}$$

where $g_i := (d-1)x_i^{d-2}$ and

$$h_i := \begin{cases} cd(d-1)x_0^{d-2} & \text{if } i = 0 \\ (d-1)(d-2)x_{i-1}x_i^{d-3} & \text{if } 1 \le i \le n \\ d(d-1)x_{n+1}^{d-2} + (d-1)(d-2)x_n x_{n+1}^{d-3} & \text{if } i = n+1. \end{cases}$$

We will subdivide Case I as follows (recall that $p \neq 2$ throughout this case):

- Case I.1: $d \not\equiv 0, 1, 2 \pmod p$ and $d \neq 3$
- Case I.2: $d \equiv 2 \pmod p$ and $p \neq 2$
- Case I.3: $d = 3$; $p \neq 2, 3$; and $n \geq 2$.

Case I.1: $d \not\equiv 0, 1, 2 \pmod p$ *and* $d \neq 3$

In this subcase, $g_1, \ldots, g_{n+1}, h_0, h_1, \ldots, h_{n+1}$ are linearly independent over $\overline{k}$. In particular, note that $e_m^\perp$ equals $S_{m-2} + T_{m+2}$, which is the $\overline{k}$-vector space spanned by all the e_i except e_{m-1}, e_m, and e_{m+1}.

Lemma 5.1. *For any* $v = (v_0, v_1, \ldots, v_{n+1}) \in \overline{k}^{n+2}$,

$$v^\perp = \bigcap_{i : v_i \neq 0} e_i^\perp.$$

Proof. Suppose $w = (w_0, w_1, \ldots, w_{n+1}) \in \overline{k}^{n+2}$. If the i-th coordinate of Aw is nonzero, then at least one of w_{i-1}, w_i, w_{i+1} is nonzero. If w_i is nonzero, then h_i occurs in the i-th coordinate of Aw and in no other coordinates. If $w_i = 0$ but $w_{i-1} \neq 0$, then then g_i occurs in the i-th coordinate of Aw and in no other coordinates. If $w_i = 0$ but $w_{i+1} \neq 0$, then then g_{i+1} occurs in the i-th coordinate of Aw and in no other coordinates. The nonzero coordinates of Aw are thus linearly independent over $\overline{k}$, since each involves a g or h not present in the other coordinates. In other words, the polynomials $\langle e_i, w \rangle$ for $i = 0, 1, \ldots, n+1$ that are nonzero are linearly independent. Thus if $\langle v, w \rangle = 0$ and $v_i \neq 0$, then $\langle e_i, w \rangle = 0$. □

Lemma 5.1 and the remark preceding it let us immediately calculate $v^\perp$ for any vector v, and also $V^\perp$ for any subspace V, since $V^\perp = \bigcap_{v \in V} v^\perp$. In particular, we obtain the following corollaries.

Corollary 5.2. *If* $v \in \overline{k}^{n+2}$ *is nonzero, then* $\operatorname{codim} v^\perp \geq 2$, *with equality if and only if* v *is a multiple of* e_0 *or* e_{n+1}.

Note that for $0 \leq m \leq n$, the $(m+1)$-dimensional subspace $S_m \subset \overline{k}^{n+2}$ has $S_m^\perp = T_{m+2}$, and $\operatorname{codim} S_m^\perp = m + 2$.

Corollary 5.3. *Suppose* $0 \leq m \leq n - 2$. *Let* V *be an* $(m+2)$*-dimensional subspace of* $\overline{k}^{n+2}$ *containing* S_m. *Then* $\operatorname{codim} V^\perp \geq m + 3$, *with equality if and only if* $V = S_{m+1}$.

Proof. Write $V = S_m + \overline{k} \cdot v$, so $V^\perp = S_m^\perp \cap v^\perp$. If v has any nonzero coordinate v_i with $m+2 \leq i \leq n$, then the condition that an element w of $S_m^\perp$ be in $v^\perp$ places at least two linear conditions on w, namely $w_i = 0$ and $w_{i+1} = 0$, so $\operatorname{codim} V^\perp \geq \operatorname{codim} S_m^\perp + 2 = m + 4$ in this case. Similarly, if $v_{n+1} \neq 0$, then the condition that an element w of $S_m^\perp$ be in $v^\perp$ places the new conditions $w_n = 0$ and $w_{n+1} = 0$ on w, so that $\operatorname{codim} V^\perp \geq m + 4$ again. The only remaining possibility is that $v_i = 0$ for all $i \geq m+2$, in which case we must have $V = S_{m+1}$ and $\operatorname{codim} V^\perp = \operatorname{codim} T_{m+3} = m + 3$. □

Corollary 5.4. *Suppose* $3 \leq m \leq n + 1$. *Let* V *be an* $(n - m + 3)$*-dimensional subspace of* $\overline{k}^{n+2}$ *containing* T_m. *Then* $\operatorname{codim} V^\perp \geq n - m + 4$, *with equality if and only if* $V = T_{m-1}$.

Proof. The proof is completely analogous to that of Corollary 5.3. □

Corollary 5.5. *The vectors* Le_0 *and* Le_{n+1} *are multiples of* e_0 *and* e_{n+1} *in some order.*

Proof. This follows from (9) and Corollary 5.2. □

We may now subdivide Case I.1 further into two subcases.

Case I.1.a: Le_0 is a multiple of e_0

Corollary 5.3 gives a characterization of the flag $S_1 \subset S_2 \subset \cdots \subset S_{n-1}$ of vector spaces containing S_0 that involves only dimensions and the $\perp$-operation. Since L preserves S_0 by assumption, we have $L(S_m) = S_m$ for $0 \le m \le n-1$. Similarly by Corollary 5.4, $L(T_m) = T_m$ for $2 \le m \le n+1$. Together, these imply that L is of the form[1]

$$L = \begin{bmatrix} * & * & 0 & \cdots & 0 & 0 & 0 \\ 0 & * & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & * & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & * & 0 \\ 0 & 0 & 0 & \cdots & 0 & * & * \end{bmatrix},$$

with zeros off the diagonal except possibly in positions ℓ_{01} and $\ell_{n+1,n}$. Since L is nonsingular, $\ell_{ii} \neq 0$ for all i, and by scaling L, we may assume $\ell_{n+1,n+1} = 1$. By equating coefficients of x_{n+1}^d in (7), we see that $\alpha = 1$. Equating coefficients of x_n^d and of $x_n^{d-1}x_{n+1}$ in

$$f(x_0, x_1, \ldots, x_n, x_{n+1}) = f(\ell_{00}x_0 + \ell_{01}x_1, \ell_{11}x_1, \ldots, \ell_{nn}x_n, \ell_{n+1,n}x_n + x_{n+1}),$$

we obtain

$$\begin{aligned} 0 &= \ell_{nn}\ell_{n+1,n}^{d-1} + \ell_{n+1,n}^d \\ 0 &= (d-1)\ell_{nn}\ell_{n+1,n}^{d-2} + d\ell_{n+1,n}^{d-1}. \end{aligned}$$

Multiply the first by $(d-1)$ and the second by $\ell_{n+1,n}$, and subtract to deduce $\ell_{n+1,n} = 0$. For $i = n, n-1, \ldots, 1$ in turn, we equate coefficients of $x_i x_{i+1}^{d-1}$ to find $\ell_{ii} = 1$. Equate coefficients of $x_0^{d-1}x_1$ and use $\ell_{00} \neq 0$ and $d \not\equiv 0 \pmod p$ to deduce $\ell_{01} = 0$. Finally equate coefficients of $x_0 x_1^{d-1}$ to deduce $\ell_{00} = 1$. Thus L is the identity, as desired.

Case I.1.b: Le_0 is a multiple of e_{n+1}

This time Corollaries 5.3 and 5.4 imply that $L(S_m) = T_{n+1-m}$ for $0 \le m \le n-1$ and $L(T_m) = S_{n+1-m}$ for $2 \le m \le n+1$, so that L is of the form

$$L = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & * & * \\ 0 & 0 & 0 & \cdots & 0 & * & 0 \\ 0 & 0 & 0 & \cdots & * & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & * & \cdots & 0 & 0 & 0 \\ 0 & * & 0 & \cdots & 0 & 0 & 0 \\ * & * & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

with nonzero entries on the reverse diagonal, and zero entries off it, except possibly at ℓ_{0n} and $\ell_{n+1,1}$. We may assume $\ell_{0,n+1} = 1$. Equating coefficients of x_n^d and of $x_n^{d-1}x_{n+1}$ in

$$f(x_0, x_1, \ldots, x_n, x_{n+1}) = f(\ell_{0n}x_n + x_{n+1}, \ell_{1n}x_n, \ldots, \ell_{n1}x_1, \ell_{n+1,0}x_0 + \ell_{n+1,1}x_1)$$

[1] Each asterisk in a matrix denotes an element of $\overline{k}$ which may or may not be zero.

we find

$$0 = c\ell_{0n}^d + \ell_{0n}\ell_{1n}^{d-1}$$
$$0 = cd\ell_{0n}^{d-1} + \ell_{1n}^{d-1}.$$

Subtracting ℓ_{0n} times the second from the first, we find $c(1-d)\ell_{0n}^d = 0$, so $\ell_{0n} = 0$. Substituting back into the second, we find $\ell_{1n} = 0$ as well. But this contradicts the nonsingularity of L.

Case I.2: $d \equiv 2 \pmod p$ *and* $p \neq 2$

We have

$$A = \begin{bmatrix} 2cg_0 & g_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ g_1 & 0 & g_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & g_2 & 0 & g_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & g_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & g_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & g_n & 0 & g_{n+1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & g_{n+1} & 2g_{n+1} \end{bmatrix},$$

and $g_0, g_1, \ldots, g_{n+1}$ are linearly independent over $\overline{k}$.

Lemma 5.6. *If* $v \in \overline{k}^{n+2}$ *is nonzero, then* $\operatorname{codim} v^\perp \geq 1$*, with equality if and only if* v *is a multiple of* e_{n+1}*.*

Proof. By Lemma 4.1, $\operatorname{codim} e_{n+1}^\perp = 1$, and the same is true for any multiple of e_{n+1}.

Now assume instead that the first nonzero coordinate v_i in v occurs for $i \leq n$. Then g_i appears exactly once in the coordinates of Av, namely in the $(i-1)$-th coordinate (or in the 0-th coordinate if $i = 0$). If $i < n$, then g_{i+1} appears in the $(i+1)$-th coordinate of Av, since $a_{i+1,i}$ is independent of the other entries of its row in A, so the span of the coordinates of Av has dimension at least 2. If $i = n$, then g_{n+1} appears in either the n-th or the $(n+1)$-th coordinate of Av, so again the span of the coordinates of Av has dimension at least 2. Thus $\operatorname{codim} v^\perp \geq 2$ by Lemma 4.1. □

Corollary 5.7. *We have* $L(T_{n+1}) = T_{n+1}$*.*

The $(n-m+2)$-dimensional space T_m satisfies $T_m^\perp = S_{m-2}$ if $2 \leq m \leq n$.

