REINTERPRETING MUMFORD

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In [Mum66], Mumford defines the group $G(\mathscr{L})$ in terms of pulling back line bundles by translations. There is an alternative interpretation of this that I find simpler and more natural.

1. Morphisms of line bundles over a morphism of varieties

For simplicity, let k be an algebraically closed field (this is not really necessary; it is just so that I can speak set-theoretically in places without losing much). A line bundle $\mathscr{L}$ on an n-dimensional k-variety X can be viewed geometrically as an $(n+1)$-dimensional variety L equipped with a morphism

$$\begin{tikzcd} L \arrow[d] \\ X \end{tikzcd}$$

whose fibers are copies of $\mathbb{A}^1$ (plus a little extra structure so that each fiber viewed as a set of k-points has the structure of a 1-dimensional vector space).

If $\mathscr{L}$ and $\mathscr{M}$ are line bundles on X, an $\mathscr{O}_X$-module homomorphism $\mathscr{L} \to \mathscr{M}$ gives rise to a diagram

$$\begin{tikzcd} L \arrow[rr] \arrow[dr] & & M \arrow[dl] \\ & X, & \end{tikzcd}$$

that is, a morphism $L \to M$ lying above the identity morphism $1_X \colon X \to X$. More generally, given $\mathscr{L}$ on X and $\mathscr{M}$ on Y, we may define a $\mathsf{morphism}$ between the corresponding geometric line bundles $L \to X$ and $M \to Y$ to be a pair (f, t) forming a commutative square

$$\begin{tikzcd} L \arrow[r, dotted, "f"] \arrow[d] & M \arrow[d] \\ X \arrow[r, dotted, "t"] & Y \end{tikzcd}$$

(and respecting the vector space structures of the fibers). One might call this a homomorphism $L \to M$ lying over $t \colon X \to Y$ instead of over 1_X.

Question 1.1. What does (f, t) mean in terms of the $\mathscr{O}_X$-module $\mathscr{L}$ and $\mathscr{O}_Y$-module $\mathscr{M}$?

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To answer this, first define t^*M as the fiber product $M \times_Y X$, so that

$$\begin{array}{c} t^*M \\ \downarrow \\ X \end{array}$$

is the geometric line bundle corresponding to $t^*\mathscr{M}$.

Answer: Giving a morphism $L \to M$ over a morphism $t\colon X \to Y$ is equivalent to giving a homomorphism of $\mathscr{O}_X$-modules $\mathscr{L} \to t^*\mathscr{M}$. This is because of the universal property of the fiber product:

$$\begin{tikzcd} L \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & t^*M \arrow[r] \arrow[d] & M \arrow[d] \\ & X \arrow[r, "t"] & Y. \end{tikzcd}$$

Remark 1.2. If (f, t) is an isomorphism (meaning that both f and t are isomorphisms), then (f, t) induces an isomorphism of the spaces of geometric sections, or equivalently an isomorphism of vector spaces $\Gamma(X, \mathscr{L}) \to \Gamma(Y, \mathscr{M})$.

2. The group $G(\mathscr{L})$

Now let X be an abelian variety over k. Given $x \in X(k)$, let $\tau_x\colon X \to X$ be translation by x. Then one can define $G(\mathscr{L})$ as the group of automorphisms (f, t) of $L \to X$ such that t is a translation τ_x for some $x \in X(k)$. (Given x, for such an automorphism f to exist over τ_x, the line bundles $\mathscr{L}$ and $\tau_x^*\mathscr{L}$ must be isomorphic, so x will automatically be in $H(\mathscr{L})$.)

One nice thing about this definition of $G(\mathscr{L})$ is that it is obvious what the group law is. Also, the action of $G(\mathscr{L})$ on $\Gamma(X, L)$ is obvious: apply Remark 1.2 to the isomorphism given by each element of $G(\mathscr{L})$ (with $X = Y$ and $\mathscr{L} = \mathscr{M}$).

3. Symmetric line bundles

Now assume in addition that $\operatorname{char} k \neq 2$. A line bundle $\mathscr{L}$ on X is **symmetric** if there is an isomorphism $\phi\colon L \to L$ lying over $[-1]\colon X \to X$. In this case, all other such ϕ arise by composing ϕ with fiber-wise multiplication by a single element of $k^\times$. This element of $k^\times$ can be chosen in a unique way to make the fiber homomorphism $\phi(0)\colon L(0) \to L(0)$ equal to the identity; in this case, ϕ is called **normalized**.

Suppose that ϕ is normalized. Then $\phi \circ \phi = 1_L$ since 1_L is the only automorphism of L lying over 1_X that acts as the identity on the fiber $L(0)$. If $x \in X(k)$ is a fixed point of $[-1]$ (that is, a 2-torsion point), then ϕ maps the fiber $L(x)$ to itself. This isomorphism $L(x) \to L(x)$ of 1-dimensional vector spaces is multiplication by a scalar; this is $e_*^{\mathscr{L}}(x) \in \{\pm 1\}$.

A line bundle $\mathscr{L}$ is **totally symmetric** if there exists $\phi\colon L \to L$ lying above $[-1]$ such that ϕ acts as the identity on the fiber above each 2-torsion point. Let $\pi\colon X \to K_X$ be the morphism to the **Kummer variety**, defined as the variety quotient of X by the order 2 group generated by $[-1]$. If $\mathscr{M}$ is a line bundle on K_X, then pulling it back in the diagram

$$\begin{tikzcd} X \arrow[rr, "{[-1]}"] \arrow[dr] & & X \arrow[dl] \\ & K_X & \end{tikzcd}$$

immediately shows that $\pi^*\mathscr{M}$ is a totally symmetric line bundle on X. This makes the converse believable too (and it is not too much work to prove).

Any isomorphism from $L \to X$ to $M \to Y$ induces an isomorphism $G(\mathscr{L}) \to G(\mathscr{M})$. If $\psi\colon L \to L$ is an isomorphism lying over $[-1]\colon X \to X$, then $(\psi, [-1])$ is an automorphism of $L \to X$, so by the previous sentence, $(\psi, [-1])$ induces an automorphism of $G(\mathscr{L})$; this is Mumford's δ_{-1}.

References

[Mum66] D. Mumford, *On the equations defining abelian varieties. I*, Invent. Math. **1** (1966), 287–354. MR0204427 (34 #4269) ↑(document)

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