Lemma 5.8. *For nonzero* $v \in \overline{k}^{n+2}$*, we have* $\operatorname{codim} v^\perp \leq 2$ *if and only if* v *is a multiple of some* e_i *or is a linear combination of* e_n *and* e_{n+1}*.*

Proof. The "if" direction is clear from Lemma 4.1. Now suppose $\operatorname{codim} v^\perp \leq 2$ and that the first nonzero v_i in v occurs for $i < n$. We must show that v is a multiple of e_i. As in the proof of Lemma 5.6, g_i appears exactly once in the coordinates of Av, namely in the $(i-1)$-th coordinate (or the 0-th coordinate if $i = 0$), and g_{i+1} appears in the $(i+1)$-th coordinate, so at least these two coordinates are linearly independent. Suppose for sake of contradiction that $v_j \neq 0$ for some $j > i$, and choose the largest such j. If $j \leq n$, then g_{j+1} appears in the $(j+1)$-th coordinate of Av but not before, so it is independent of the $(i-1)$-th and $(i+1)$-th coordinates, and the span is of dimension at least 3, as desired. If $j = n+1$, then g_{n+1} appears in the n-th coordinate of Av and not before, so we are again done, unless $i + 1 = n$.

To handle the remaining case $i = n-1$, $j = n+1$ we break into cases according as $v_n = 0$ or not. If $v_n = 0$, then g_{n-1} appears only in the $(n-2)$-th coordinate of Av, g_n appears only in the n-th coordinate of Av, and g_{n+1} appears in the $(n+1)$-th coordinate of Av, so these three coordinates are independent. If $v_n \neq 0$, then g_{n-1} appears only in the $(n-2)$-th coordinate of Av, a pure multiple of g_n occurs in the $(n-1)$-th coordinate, and a non-pure combination of g_n and g_{n+1} occurs in the n-th coordinate, so again the span of the coordinates is at least 3-dimensional. Hence $\operatorname{codim} v^\perp \geq 3$. □

Corollary 5.9. *We have $L(T_n) = T_n$.*

Proof. Lemma 5.8 shows that T_n is the only 2-dimensional subspace consisting entirely of vectors v for which $\operatorname{codim} v^\perp \leq 2$. □

Lemma 5.10. *For $0 \leq i \leq n-1$, Le_i is a multiple of e_i.*

Proof. By Lemma 5.8 and Corollary 5.9, L acts on $e_0, e_1, \ldots, e_{n-1}$ by scaling them independently and then permuting them. Equating coefficients of x_0^d in (7) we see that Le_0 must be a multiple of e_0. By induction on i for $1 \leq i \leq n-1$, equating coefficients of $x_{i-1}x_i^{d-1}$ shows that Le_i must be a multiple of e_i. □

Lemma 5.10 and Corollaries 5.7 and 5.9 together imply that L is of the form

$$L = \begin{bmatrix} * & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & * & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & * & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & * & 0 \\ 0 & 0 & 0 & \cdots & 0 & * & * \end{bmatrix}.$$

The argument at the end of Case I.1.a now implies that L is (a scalar multiple of) the identity.

Case I.3: $d = 3$; $p \neq 2, 3$; and $n \geq 2$

We have

$$A = 2\begin{bmatrix} 3cx_0 & x_1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ x_1 & x_0 & x_2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & x_2 & x_1 & x_3 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & x_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x_{n-3} & x_{n-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & x_{n-1} & x_{n-2} & x_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & x_n & x_{n-1} & x_{n+1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & x_{n+1} & x_n + 3x_{n+1} \end{bmatrix}.$$

Lemma 5.11. *If $n > 2$, then $\operatorname{codim} v^\perp \leq 2$ if and only if v is a multiple of one of the following:*

$$e_0, \qquad e_0 \pm \sqrt{3c}e_1, \qquad 3e_n - e_{n+1}, \qquad e_{n+1}.$$

If $n = 2$, the same is true, except that multiples of $e_0 \pm \sqrt{3c}(e_1 + 3e_3)$ are also possible.

Proof. We may assume $v \neq 0$. Let j be the largest integer such that v_j is nonzero.

If $j = 0$, then v is a multiple of e_0, and $\operatorname{codim} v^\perp = 2$.

If $j = 1$, then to have $\operatorname{codim} v^\perp \leq 2$, we must have $v_0 \neq 0$, since $\operatorname{codim} e_1^\perp = 3$. Assume $v_0 = 1$. Then

$$Av = 2(3cx_0 + v_1x_1, v_1x_0 + x_1, v_1x_2, 0, 0, \ldots, 0).$$

In order for the span of the coordinates to have dimension at most 2, the first two coordinates must be dependent. By looking at the coefficients of x_1, we see that this would imply

$$3cx_0 + v_1x_1 = v_1(v_1x_0 + x_1),$$

which holds if and only if $v_1 = \pm\sqrt{3c}$.

If $2 \leq j \leq n$, then x_{j+1} appears in the $(j+1)$-th coordinate of Av but not before, and the j-th coordinate is a nonzero combination of x_{j-1} and x_j, so if $\operatorname{codim} v^\perp \leq 2$, then the 0-th, 1-st, ..., $(j-1)$-th coordinates of Av are all multiples of the j-th coordinate. In particular, x_{j-2} does not appear in the $(j-1)$-th coordinate of Av, so $v_{j-1} = 0$. Thus the j-th coordinate of Av is a multiple of x_{j-1}. But the $(j-1)$-th coordinate involves x_j, so it cannot be a multiple of the j-th coordinate, a contradiction.

Finally we have the case $j = n+1$. Suppose $\operatorname{codim} v^\perp \leq 2$. If $v_n \neq 0$, then x_{n-1} appears in the n-th coordinate of Av and not afterwards, and the $(n+1)$-th coordinate is nonzero, so these coordinates already span a 2-dimensional space, and all others must be dependent on them. In this case, all coordinates must be combinations of x_{n-1}, x_n, and x_{n+1} only. If furthermore $0 \leq i < n$ and $v_i \neq 0$, we get a contradiction by observing that x_{i-1} (x_0 if $i = 0$) appears in the i-th coordinate of Av. Thus, from our assumption $v_n \neq 0$ we deduce that v is a combination of e_n and e_{n+1} in which both appear. The $(n-1)$-th coordinate of Av is a nonzero multiple of x_n, and this can be in the span of the n-th and $(n+1)$-th coordinates only if the $(n+1)$-th coordinate also is a multiple of x_n, which happens if and only if v is a multiple of $3e_n - e_{n+1}$.

Thus from now on, we may assume $j = n+1$ *and* $v_n = 0$. If $v_{n-1} \neq v_{n+1}/3$, then the last two coordinates of Av are independent, so in order to have $\operatorname{codim} v^\perp \leq 2$, all other coordinates must be combinations of these last two. In particular, they would all be combinations of x_n and x_{n+1} only. For $0 \leq i \leq n-1$, the non-appearance of x_{i-1} (of x_0 if $i = 0$) in the i-th coordinate of Av then forces $v_i = 0$, so that v is a multiple of e_{n+1}, and in this case $\operatorname{codim} v^\perp = 2$.

Finally we have the case $v_n = 0$, $v_{n-1} = v_{n+1}/3 \neq 0$. The $(n-1)$-th coordinate of Av is a combination of x_{n-1} and x_{n-2} in which the latter appears, and the $(n+1)$-th coordinate is a multiple of $x_n + 3x_{n+1}$. These already span a 2-dimensional space, so if $\operatorname{codim} v^\perp \leq 2$, all other coordinates must be combinations of x_{n-2}, x_{n-1}, x_n, and x_{n+1}. Suppose that $n > 2$. Then for $0 \leq i \leq n-2$, the non-appearance of x_{i-1} (of x_0 if $i = 0$) in the i-th coordinate of Av forces $v_i = 0$. The $(n-2)$-th, $(n-1)$-th, and $(n+1)$-th coordinates of Av are now nonzero multiples of x_{n-1}, x_{n-2}, and $x_n + 3x_{n+1}$, respectively, so there are independent, and $\operatorname{codim} v^\perp \geq 3$.

We are left with the case $n = 2$, $v_2 = 0$, $v_1 = v_3/3$. If $v_1 \neq \pm\sqrt{3c}v_0$, then the 0-th and 1-st coordinates of Av are independent, and neither involves x_3, so the last coordinate is independent of both of them, yielding $\operatorname{codim} v^\perp \geq 3$. Otherwise, if $v_1 = \pm\sqrt{3c}v_0$, then v is a nonzero multiple of $e_0 \pm \sqrt{3c}(e_1 + 3e_3)$, and we check that in this case $\operatorname{codim} v^\perp = 2$. □

We next subdivide Case I.3 according as $n = 2$ or $n > 2$.

Case I.3.a: $n > 2$

Corollary 5.12. *We have $L(S_1) = S_1$ and $L(T_n) = T_n$.*

Proof. By (8), L must permute the five lines generated by the vectors listed in Lemma 5.11. The only 2-dimensional subspace of $\overline{k}^{n+2}$ containing three of these five lines in S_1, so $L(S_1) = S_1$. The subspace spanned by the other two lines is T_n, so $L(T_n) = T_n$. □

Lemma 5.13. *The vectors Le_n and Le_{n+1} are nonzero multiples of e_n and e_{n+1}, respectively.*

Proof. By Corollary 5.12, we know $L(T_n) = T_n$. Hence $y_0, y_1, \ldots, y_{n-1}$ are linear combinations of $x_0, x_1, \ldots, x_{n-1}$ only.

Substituting $x_0 = x_1 = \cdots = x_{n-1} = 0$ in (7), we find

$$(x_n + x_{n+1})x_{n+1}^2 = \alpha(z_n + z_{n+1})z_{n+1}^2, \tag{10}$$

where z_i denotes the part of the linear form y_i involving x_n and x_{n+1}. By unique factorization, this implies that z_{n+1} is a nonzero scalar multiple of x_{n+1}. Without loss of generality, we may assume $z_{n+1} = x_{n+1}$; i.e. $\ell_{n+1,n+1} = 1$. Equating coefficients of x_{n+1}^3 in (10), we obtain $\alpha = 1$. Now (10) implies $z_n = x_n$. This gives the desired result. □

Corollary 5.14. *We have $L(S_{n-2}) = S_{n-2}$ and $L(S_{n-1}) = S_{n-1}$.*

Proof. This follows from Lemma 5.13, since $e_n^\perp = S_{n-2}$ and $e_{n+1}^\perp = S_{n-1}$. □

Lemma 5.15. *For $1 \le m \le n+1$, $T_m^\perp = S_{m-2}$.*

Proof. We use backwards induction on m. Clearly $T_{n+1}^\perp = S_{n-1}$. For $1 \le m \le n$,

$$T_m^\perp = T_{m+1}^\perp \cap e_m^\perp = S_{m-1} \cap (S_{m-2} + T_{m+2}) = S_{m-2}.$$

□

Lemma 5.16. *For $2 \le m \le n+1$, Le_m is a multiple of e_m.*

Proof. We know it already for $m = n+1$ and $m = n$. We use backwards induction on m. Suppose $2 \le m \le n-1$, and that $Le_{m'}$ is a multiple of $e_{m'}$ for $m' > m$. Then T_{m+1} and T_{m+2} are each preserved by L, and so are $S_{m-1} = T_{m+1}^\perp$ and $S_m = T_{m+2}^\perp$ by Lemma 5.15. Hence if $v = Le_m$, then $v \in S_m$, since $e_m \in S_m$. Also $v \notin S_{m-1}$, since otherwise $L(S_m) \subset S_{m-1}$, and L would not be invertible. Moreover $v^\perp \cap S_{m-1}$ has codimension 1 in S_{m-1}, since $e_m^\perp \cap S_{m-1}$ has codimension 1 in S_{m-1}. In other words, the span of the 0-th, 1-st, ..., $(m-1)$-th coordinates of Av is 1-dimensional. But x_m appears in the $(m-1)$-th coordinate of Av (since $v \in S_m \setminus S_{m-1}$), and not before, so the 0-th, 1-st, ..., $(m-2)$-th coordinates must all be zero. This forces $v_0 = v_1 = \cdots = v_{m-1} = 0$, so $v = Le_m$ is a multiple of e_m. □

We have $S_0 = S_1 \cap e_2^\perp$, so S_0 also is fixed by L. Putting this together with Corollary 5.12 and Lemma 5.16, we see that L is of the form

$$L = \begin{bmatrix} * & * & 0 & \cdots & 0 & 0 & 0 \\ 0 & * & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & * & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & * & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & * \end{bmatrix}.$$

The argument at the end of Case I.1.a now implies that L is (a scalar multiple of) the identity.

Case I.3.b: $n = 2$

The form defining X is

$$f := cx_0^3 + x_0x_1^2 + x_1x_2^2 + x_2x_3^2 + x_3^3.$$

For future convenience, we will make the change of coordinates

$$(x_0, x_1, x_2, x_3) \mapsto (x_0, x_1, x_3, -x_2 - x_3/3),$$

and for the rest of Case I.3.b, we will work with the new f, which is

$$f := cx_0^3 + x_0x_1^2 - x_2^3 + \left(\frac{3x_1 + x_2}{3}\right)x_3^2 + \frac{2x_3^3}{27}.$$

The new A is

$$A = 2 \begin{bmatrix} 3cx_0 & x_1 & 0 & 0 \\ x_1 & x_0 & 0 & x_3 \\ 0 & 0 & -3x_2 & \frac{1}{3}x_3 \\ 0 & x_3 & \frac{1}{3}x_3 & x_1 + \frac{1}{3}x_2 + \frac{2}{9}x_3 \end{bmatrix}.$$

The set W of vectors $v \in \overline{k}^4$ such that $\operatorname{codim} v^\perp \le 2$ is the union of seven lines, the transform of those generated by the seven vectors in Lemma 5.11. They are the lines E_1, E_2, …, E_7 generated by e_0, $e_0 + \sqrt{3c}e_1$, $e_0 - \sqrt{3c}e_1$, e_3, e_2, $e_0 + \sqrt{3c}(e_1 - 3e_2)$, and $e_0 - \sqrt{3c}(e_1 - 3e_2)$, respectively. By (8), L must permute the E_i.

There are four 2-dimensional subspaces of $\overline{k}^4$ containing exactly three of these lines, namely

$$\begin{aligned} W_1 &:= S_1 \supset E_1, E_2, E_3 \\ W_2 &:= \overline{k}e_0 + \overline{k}(e_1 - 3e_2) \supset E_1, E_6, E_7 \\ W_3 &:= \overline{k}(e_0 + \sqrt{3c}e_1) + \overline{k}e_2 \supset E_2, E_5, E_6 \\ W_4 &:= \overline{k}(e_0 - \sqrt{3c}e_1) + \overline{k}e_2 \supset E_3, E_5, E_7. \end{aligned}$$

The only E_i not contained in any W_j is E_4, so $L(E_4) = E_4$. The span of the other six E_i is S_2, so $L(S_2) = S_2$.

We now know that L has the form

$$L = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}.$$

Without loss of generality we may assume $\ell_{33} = 1$. Equating coefficients of x_3^3 in (7), we find $\alpha = 1$. By viewing both sides of (7) as polynomials in x_3, we see that the forms $cx_0^3 + x_0x_1^2 - x_2^3$ and $3x_1 + x_2$ are each preserved by L.

If $\gamma \in \overline{k}^*$ and the plane cubic

$$(11) \qquad (cx_0^3 + x_0x_1^2 - x_2^3) + \gamma\,(3x_1 + x_2)^3 = 0$$

has a unique singularity[2], then that singularity is preserved by the automorphism induced by L. A short calculation shows that the singularities on these curves are at the points $(x_0 : x_1 : x_2) = (-9s^2 : 2 : 2s)$, where $s \in \overline{k} \setminus \{0, -3\}$ satisfies $c = -\frac{4}{243s^4}$ and $\gamma = \left(\frac{s}{3+s}\right)^2$.

For $c \neq -2^2 3^{-9}, 2^4 3^{-9}$, we find that there are four possibilities for s, giving rise to four distinct values of γ for which the curve has a unique singularity. The four distinct points so obtained are in general position in $\mathbf{P}^2$, since they lie on the conic $2x_0x_1 + 9x_2^2 = 0$. Hence an automorphism of $\mathbf{P}^2$ that fixes them is trivial. Together with the fact that L preserves the form $3x_1 + x_2$ (and not just up to scalar multiple), this implies that the upper left 3×3 block of L is the identity, so L is the identity.

If $c = -2^2 3^{-9}$, then we may assume $p \neq 5$ in addition to $p \neq 2, 3$, since $c_{\text{bad}} = -2^6 3^{-9}$ coincides with this c in characteristic 5. We dehomogenize the cubic

$$-2^2 3^{-9} x_0^3 + x_0x_1^2 - x_2^3 = 0$$

by setting $x = x_2/x_0$ and $y = x_1/x_0$, to obtain the elliptic curve in Weierstrass form

$$E : y^2 = x^3 + 2^2 3^{-9}.$$

[2]There is automatically at most one singularity if the cubic is irreducible.

(As usual, we choose the point at infinity as origin O on E, to make E an algebraic group.) Then L induces an automorphism σ of $\mathbf{P}^2$ preserving E. The automorphism σ also preserves the line $3y + x = 0$, which is tangent to E at $P := (2/27, -2/81)$ and meets E again at $[-2]P = (-1/27, 1/81)$. Hence $\sigma(P) = P$. The action of σ on E is the composition of an automorphism η of E as an elliptic curve (i.e. fixing O), and a translation on E. Since σ preserves the class of a line section, which is the class of the divisor $3 \cdot O$, the translation must be a translation by a 3-torsion point T. It follows that η fixes $[3]P = (-2/81, 10/729)$. The six automorphisms of E have the form $(x, y) \mapsto (\pm x, \omega y)$, where $\omega^3 = 1$, but $x([3]P)$ and $y([3]P)$ are finite and nonzero in k, so η must be the identity. Since $\sigma(P) = P$, it then follows that $T = O$. Thus σ fixes E pointwise, and hence is the identity. Since L does not scale $3x_1 + x_2$, this implies that L is the identity.

The last case $c = 2^4 3^{-9}$ (in which two of the four s-values, namely $-3/2 + 3i/2$ and $-3/2 - 3i/2$ give rise to the same γ) was ruled out by assumption at the very beginning, so we are done with Case I.3.b, and indeed we are done with all of Case I.

6. Controlling the automorphisms: Case II

We will subdivide Case II as follows (recall that $p \neq 2$ throughout this case):

- Case II.1: $d \equiv 0 \pmod{p}$; $d \neq 3$; and $p \neq 2, 3$
- Case II.2: $d \equiv 0 \pmod{p}$; $d \neq 3$; and $p = 3$
- Case II.3: $d = 3$, $p = 3$, and $n \geq 2$.

Case II.1: $d \equiv 0 \pmod{p}$; $d \neq 3$; and $p \neq 2, 3$

We have

$$A = \begin{bmatrix} h_0 & g_1 & 0 & 0 & \cdots & 0 & 0 \\ g_1 & h_1 & g_2 & 0 & \cdots & 0 & 0 \\ 0 & g_2 & h_2 & g_3 & \cdots & 0 & 0 \\ 0 & 0 & g_3 & h_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & h_n & g_{n+1} \\ 0 & 0 & 0 & 0 & \cdots & g_{n+1} & h_{n+1} \end{bmatrix}$$

where

$$g_i := \begin{cases} -x_1^{d-2} - 4x_0 x_1^{d-3} & \text{if } i = 1 \\ -x_i^{d-2} & \text{if } 2 \leq i \leq n+1 \end{cases}$$

and

$$h_i := \begin{cases} 2x_1^{d-2} & \text{if } i = 0 \\ 2x_0 x_1^{d-3} + 6x_0^2 x_1^{d-4} & \text{if } i = 1 \\ 2x_{i-1} x_i^{d-3} & \text{if } 2 \leq i \leq n+1. \end{cases}$$

The polynomials $g_1, \ldots, g_{n+1}, h_0, \ldots, h_{n+1}$ are linearly independent, and hence the same proof as in Case I.1 shows that L must be of the form

$$L = \begin{bmatrix} * & * & 0 & \cdots & 0 & 0 & 0 \\ 0 & * & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & * & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & * & 0 \\ 0 & 0 & 0 & \cdots & 0 & * & * \end{bmatrix},$$

or of the form

$$L = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & * & * \\ 0 & 0 & 0 & \cdots & 0 & * & 0 \\ 0 & 0 & 0 & \cdots & * & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & * & \cdots & 0 & 0 & 0 \\ 0 & * & 0 & \cdots & 0 & 0 & 0 \\ * & * & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

The second case is easily ruled out, since equating coefficients of $x_n^{d-2}x_{n+1}^2$ in

$$f(x_0, x_1, \ldots, x_n, x_{n+1}) = f(\ell_{0n}x_n + \ell_{0,n+1}x_{n+1}, \ell_{1n}x_n, \ldots, \ell_{n1}x_1, \ell_{n+1,0}x_0 + \ell_{n+1,1}x_1)$$

yields $0 = \ell_{1n}^{d-2}\ell_{0,n+1}^2$, which contradicts the nonsingularity of L.

In the first case, since L is nonsingular, $\ell_{ii} \neq 0$ for all i. By scaling L, we may assume $\ell_{n+1,n+1} = 1$. By equating coefficients of x_{n+1}^d in (7), we see that $\alpha = 1$. Equating coefficients of $x_n^{d-1}x_{n+1}$ in

$$f(x_0, x_1, \ldots, x_n, x_{n+1}) = f(\ell_{00}x_0 + \ell_{01}x_1, \ell_{11}x_1, \ldots, \ell_{nn}x_n, \ell_{n+1,n}x_n + x_{n+1}),$$

we obtain

$$0 = (d-1)\ell_{nn}\ell_{n+1,n}^{d-2}.$$

Since $d-1$ and ℓ_{nn} are nonzero in $\overline{k}$, we have $\ell_{n+1,n} = 0$. For $i = n, n-1, \ldots, 1$ in turn, we equate coefficients of $x_i x_{i+1}^{d-1}$ to find $\ell_{ii} = 1$. Equate coefficients of $x_0^2 x_1^{d-2}$, of $x_0 x_1^{d-1}$ and of x_1^d to obtain

$$\begin{aligned} \ell_{00}^2 &= 1 \\ 2\ell_{00}\ell_{01} + \ell_{00} &= 1 \\ c\ell_{01}^d + \ell_{01}^2 + \ell_{01} &= 0. \end{aligned}$$

The first two equations yield the possibilities $(1, 0)$ and $(-1, -1)$ for (ℓ_{00}, ℓ_{01}), but only $(1, 0)$ is consistent with the third equation. Thus L is the identity.

Case II.2: $d \equiv 0 \pmod{p}$; $d \neq 3$; and $p = 3$

When $p = 3$, there is a single linear relation between the g's and h's (as defined in Case II.1), namely $g_1 = h_0 + h_1$.

Lemma 6.1. *For nonzero $v \in \overline{k}^{n+2}$, we have $\operatorname{codim} v^\perp \geq 2$, with equality if and only if v is a multiple of e_0, $e_0 + e_1$, or e_{n+1}.*

Proof. The values of $\operatorname{codim} v^\perp$ will be exactly the same as in Case I.1 except possibly for v's for which the appearances of g_1, h_0, h_1 in the coordinates of Av are dependent due to the new relation between them. This happens when $v_0 h_0 + v_1 g_1$ is a scalar multiple of $v_0 g_1 + v_1 h_1$ and both are nonzero. Using $g_1 = h_0 + h_1$, we see that this holds exactly when $v_0 = v_1 \neq 0$. We may assume this from now on, since otherwise the inequality and the equality cases are the same as in Case I.1.

Let j be the largest integer such that $v_j \neq 0$. If $j \leq 1$, then v is a multiple of $e_0 + e_1$ and we are done. If $j > 1$, then g_j appears in the $(j-1)$-th coordinate of Av but not before, and h_j appears in the j-th coordinate of Av but not before, and the 0-th coordinate of Av is nonzero, so these three coordinates are linearly independent, and $\operatorname{codim} v^\perp \geq 3$. ☐

Corollary 6.2. *The vector Le_{n+1} is a multiple of e_{n+1}, and $L(S_1) = S_1$.*

Proof. We have

$$\begin{aligned}\langle e_0, e_0 + e_1\rangle &= h_0 + g_1 = x_1^{d-2} - x_0 x_1^{d-3},\\ \langle e_0, e_{n+1}\rangle &= 0,\\ \langle e_0 + e_1, e_{n+1}\rangle &= 0,\end{aligned}$$

unless $n = 1$, in which case $\langle e_0 + e_1, e_{n+1}\rangle = -x_{n+1}^{d-2}$ instead. If $n > 1$, the multiples of e_{n+1} are distinguished from the multiples of e_0 and $e_0 + e_1$ by the fact that they pair to give zero with the latter two, so L maps e_{n+1} to itself, and fixes the subspace S_1 generated by the multiples of the other two. If $n = 1$, then the multiples of e_{n+1} are distinguished by the fact that they pair with multiples of e_0 or $e_0 + e_1$ to give perfect $(d-2)$-th powers always, so the result again follows. □

Any easy induction on m proves that for $0 \le m \le n$, $S_m^\perp = T_{m+2}$, which is of codimension $m+2$.

Lemma 6.3. *Suppose $1 \le m \le n-2$. Let V be an $(m+2)$-dimensional subspace of $\overline{k}^{n+2}$ containing S_m. Then $\operatorname{codim} V^\perp \ge m+3$, with equality if and only if $V = S_{m+1}$.*

Proof. Write $V = S_m + \overline{k} \cdot v$, so

$$V^\perp = S_m^\perp \cap v^\perp = T_{m+2} \cap v^\perp.$$

If v has any nonzero coordinate v_i with $m+2 \le i \le n$, then the condition that an element w of T_{m+2} be in $v^\perp$ places at least two linear conditions on w, namely $w_i = 0$ and $w_{i+1} = 0$, so $\operatorname{codim} V^\perp \ge \operatorname{codim} T_{m+2} + 2 = m+4$ in this case. Similarly, if $v_{n+1} \ne 0$, then the condition that an element w of $S_m^\perp$ be in $v^\perp$ places the new conditions $w_n = 0$ and $w_{n+1} = 0$ on w, so that $\operatorname{codim} V^\perp \ge m+4$ again. The only remaining possibility is that $v_i = 0$ for all $i \ge m+2$, in which case we must have $V = S_{m+1}$ and $\operatorname{codim} V^\perp = \operatorname{codim} T_{m+3} = m+3$. □

Corollary 6.4. *We have $L(S_m) = S_m$ for $1 \le m \le n-1$.*

Lemma 6.5. *We have $L(T_m) = T_m$ for $2 \le m \le n+1$.*

Proof. Suppose $n = 1$. Then the needed fact $L(T_2) = T_2$ follows from the first half of Corollary 6.2.

Suppose $n \ge 2$. Using $S_m^\perp = T_{m+2}$ and Corollary 6.4 proves the result for all the required m except $m = 2$. We know that $L(T_2)$ is an n-dimensional subspace of $\overline{k}^{n+2}$ containing $L(T_3) = T_3$ such that $\operatorname{codim} L(T_2)^\perp = n+1$. Write $L(T_2) = T_3 + \overline{k} \cdot v$, where $v_i = 0$ for $i \ge 3$. Since $T_3^\perp = S_1$, which has codimension n, in order to have $\operatorname{codim} L(T_2)^\perp = n+1$, the first two coordinates of Av must be linearly dependent. This is possible only if v is a multiple of $e_0 + e_1$ or a multiple of e_2. But $L(T_2) \cap S_1 = L(T_2 \cap S_1) = \{0\}$, so $e_0 + e_1 \notin L(T_2)$. Thus $L(T_2) = T_3 + \overline{k} \cdot e_2 = T_2$. □

Corollary 6.6. *We have $L(S_m) = S_m$ for $0 \le m \le n-1$.*

Proof. The new result, $L(S_0) = S_0$, follows from $L(T_2) = T_2$ and $T_2^\perp = S_0$. □

By Lemma 6.5 and Corollary 6.6, L is of the form

$$L = \begin{bmatrix} * & * & 0 & \cdots & 0 & 0 & 0 \\ 0 & * & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & * & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & * & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & * & 0 \\ 0 & 0 & 0 & \cdots & 0 & * & * \end{bmatrix}.$$

Repeating the argument at the end of Case II.1 completes the proof in this case.

Case II.3: $d=3$, $p=3$, $n\geq 2$

In this case we have

$$A=-\begin{bmatrix} x_1 & x_0+x_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ x_0+x_1 & x_0 & x_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & x_2 & x_1 & x_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & x_3 & x_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & x_{n-2} & x_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & x_n & x_{n-1} & x_{n+1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & x_{n+1} & x_n \end{bmatrix}.$$

Lemma 6.7. *For nonzero $v\in\overline{k}^{n+2}$, we have $\operatorname{codim} v^{\perp}\geq 2$, with equality if and only if v is a multiple of e_0, e_0+e_1, or e_{n+1}.*

Proof. Let i be the smallest integer such that v_i is nonzero. Let j be the largest integer such that v_j is nonzero.

If $i=0$ and $j=0$, then v is a multiple of e_0, and $\operatorname{codim} v^{\perp}=2$.

If $i=0$ and $j=1$, then we may assume $v=e_0+\gamma e_1$ for some $\gamma\in\overline{k}^*$. If $\operatorname{codim} v^{\perp}\leq 2$, then the first two coordinates of Av must be linearly dependent, which implies $\gamma^2+\gamma+1=0$, which yields $\gamma=1$ (since we are in characteristic 3). Hence v is a multiple of e_0+e_1.

If $i=0$ and $2\leq j\leq n$, then x_{j+1} appears only in the $(j+1)$-th coordinate of Av, x_j appears in the $(j-1)$-th coordinate of Av and not before, and the 0-th coordinate of Av is nonzero, so these three coordinates are independent, and $\operatorname{codim} v^{\perp}\geq 3$.

If $i=0$ and $j=n+1$, then we branch according as v_n is zero or not. If $v_n=0$, then the $(n+1)$-th coordinate of Av is a nonzero multiple of x_n, the n-th coordinate of Av is a combination of x_n and x_{n+1} in which x_{n+1} appears, and the 0-th coordinate is a nonzero combination of x_0 and x_1, so these three coordinates are independent, and $\operatorname{codim} v^{\perp}\geq 3$. If $v_n\neq 0$ and $n>2$, then the $(n+1)$-th coordinate is a nonzero combination of x_n and x_{n+1}, the 0-th coordinate is a nonzero combination of x_0 and x_1, and the n-th coordinate involves x_{n-1}, which appears in neither the 0-th nor the $(n+1)$-th coordinate, so these three coordinates are independent, and $\operatorname{codim} v^{\perp}\geq 3$. Finally suppose $v_n\neq 0$ and $n=2$. The 0-th coordinate of Av is a nonzero combination of x_0 and x_1, and the 3-rd coordinate of Av is a nonzero combination of x_2 and x_3, so these two coordinates are independent. If moreover $\operatorname{codim} v^{\perp}\leq 2$, then the 2-nd coordinate must be a linear combination of the 0-th and 3-rd. The 0-th coordinate must appear in this combination since x_1 appears in the 2-nd coordinate of Av. But x_0 does not appear in the 2-nd coordinate, so x_0 cannot appear in the 0-th coordinate, and this implies $v_1=0$. Then x_3 appears while x_2 does not appear in the 2-nd coordinate, making it impossible for the 2-nd coordinate to be a combination of the 0-th and 3-rd coordinates.

If $i\geq 1$ and $j\leq n$, then the $(i-1)$-th coordinate of Av is nonzero, and the i-th coordinate is not a multiple of it, so these two coordinates are independent. Also, the $(j+1)$-th coordinate of Av is a multiple of x_{j+1}, which does not appear anywhere else in Av, so the $(i-1)$-th, i-th, and $(j+1)$-th coordinates are independent, and $\operatorname{codim} v^{\perp}\geq 3$.

If $1\leq i\leq n-1$ and $j=n+1$, then the $(i-1)$-th coordinate of Av is a nonzero multiple of x_i (or of x_0+x_1 if $i=1$), the i-th coordinate of Av is a nonzero combination of x_{i-1} and x_{i+1}, and the n-th coordinate of Av involves x_{n+1}, which does not appear earlier, so these three coordinates are independent, and $\operatorname{codim} v^{\perp}\geq 3$.

If $i = n$ and $j = n+1$, then x_{n-1} appears only in the n-th coordinate of Av, the $(n-1)$-th coordinate is a nonzero multiple of x_n, and the $(n+1)$-th coordinate is a combination of x_n and x_{n+1} in which both appear, so these three coordinates are independent, and $\operatorname{codim} v^\perp \geq 3$.

If $i = n+1$ and $j = n+1$, then v is a multiple of e_{n+1}, and $\operatorname{codim} v^\perp = 2$. □

Corollary 6.8. *The vector Le_{n+1} is a multiple of e_{n+1}, and $L(S_1) = S_1$.*

Proof. Since $n \geq 2$, we have

$$\begin{aligned} \langle e_0, e_0 + e_1 \rangle &= -x_0 - 2x_1, \\ \langle e_0, e_{n+1} \rangle &= 0, \\ \langle e_0 + e_1, e_{n+1} \rangle &= 0, \end{aligned}$$

so the multiples of e_{n+1} are distinguished from the multiples of e_0 and $e_0 + e_1$ by the fact that they pair to give zero with the latter two. Thus L maps e_{n+1} to itself, and fixes the subspace S_1 generated by the multiples of the other two. □

Lemma 6.9. *Suppose $1 \leq m \leq n-2$. Let V be an $(m+2)$-dimensional subspace of $\overline{k}^{n+2}$ containing S_m. Then $\operatorname{codim} V^\perp \geq m+3$, with equality if and only if $V = S_{m+1}$.*

Proof. Write $V = S_m + \overline{k} \cdot v$, so

$$V^\perp = S_m^\perp \cap v^\perp = T_{m+2} \cap v^\perp.$$

We may assume $v_i = 0$ for $i \leq m$. We must show that the codimension of $T_{m+2} \cap v^\perp$ in T_{m+2} is at least 1, with equality if and only if v is a nonzero multiple of e_{m+1}. This is the same as showing that the span of the $(m+2)$-th, …, $(n+1)$-th coordinates of Av is of dimension at least 1, with equality if and only if v is a nonzero multiple of e_{m+1}.

Let j be the largest integer such that v_j is nonzero. If $j = m+1$, then v is a nonzero multiple of e_{m+1}, the $(m+2)$-th coordinate of Av is a nonzero multiple of x_{m+2}, and all later coordinates are zero, so we have equality, as desired.

If $m+2 \leq j \leq n$, then the $(j+1)$-th coordinate of Av is a nonzero multiple of x_{j+1}, but the j-th coordinate of Av involves x_{j-1}, so the span is of dimension at least 2.

If $j = n+1$ and $v_n = 0$, then the $(n+1)$-th coordinate of Av is a nonzero multiple of x_n, but the n-th coordinate involves x_{n+1}, so the span is of dimension at least 2.

If $j = n+1$ and $v_n \neq 0$, then x_{n-1} appears in the n-th coordinate of Av, and the $(n+1)$-th coordinate of Av is nonzero but does not involve x_{n-1}, so again the span is of dimension at least 2. □

The rest of the proof of this case is exactly analogous to the corresponding final section of the proof in Case II.2, from Corollary 6.4 on.

7. Controlling the automorphisms: Case III

We have

$$A = \begin{bmatrix} 0 & g_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ g_1 & 0 & g_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & g_2 & 0 & g_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & g_3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & g_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & g_n & 0 & g_{n+1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & g_{n+1} & 0 \end{bmatrix},$$

where

$$g_i := \begin{cases} x_0^{d-2} + x_1^{d-2} & \text{if } i = 1 \\ x_i^{d-2} & \text{if } 2 \leq i \leq n+1. \end{cases}$$

Note that $g_1, \ldots, g_{n+1}$ are linearly independent over $\overline{k}$.

Lemma 7.1. *Suppose $n \geq 4$. If $v \in \overline{k}^{n+2}$ is nonzero and $\operatorname{codim} v^{\perp} \leq 2$ then v is a multiple of some e_i, or v is a combination of e_0 and e_1, or a combination of e_0 and e_2, or a combination of e_n and e_{n+1}, or a combination of e_{n-1} and e_{n+1}, or a combination of e_0 and e_{n+1}.*

If $n = 2$ or $n = 3$, then the same result holds, except that combinations of e_0, e_{n-1}, e_{n+1} and combinations of e_0, e_2, e_{n+1} are also possible.

For all $n \geq 2$, only the multiples of e_0 and the multiples of e_{n+1} satisfy $\operatorname{codim} v^{\perp} = 1$.

Proof. It is clear that the listed v's satisfy $\operatorname{codim} v^{\perp} \leq 2$. Now suppose $\operatorname{codim} v^{\perp} \leq 2$.

Let i be the smallest integer such that v_i is nonzero. Let j be the largest integer such that v_j is nonzero.

If $i = 0$ and $j \leq 1$, then v is a combination of e_0 and e_1. If v is a multiple of e_0, then $\operatorname{codim} v^{\perp} = 1$; otherwise, the 0-th and 2-nd coordinates of Av are independent and $\operatorname{codim} v^{\perp} = 2$.

If $i = 0$ and $j = 2$, then $v_1 = 0$, since otherwise, the 0-th coordinate of Av is a multiple of g_1, the 2-nd coordinate of Av is a multiple of g_2, and the 3-rd coordinate of Av is a multiple of g_3, which makes $\operatorname{codim} v^{\perp} \geq 3$. Hence v is a combination of e_0 and e_2. The 0-th and 2-nd coordinates of Av are independent unless v is a multiple of e_0, in which case $\operatorname{codim} v^{\perp} = 1$.

If $i = 0$ and $3 \leq j \leq n$, then g_{j+1} appears in the $(j+1)$-th coordinate of Av but not before, g_j appears in the $(j-1)$-th coordinate of Av but not before, and the 1-st coordinate of Av is nonzero, so these three coordinates are independent, and $\operatorname{codim} v^{\perp} \geq 3$.

If $i = 0$ and $j = n+1$, then the 1-st and n-th coordinates of Av are nonzero and independent because x_{n+1} appears only in the latter. Thus $\operatorname{codim} v^{\perp} \geq 2$. Hence if $\operatorname{codim} v^{\perp} \leq 2$, then every other coordinate of Av must be a combination of the 1-st and n-th. In particular, each nonzero coordinate of Av involves g_1 or g_{n+1}, so the 2-nd, 3-rd, ..., $(n-1)$-th coordinates of Av must be zero. If $n \geq 4$ this forces $v_1 = v_2 = \cdots = v_n = 0$, as desired. If $n = 3$, then the vanishing of the 2-nd coordinate of Av forces only $v_1 = v_3 = 0$, so that v is a combination of e_0, e_2, and e_4, as desired. Finally, if $n = 2$, then either $v_1 = 0$ or $v_2 = 0$, since if all v_i were nonzero, then for $m = 1, 2, 3$, the term g_m occurs in the m-th coordinate of Av but not afterwards, making $\operatorname{codim} v^{\perp} \geq 3$. Thus v is a combination of e_0, e_1, and e_3, or a combination of e_0, e_2, and e_3, as desired.

We have now completely finished the case $i = 0$, and symmetrical considerations prove all cases in which $j = n+1$. Therefore, from now on, we assume $1 \leq i \leq j \leq n$. If $i = j$, then v is a multiple of e_i, and $\operatorname{codim} v^{\perp} = 2$, as desired. Otherwise, if $1 \leq i < j \leq n$, then g_i appears only in the $(i-1)$-th coordinate of Av, g_{j+1} appears only in the $(j+1)$-th coordinate of Av, and g_{i+1} appears in the $(i+1)$-th coordinate of Av, so these three coordinates are independent, and $\operatorname{codim} v^{\perp} \geq 3$. $\square$

Lemma 7.2. *The matrix L is of the form*

$$L = \begin{bmatrix} * & * & * & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & * & * & * \end{bmatrix}$$

or of the form

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & * & * & * \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & * & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & * & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ * & * & * & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In other words, either L has nonzero entries on the diagonal and zeros elsewhere except possibly at ℓ_{01}, ℓ_{02}, $\ell_{n+1,n-1}$, $\ell_{n+1,n}$, or L has nonzero entries on the reverse diagonal and zeros elsewhere except possibly at $\ell_{0,n-1}$, $\ell_{0,n}$, $\ell_{n+1,1}$, $\ell_{n+1,2}$.

Proof. If $n = 1$, then the nonzero $v \in \overline{k}^{n+2}$ for which $\operatorname{codim} v^\perp = 1$ are exactly the combinations of e_0 and e_2, so L must preserve the subspace $S_0 + T_2$; i.e., L must have the form

$$L = \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{bmatrix},$$

and this is what Lemma 7.2 is claiming in this case.

For $n \geq 2$, Lemma 7.1 implies that L maps e_0 and e_{n+1} to themselves or interchanges them, up to scalar multiple. By symmetry, we may assume that Le_0 is a multiple of e_0, and that Le_{n+1} is a multiple of e_{n+1}. (The possibilities where Le_0 is a multiple of e_{n+1} will give rise to the mirror reflections of the possibilities for L in the first case.) The subset $W := \{v : \operatorname{codim} v^\perp \leq 2\}$ of $\overline{k}^{n+2}$ is preserved by L, and Lemma 7.1 gives an explicit description of W.

If $n = 2$, then $e_1 \in e_3^\perp$, so $Le_1 \in (Le_3)^\perp = e_3^\perp = S_1 + T_3$. Similarly $Le_2 \in e_0^\perp = S_0 + T_2$. This completes the proof in the case $n = 2$.

If $n = 3$, then the subspace V of $\overline{k}^{n+2}$ generated by e_0, e_2, and e_4 is preserved by L, since by Lemma 7.1 it is the only 3-dimensional subspace contained in W. Also, L preserves S_1 (resp. T_3), since by Lemma 7.1 this is the only 2-dimensional subspace that contains $\overline{k}e_0$ (resp. $\overline{k}e_4$), that is not contained in V, and that is contained in W. These restrictions together imply that L has the desired shape.

From now on, we assume $n \geq 4$. By Lemma 7.1, the 2-dimensional subspaces containing $\overline{k}e_0$ and contained in W are $S_1 = \overline{k}e_0 + \overline{k}e_1$ and $R := \overline{k}e_0 + \overline{k}e_2$. Hence L preserves $\{S_1, R\}$, and preserves their sum, which is S_2. Similarly L preserves T_{n-1}. It then follows from Lemma 7.1 that L permutes e_3, e_4, $\ldots$, e_{n-2} up to scalar multiple, since these (and their multiples) are the only vectors of W outside $S_2 + T_{n-1}$.

We next prove by induction on i that $L(S_i) = S_i$ for $2 \le i \le n-2$, and that Le_i is a multiple of e_i for $3 \le i \le n-2$. The base case $L(S_2) = S_2$ is already known. Suppose $3 \le i \le n-2$, and $L(S_{i-1}) = S_{i-1}$, and $L(e_j)$ is a multiple of e_j for $3 \le j \le i-1$. We know already that Le_i is a multiple of some e_k, $k \ge i$, but the only such e_k that can pair with some vector in S_{i-1} to give something nonzero is e_i, so Le_i must be a multiple of e_i. Hence also $L(S_i) = S_i$, which completes the induction step.

In particular, we now know that $L(T_3) = T_3$. The subspaces S_1 and R can be distinguished using the fact that only the latter contains elements that can pair with some vector in T_3 to give something nonzero, so $L(S_1) = S_1$ and $L(R) = R$. Similarly we deduce that $L(T_n) = T_n$ and that L preserves the subspace $R' := \overline{k}e_{n-1} + \overline{k}e_{n+1}$. The restrictions we have deduced, taken together, imply that L has the desired shape. □

Case III.1: $n = 1$

Equation (7) becomes

$$(12)\quad x_0^{d-1}x_1 + x_0x_1^{d-1} + x_1x_2^{d-1} + x_2^d$$
$$= \alpha\Big[(\ell_{00}x_0 + \ell_{01}x_1 + \ell_{02}x_2)^{d-1}\ell_{11}x_1 + (\ell_{00}x_0 + \ell_{01}x_1 + \ell_{02}x_2)(\ell_{11}x_1)^{d-1}$$
$$+ \ell_{11}x_1(\ell_{20}x_0 + \ell_{21}x_1 + \ell_{22}x_2)^{d-1} + (\ell_{20}x_0 + \ell_{21}x_1 + \ell_{22}x_2)^d\Big].$$

Equating coefficients of x_0^d yields $0 = \alpha\ell_{20}^d$, so $\ell_{20} = 0$. Since L is nonsingular, $\ell_{00} \ne 0$. Equating coefficients of $x_0^{d-2}x_1^2$ yields $0 = \alpha\ell_{00}^{d-2}\ell_{01}\ell_{11}$, but $\alpha, \ell_{00}, \ell_{11}$ must all be nonzero, so $\ell_{01} = 0$. Equating coefficients of $x_0^{d-2}x_1x_2$ yields $0 = \alpha\ell_{00}^{d-2}\ell_{02}\ell_{11}$, so $\ell_{02} = 0$. Equating coefficients of $x_1^{d-1}x_2$ yields $0 = \alpha\ell_{11}\ell_{21}^{d-2}\ell_{22}$, and $\ell_{11}, \ell_{22} \ne 0$ by nonsingularity, so $\ell_{21} = 0$. We now know that L is diagonal. Without generality assume $\ell_{22} = 1$. Equating coefficients of x_2^d in (12) shows $\alpha = 1$. Equating coefficients of $x_1x_2^{d-1}$ shows $\ell_{11} = 1$. Equating coefficients of $x_0x_1^{d-1}$ shows $\ell_{00} = 1$. Thus L is the identity.

Case III.2: $n \ge 2$

Equating coefficients of x_0^d in (7) rules out the second possibility in Lemma 7.2, so L is of the form

$$L = \begin{bmatrix} * & * & * & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & * & * & * \end{bmatrix},$$

and (7) becomes

$$(13)\quad f(x_0, x_1, \ldots, x_{n+1})$$
$$= \alpha f(\ell_{00}x_0 + \ell_{01}x_1 + \ell_{02}x_2, \ell_{11}x_1, \ldots, \ell_{nn}x_n, \ell_{n+1,n-1}x_{n-1} + \ell_{n+1,n}x_n + \ell_{n+1,n+1}x_{n+1}).$$

Note that $\ell_{ii} \ne 0$ for all i, since L is nonsingular. Equating coefficients of $x_0^{d-2}x_1^2$ in (13) yields $0 = \alpha\ell_{00}^{d-2}\ell_{01}\ell_{11}$ so $\ell_{01} = 0$. Equating coefficients of $x_0^{d-2}x_1x_2$ yields $0 = \alpha\ell_{00}^{d-2}\ell_{02}\ell_{11}$ so $\ell_{02} = 0$.

Equating coefficients of $x_n^{d-1}x_{n+1}$ yields $0 = \alpha \ell_{nn} \ell_{n+1,n}^{d-2} \ell_{n+1,n+1}$ so $\ell_{n+1,n} = 0$. Equating coefficients of $x_{n-1}^{d-2} x_n x_{n+1}$ yields $0 = \alpha \ell_{nn} \ell_{n+1,n-1}^{d-2} \ell_{n+1,n+1}$ so $\ell_{n+1,n-1} = 0$. We now know that L is diagonal. Without loss of generality assume $\ell_{n+1,n+1} = 1$. Equating coefficients of x_{n+1}^d in (13) shows $\alpha = 1$. We now prove $\ell_{ii} = 1$ for all i by backwards induction, by equating coefficients of $x_i x_{i+1}^{d-1}$. Thus L is the identity.

8. Controlling the automorphisms: Case IV

The matrix A will have $\lfloor \frac{n+1}{2} \rfloor$ nonzero 2×2 blocks along the diagonal, and zeros elsewhere. For odd i, define $g_i := -2x_{i-1}x_i^{d-3}$. Also define

$$h_i := \begin{cases} 2x_{i+1}^{d-2} & \text{if } i \text{ is even} \\ 2x_{i-1}^2 x_i^{d-4} & \text{if } i \text{ is odd.} \end{cases}$$

(Note that $d \geq 4$ in Case IV.) The g's and h's are linearly independent over $\overline{k}$.

Case IV.1: n is odd

We have

$$A = \begin{bmatrix} h_0 & g_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ g_1 & h_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & h_2 & g_3 & \cdots & 0 & 0 & 0 \\ 0 & 0 & g_3 & h_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & h_{n-1} & g_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & g_n & h_n & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}.$$

Clearly Le_{n+1} is a multiple of e_{n+1}, because only the multiples of e_{n+1} pair under $\langle\ ,\ \rangle$ with all vectors to give zero. For $i = 0, 2, 4, \ldots, n-1$, let V_i be the $\overline{k}$-vector space spanned by e_i, e_{i+1}, and e_{n+1}. It is clear that V_0, V_2, V_4, $\ldots$, V_{n-1} are the only 3-dimensional subspaces V of $\overline{k}^{n+2}$ such that $\operatorname{codim} V^\perp = 2$. Thus $L(V_i) = V_{\pi(i)}$ for some permutation π of $\{0, 2, 4, \ldots, n-1\}$. In other words, L has the form of a permutation matrix, except with 2×2 blocks, and with an added row at the bottom with potentially nonzero entries, and with zeros in an added final column on the right (except for the lower right corner, which must be nonzero).

If we view both sides of (7) as polynomials in x_{n+1} and equate coefficients of x_{n+1}^{d-1}, we find that y_n is a nonzero multiple of x_n. Thus $L(V_{n-1}) = V_{n-1}$. If we instead equate coefficients of x_{n+1}, we find that y_0 is a nonzero multiple of x_0. Thus $L(V_0) = V_0$.

We now prove by backwards induction that y_i is a nonzero multiple of x_i for $i = n-1, n-2, \ldots, 1$. (We already know it for $i = n$ and $i = 0$.) First suppose i is even. By assumption, y_{i+1} is a multiple of x_{i+1}, so $\pi(i) = i$. It follows that y_i is a linear combination of x_i and x_{i+1}. Moreover, x_i occurs in this combination, since otherwise L would be singular. Suppose $\ell_{i,i+1} \neq 0$. Then for each $j < i$, equating coefficients of $x_j x_{i+1}^{d-1}$ in (7) yields

$$0 = \alpha \ell_{i-1,j} \ell_{i,i+1}^{d-1}$$

so $\ell_{i-1,j} = 0$. The block form of L implies $\ell_{i-1,j} = 0$ for $j \geq i$ as well, so L has a row of zeros, which is a contradiction. Thus $\ell_{i,i+1}$ must have been zero, and hence y_i is a (nonzero) multiple of x_i.

Next suppose i is odd, $1 \le i \le n-2$. For $j < i$, equating coefficients of $x_j x_{i+1}^{d-1}$ in (7) yields $0 = \alpha \ell_{i,j} \ell_{i+1,i+1}^{d-1}$, and $\ell_{i+1,i+1}$ is nonzero (since y_{i+1} is a nonzero multiple of x_{i+1}), so $\ell_{i,j} = 0$ for $j < i$. On the other hand, the block form of L implies $\ell_{i,j} = 0$ for $j > i$ also, so y_i is a (nonzero) multiple of x_i.

Equating coefficients of x_n^d in (7) yields $0 = \alpha \ell_{nn} \ell_{n+1,n}^{d-1}$, so $\ell_{n+1,n} = 0$. For each $j < n$, equating coefficients of $x_j^{d-1} x_n$ shows that $0 = \alpha \ell_{nn} \ell_{n+1,j}^{d-1}$, so $\ell_{n+1,j} = 0$. Thus y_{n+1} is a (nonzero) multiple of x_{n+1}.

We now know that L is diagonal. We may assume $\ell_{00} = 1$. Equating coefficients of x_0^d in (7) shows $\alpha = 1$. Equating coefficients of $x_{n+1} x_0^{d-1}$ in (7) shows $\ell_{n+1,n+1} = 1$. We can now show $\ell_{ii} = 0$ for $i = n, n-1, \dots, 1$ as well, by backwards induction: equating coefficients of $x_i x_{i+1}^{d-1}$ in (7) yields $\ell_{ii} \ell_{i+1,i+1}^{d-1} = 1$, so if $\ell_{i+1,i+1} = 1$, then $\ell_{ii} = 1$. Thus L is the identity.

Case IV.2: n is even

The matrix A has the same form as in Case IV.1 except that it ends with two final rows of zeros and two final columns of zeros, instead of only one of each.

The subspace of v in $\overline{k}^{n+2}$ such that $\langle v, w \rangle = 0$ for all w in $\overline{k}^{n+2}$ is T_n, so $L(T_n) = T_n$. For $i = 0, 2, 4, \dots, n-1$, let V_i be the $\overline{k}$-vector space spanned by e_i, e_{i+1}, e_n, and e_{n+1}. It is clear that V_0, V_2, V_4, …, V_{n-2} are the only 4-dimensional subspaces V of $\overline{k}^{n+2}$ such that $\operatorname{codim} V^\perp = 2$. Thus $L(V_i) = V_{\pi(i)}$ for some permutation π of $\{0, 2, 4, \dots, n-2\}$. In other words, L has the form of a permutation matrix, except with 2×2 blocks, and with two added rows at the bottom with potentially nonzero entries, and with zeros in two added final columns on the right (except for the lower right 2×2 block, which may have nonzero entries).

If we substitute $x_0 = x_1 = \cdots = x_{n-1} = 0$ in (7), we obtain

$$x_n x_{n+1}^{d-1} = \alpha(\ell_{nn} x_n + \ell_{n,n+1} x_{n+1})(\ell_{n+1,n} x_n + \ell_{n+1,n+1} x_{n+1})^{d-1}.$$

By unique factorization, $\ell_{nn} x_n + \ell_{n,n+1} x_{n+1}$ is a nonzero multiple of x_n, and $\ell_{n+1,n} x_n + \ell_{n+1,n+1} x_{n+1}$ is a nonzero multiple of x_{n+1}. Hence the lower right 2×2 block of L is diagonal, with nonzero entries on the diagonal.

View both sides of (7) as polynomials in x_{n+1}. Equating coefficients of x_{n+1}^{d-1} shows that y_n is a nonzero multiple of x_n. Equating coefficients of x_{n+1} shows that y_0^{d-1} is a nonzero multiple of x_0^{d-1}, so y_0 is a nonzero multiple of x_0.

Now view both sides of (7) as polynomials in x_n. Equating coefficients of x_n shows that y_{n+1}^{d-1} is a nonzero multiple of x_{n+1}^{d-1}, so y_{n+1} is a nonzero multiple of x_{n+1}. Equating coefficients of x_n^{d-1} shows that y_{n-1} is a nonzero multiple of x_{n-1}.

The same backwards induction on i as in Case IV.1 now shows that y_i is a nonzero multiple of x_i for all i. (We already know it for $i = 0, n-1, n, n+1$.) Thus L is diagonal. We deduce that L is (a scalar multiple of) the identity as in the end of Case IV.1.

9. Controlling the automorphisms: Case V

Note that $d \ge 5$ in Case V. We have

$$A = \begin{bmatrix} 0 & x_1^{d-3} x_2 & x_1^{d-2} \\ x_1^{d-3} x_2 & 0 & x_0 x_1^{d-3} \\ x_1^{d-2} & x_0 x_1^{d-3} & 0 \end{bmatrix}.$$

The greatest common divisor of the entries of A is x_1^{d-3}, so y_1^{d-3} must be a nonzero multiple of x_1^{d-3}. Hence y_1 is a nonzero multiple of x_1. Without loss of generality we may assume $y_1 = x_1$.

Then (7) becomes

$$f(x_0, x_1, x_2) = \alpha f(\ell_{00}x_0 + \ell_{01}x_1 + \ell_{02}x_2, x_1, \ell_{20}x_0 + \ell_{21}x_1 + \ell_{22}x_2). \tag{14}$$

If we set $x_1 = 0$ and use the definition of f, we obtain

$$x_2 x_0^{d-1} = \alpha(\ell_{20}x_0 + \ell_{22}x_2)(\ell_{00}x_0 + \ell_{02}x_2)^{d-1}.$$

By unique factorization, we deduce that $\ell_{20} = 0$ and $\ell_{02} = 0$. Now (14) becomes

$$\begin{aligned}(15)\quad & \alpha^{-1}(x_0x_1^{d-2}x_2 + x_0x_1^{d-1} + x_1x_2^{d-1} + x_2x_0^{d-1} + x_1^2x_2^{d-2})\\ &= (\ell_{00}x_0 + \ell_{01}x_1)x_1^{d-2}(\ell_{21}x_1 + \ell_{22}x_2) + (\ell_{00}x_0 + \ell_{01}x_1)x_1^{d-1} + x_1(\ell_{21}x_1 + \ell_{22}x_2)^{d-1}\\ &\quad + (\ell_{21}x_1 + \ell_{22}x_2)(\ell_{00}x_0 + \ell_{01}x_1)^{d-1} + x_1^2(\ell_{21}x_1 + \ell_{22}x_2)^{d-2}\end{aligned}$$

Equating coefficients of $x_0^{d-1}x_1$ yields $0 = \ell_{21}\ell_{00}^{d-1}$. The nonsingularity of L guarantees $\ell_{00} \neq 0$, so $\ell_{21} = 0$. Equating coefficients of x_1^d yields $0 = \ell_{01}$, so L is diagonal.

Equating coefficients of $x_1x_2^{d-1}$ and of $x_1^2x_2^{d-2}$ yields

$$\alpha^{-1} = \ell_{22}^{d-1},$$
$$\alpha^{-1} = \ell_{22}^{d-2}.$$

Dividing, we find $\ell_{22} = 1$, and then $\alpha = 1$. Equating coefficients of $x_0x_1^{d-1}$ now shows $\ell_{00} = 1$. Thus L is the identity.

10. Controlling the automorphisms: Case VI

Let $m = 3\lfloor \frac{n+2}{3} \rfloor$. For $i = 0, 3, 6, \ldots, m-3$, define $f_i := x_{i+1}^{d-3}x_{i+2}$, $g_i := x_{i+1}^{d-2}$, and $h_i := x_i x_{i+1}^{d-3}$. We have

$$A = \begin{bmatrix} 0 & f_0 & g_0 & 0 & 0 & 0 & \cdots \\ f_0 & 0 & h_0 & 0 & 0 & 0 & \cdots \\ g_0 & h_0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & f_3 & g_3 & \cdots \\ 0 & 0 & 0 & f_3 & 0 & h_3 & \cdots \\ 0 & 0 & 0 & g_3 & h_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

in which there are $\lfloor \frac{n+2}{3} \rfloor$ 3×3 blocks along the diagonal, and zeros elsewhere. (There will be $(n+2-m)$ rows of zeros at the bottom, and also $(n+2-m)$ columns of zeros at the right.) Note that $f_0, g_0, h_0, f_3, g_3, h_3, \ldots$ are linearly independent over $\overline{k}$.

The set of v in $\overline{k}^{n+2}$ such that $\langle v, w \rangle = 0$ for all w in $\overline{k}^{n+2}$ is T_m, so $L(T_m) = T_m$. Note that $\dim T_m = (n+2) \bmod 3 = n+2-m$. For $i = 0, 3, 6, \ldots, m-3$, let V_i be the $(n+2-m)+3$-dimensional vector space spanned by T_m, e_i, e_{i+1}, and e_{i+2}.

Lemma 10.1. *If* $\operatorname{codim} v^\perp \leq 3$, *then* $v \in V_i$ *for some* i.

Proof. If v is not contained in any V_i, then there are at least two distinct $i, j \in \{0, 3, 6, \ldots, m-3\}$ such that v equals a nonzero combination w of e_i, e_{i+1}, e_{i+2}, plus a nonzero combination w' of e_j, e_{j+1}, e_{j+2}, plus an element of T_m. Any nonzero combination of the three columns A_i, A_{i+1}, A_{i+2} will have entries spanning a vector space of dimension at least 2, because there will be at least two nonzero entries, and there will be one form f_i, g_i, or h_i that appears in some but not all of these nonzero entries. The span of the nonzero entries of this combination does not intersect the span of the entries of a nonzero combination of A_j, A_{j+1}, A_{j+2}, so we see that $\operatorname{codim} v^\perp \geq 2 + 2 = 4$. □

Corollary 10.2. *We have* $L(V_i) = V_{\pi(i)}$ *for some permutation* π *of* $\{0, 3, 6, \ldots, m-3\}$.

Proof. By Lemma 10.1, the V_i are the only $((n+2-m)+3)$-dimensional subspaces V such that $\operatorname{codim} V^{\perp} = 3$. □

Corollary 10.3. *For $i = 0, 3, 6, \ldots, m-3$, each of y_i, y_{i+1}, y_{i+2} is a linear combination of x_j, x_{j+1}, x_{j+2}, where $j = \pi^{-1}(i)$.*

Proof. This is a direct consequence of Corollary 10.2 and the fact $L(T_m) = T_m$. □

Before proceeding further, we subdivide Case VI as follows.

- Case VI.1: $n \equiv 0 \pmod 3$
- Case VI.2: $n \equiv 1 \pmod 3$ and $n \geq 4$
- Case VI.3: $n \equiv 2 \pmod 3$.

(Also, remember that throughout Case VI, $p = 2$, d is odd, and $n > 1$.)

Case VI.1: $n \equiv 0 \pmod 3$

We have $m = n$. Each of $y_0, y_1, \ldots, y_{n-1}$ is a linear combination of $x_0, x_1, \ldots, x_{n-1}$, by Corollary 10.3. Thus if we substitute $x_0 = x_1 = \cdots = x_{n-1} = 0$ in (7), we obtain

$$x_n x_{n+1}^{d-1} = \alpha(\ell_{n,n}x_n + \ell_{n,n+1}x_{n+1})(\ell_{n+1,n}x_n + \ell_{n+1,n+1}x_{n+1})^{d-1}.$$

By unique factorization, $\ell_{n,n+1} = \ell_{n+1,n} = 0$.

If we consider both sides of (7) as polynomials in x_{n+1} and equate coefficients of x_{n+1}^{d-1}, we deduce that y_n is a multiple of x_n. In particular, the only y_i in which x_n appears is y_n. If we consider both sides of (7) as polynomials in x_n and equate coefficients of x_n, we deduce that y_{n+1}^{d-1} is a multiple of x_{n+1}^{d-1}, so y_{n+1} is a multiple of x_{n+1}.

If we again consider both sides of (7) as polynomials in x_{n+1}, but this time equate coefficients of x_{n+1}, we deduce that y_0^{d-1} is a multiple of x_0^{d-1}, so y_0 is a multiple of x_0. Corollary 10.3 implies $\pi(0) = 0$.

Similarly if we consider both sides of (7) as polynomials in x_n and equate coefficients of x_n^{d-1}, we deduce that y_{n-1} is a multiple of x_{n-1}, and $\pi(m-3) = m-3$.

We now prove $\pi(i)$ for all $i = 0, 3, 6, \ldots, m-3$ by induction. Suppose $i \geq 3$, and we know $\pi(j) = j$ for $j < i$. By Corollary 10.3, the only y-monomial of $f(y_0, y_1, \ldots, y_{n+1})$ whose expansion can contain $x_{i-1}x_i^{d-1}$ is $y_{i-1}y_i^{d-1}$. It follows that y_i must involve x_i, so $\pi(i) = i$ by Corollary 10.3.

Fix $i \in \{3, 6, 9, \ldots, m-6\}$. If we expand $f(y_0, y_1, \ldots, y_{n+1})$ and discard all monomials unless they involve both one of x_{i-3}, x_{i-2}, x_{i-1} and one of x_i, x_{i+1}, x_{i+2}, then what remains, by Corollary 10.3, is exactly the expansion of $y_{i-1}y_i^{d-1}$. Hence $y_{i-1}y_i^{d-1}$ is a multiple of $x_{i-1}x_i^{d-1}$, and by unique factorization, we see that y_{i-1} is a multiple of x_{i-1} and y_i is a multiple of x_i.

We now know that L is of the form

$$L = \begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & * \end{bmatrix},$$

and the diagonal entries must be nonzero, since L is nonsingular.

Equating coefficients of $x_0 x_1^{d-1}$ in (7) yields

$$0 = \alpha \ell_{10} \ell_{11}^{d-1}$$

so $\ell_{10} = 0$. Equating coefficients of $x_1^{d-1} x_2$ yields

$$0 = \alpha \ell_{11}^{d-1} \ell_{12}$$

so $\ell_{12} = 0$.

Let i be a positive multiple of 3. Equating coefficients of x_i^d yields

$$0 = \alpha \ell_{i,i} \ell_{i+1,i}^{d-1},$$

so $\ell_{i+1,i} = 0$. Similarly, equating coefficients of x_{i+2}^d yields

$$0 = \alpha \ell_{i+1,i+2} \ell_{i+2,i+2}^{d-1}$$

so $\ell_{i+1,i+2} = 0$.

We now know that L is diagonal. Without loss of generality suppose $\ell_{11} = 1$. Equating coefficients of x_1^d in (7) shows $\alpha = 1$. Equating coefficients of $x_{n+1} x_0^{d-1}$ shows $\ell_{n+1,n+1} = \ell_{00}^{1-d}$. Equating coefficients of $x_n x_{n+1}^{d-1}$ shows

$$\ell_{n,n} = \ell_{n+1,n+1}^{1-d} = \ell_{00}^{(1-d)^2}.$$

By backwards induction on i, we show

$$\ell_{i,i} = \ell_{00}^{(1-d)^{n+2-i}} \tag{16}$$

for all $i \geq 1$. In particular,

$$1 = \ell_{11} = \ell_{00}^{(1-d)^{n+1}}.$$

On the other hand, equating coefficients of $x_0 x_1^{d-3} x_2$, we find

$$1 = \ell_{00} \ell_{22} = \ell_{00}^{1+(1-d)^n}.$$

Since the exponents $(1-d)^{n+1}$ and $1 + (1-d)^n$ are relatively prime, it follows that $\ell_{00} = 1$, and then by (16), $\ell_{i,i} = 1$ for all i. Thus L is the identity.

Case VI.2: $n \equiv 1 \pmod 3$ *and* $n \geq 4$

We have $m = n + 2$. In what follows, subscripts are to be considered modulo m. Suppose $i \in \{0, 3, 6, \ldots, m-3\}$. Because of Corollary 10.3, the only y-monomials in $f(y_0, y_1, \ldots, y_{n+1})$ whose expansions could possibly contain $x_{i-1} x_i^{d-1}$ are those of the form $y_{j-1} y_j^{d-1}$ for some $j \in \{0, 3, 6, \ldots, m-3\}$. Moreover, for fixed i, at most one of these y-monomials can contribute an $x_{i-1} x_i^{d-1}$ term. On the other hand, by (7), $x_{i-1} x_i^{d-1}$ must appear in one of them, since it appears in $f(x_0, x_1, \ldots, x_{n+1})$. Suppose it appears in $y_{j-1} y_j^{d-1}$. Then, again by Corollary 10.3, the monomials in the expansion of $y_{j-1} y_j^{d-1}$ are exactly those monomials in the expansion of $f(y_0, y_1, \ldots, y_{n+1})$ involving both one of x_{i-3}, x_{i-2}, x_{i-1} and one of x_i, x_{i+1}, x_{i+2}. By (7) it then follows that $y_{j-1} y_j^{d-1}$ is a multiple of $x_{i-1} x_i^{d-1}$. By unique factorization, we deduce that y_{j-1} is a multiple of x_{j-1} and y_j is a multiple of x_j. By Corollary 10.3, it follows that $\pi(i) = j$ and $\pi(i-3) = j-3$. (We should identify -3 with $m-3$ when necessary.) Thus π acts as a rotation of $\{0, 3, 6, \ldots, m-3\}$, and there exists an integer r divisible by 3, determined up to a multiple of m, such that if $j \not\equiv 1 \pmod 3$, then y_j is a multiple of x_{j+r}. If $j \equiv 1 \pmod 3$, then by Corollary 10.3, y_j is a combination of x_{j+r-1}, x_{j+r}, and x_{j+r+1}.

It follows that the only y-monomial on the right hand side of (7) whose expansion could contain x_1^d is y_1^d. Thus $r \equiv 0 \pmod m$.

Equating coefficients of x_0^d in (7), we find $\ell_{10} = 0$. Equating coefficients of $x_0 x_2^{d-1}$, we find $\ell_{12} = 0$. Thus y_1 is a multiple of x_1.

Now suppose $j \in \{4, 7, 10, \ldots, m-2\}$. Equating coefficients of x_{j-1}^d in (7), we deduce that $\ell_{j,j-1} = 0$. Equating coefficients of x_{j+1}^d, we deduce that $\ell_{j,j+1} = 0$.

We now know that L is diagonal. The same proof as at the end of Case VI.1 shows that L is (a scalar multiple of) the identity.

Case VI.3: $n \equiv 2 \pmod 3$

We have $m = n + 1$. Since T_m is the one-dimensional vector space generated by e_{n+1}, we know that Le_{n+1} is a multiple of e_{n+1}. In other words, the only y_i that involves x_{n+1} is y_{n+1}.

If we view both sides of (7) as polynomials in x_{n+1}, and equate coefficients of x_{n+1}^{d-1} in (7), we deduce that y_n is a multiple of x_n. Similarly, equating coefficients of x_{n+1} shows that y_0 is a multiple of x_0. In particular, we have $\pi(0) = 0$ and $\pi(m-3) = m-3$.

We now show $\pi(i) = i$ for all $i \in \{0, 3, 6, \ldots, m-3\}$ by induction on i. Suppose $i \geq 3$, and we know $\pi(j) = j$ for $j < i$. By Corollary 10.3, the only y-monomial in the right hand side of (7) whose expansion can contain $x_{i-1} x_i^{d-1}$ is $y_{i-1} y_i^{d-1}$. It follows that y_i involves x_i, and $\pi(i) = i$, as desired. In fact, the monomials in the expansion of $y_{i-1} y_i^{d-1}$ are exactly those monomials in the expansion of $f(y_0, y_1, \ldots, y_{n+1})$ involving both one of x_{i-3}, x_{i-2}, x_{i-1} and one of x_i, x_{i+1}, x_{i+2}, and in which the exponents of x_i, x_{i+1}, x_{i+2} are even. By (7) it then follows that $y_{i-1} y_i^{d-1}$ is a multiple of $x_{i-1} x_i^{d-1}$. By unique factorization, we deduce that y_{i-1} is a multiple of x_{i-1} and y_i is a multiple of x_i.

We now know that L is of the form

$$L = \begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & * & 0 \\ * & * & * & * & * & * & \cdots & * & * & * & * \end{bmatrix},$$

and the diagonal entries must be nonzero, since L is nonsingular.

Equating coefficients of $x_0 x_1^{d-1}$ in (7), we find $0 = \alpha \ell_{10} \ell_{11}^{d-1}$, so $\ell_{10} = 0$. Equating coefficients of x_0^d, we find $0 = \alpha \ell_{n+1,0} \ell_{00}^{d-1}$, so $\ell_{n+1,0} = 0$. Thus y_0 is the only y_i that involves x_0.

If we view both sides of (7) as polynomials in x_0, and equate coefficients of x_0^{d-1}, we deduce that y_{n+1} is a multiple of x_{n+1}.

Equating coefficients of $x_1^{d-1} x_2$, we find $0 = \alpha \ell_{11}^{d-1} \ell_{12}$, so $\ell_{12} = 0$.

Now suppose $j \in \{4, 7, 10, \ldots, m-2\}$. Equating coefficients of x_{j-1}^d in (7), we deduce that $\ell_{j,j-1} = 0$. Equating coefficients of x_{j+1}^d, we deduce that $\ell_{j,j+1} = 0$.

We now know that L is diagonal. The same proof as at the end of Case VI.1 shows that L is (a scalar multiple of) the identity.

11. The automorphism group scheme

Finally, we consider the *automorphism group scheme* $\mathbf{Aut}\,\overline{X}$ of a smooth hypersurface $\overline{X}$ over $\overline{k}$. One can recover $\operatorname{Aut}\overline{X}$ as the group of $\overline{k}$-points of $\mathbf{Aut}\,\overline{X}$, but the triviality of $\mathbf{Aut}\,\overline{X}$ cannot be deduced immediately from the triviality of $\operatorname{Aut}\overline{X}$, because *a priori* $\mathbf{Aut}\,\overline{X}$ could be non-reduced. Fortunately, it is usually reduced:

Theorem 11.1. *If X is a smooth hypersurface in $\mathbf{P}^{n+1}$ of degree d, where $n \ge 1$, $d \ge 3$, and (n,d) does not equal $(1,3)$, then the connected component of the identity of $\mathbf{Aut}\,\overline{X}$ is trivial.*

Proof. Let $T_{\overline{X}}$ denote the tangent sheaf of $\overline{X}$ over $\overline{k}$. Under the hypotheses on (n,d), we have $H^0(\overline{X}, T_{\overline{X}}) = 0$ by [KS99, Theorem 11.5.2]. Thus the tangent space at the identity of $\mathbf{Aut}\,\overline{X}$ is trivial, so the connected component of the identity of $\mathbf{Aut}\,\overline{X}$ is trivial. □

Combining Corollary 1.9 and Theorem 11.1, we obtain:

Corollary 11.2. *For any field k and integers $n \ge 1$, $d \ge 3$ with (n,d) not equal to $(1,3)$ or $(2,4)$, there exists a smooth hypersurface X over k of degree d in $\mathbf{P}^{n+1}$ such that $\mathbf{Aut}\,\overline{X}$ is trivial.*

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References

[BT61] Raoul Bott and John Tate, 1961 (approximately). Colloquium lecture, University of Pennsylvania.

[Cha78] H. C. Chang, *On plane algebraic curves*, Chinese J. Math. **6** (1978), no. 2, 185–189.MR 80f:14012

[HS81] Alan Howard and Andrew John Sommese, *On the orders of the automorphism groups of certain projective manifolds*, Manifolds and Lie groups (Notre Dame, Ind., 1980), 1981, pp. 145–158.MR 84e:14011

[KS99] Nicholas M. Katz and Peter Sarnak, *Random matrices, Frobenius eigenvalues, and monodromy*, American Mathematical Society Colloquium Publications, vol. 45, American Mathematical Society, Providence, RI, 1999.MR 2000b:11070

[MM63] Hideyuki Matsumura and Paul Monsky, *On the automorphisms of hypersurfaces*, J. Math. Kyoto Univ. **3** (1963/1964), 347–361.MR 29 #5819

[OS77] Peter Orlik and Louis Solomon, *Singularities. II. Automorphisms of forms*, Math. Ann. **231** (1977/78), no. 3, 229–240.MR 57 #16292b

[Poo00a] Bjorn Poonen, *Varieties without extra automorphisms. I. Curves*, Math. Res. Lett. **7** (2000), no. 1, 67–76.MR 2001g:14052a

[Poo00b] ———, *Varieties without extra automorphisms. II. Hyperelliptic curves*, Math. Res. Lett. **7** (2000), no. 1, 77–82.MR 2001g:14052b

[Sch38] Hermann Ludwig Schmid, *Über die Automorphismen eines algebraischen Funktionenkörpers von Primzahlcharakteristik*, J. Reine Angew. Math. **179** (1938), 5–15.

[Sil92] Joseph H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 106, Springer-Verlag, New York, 1992. Corrected reprint of the 1986 original.MR 95m:11054

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