THE VALUATION OF THE DISCRIMINANT OF A HYPERSURFACE

BJORN POONEN AND MICHAEL STOLL

Abstract. Let $R$ be a discrete valuation ring, with valuation $v : R \to \mathbb{Z} \cup \{\infty\}$ and residue field $k$. Let $H$ be a hypersurface $\text{Proj} \, R[x_0, \ldots, x_n]/(f)$. Let $H_k$ be the special fiber, and let $(H_k)_{\text{sing}}$ be its singular subscheme. Let $\Delta(f)$ be the discriminant of $f$. We use Zariski’s main theorem and degeneration arguments to prove that $v(\Delta(f)) = 1$ if and only if $H$ is regular and $(H_k)_{\text{sing}}$ consists of a nondegenerate double point over $k$. We also give lower bounds on $v(\Delta(f))$ when $H_k$ has multiple singularities or a positive-dimensional singularity.

1. Introduction

Throughout the paper, $R$ denotes a discrete valuation ring, with valuation $v : R \to \mathbb{Z} \cup \{\infty\}$, maximal ideal $m = (\pi)$, and residue field $k$ (except in a few places where $k$ is an arbitrary field).

Let $E \subset \mathbb{P}^2_R$ be defined by a Weierstrass equation, with generic fiber an elliptic curve. If the discriminant of the equation has valuation 1, then $E$ is regular and the singular locus of its special fiber consists of a node; this follows from Tate’s algorithm [Tat75], for example; see also [Sil94, Lemma IV.9.5(a)]. Our main theorem (Theorem 1.1) generalizes this to hypersurfaces of arbitrary degree and dimension (terminology will be explained later).

**Theorem 1.1.** Let $f \in R[x_0, \ldots, x_n]$ be a homogeneous polynomial. Let $\Delta(f)$ be its discriminant. Let $H = \text{Proj} \, R[x_0, \ldots, x_n]/(f)$. Then the following are equivalent:

(i) $v(\Delta(f)) = 1$;

(ii) $H$ is regular, and $(H_k)_{\text{sing}}$ consists of a nondegenerate double point in $H(k)$.

We also prove that if $(H_k)_{\text{sing}}$ consists of $r$ isolated closed points, then $v(\Delta(f)) \geq r$ (Theorem 6.2). If $\dim (H_k)_{\text{sing}} \geq 1$, we show that $H_k$ is a limit of hypersurfaces whose singular subscheme is finite but with many points, and we combine this and an argument using the Greenberg functor to deduce that $v(\Delta(f)) \geq \max(\lfloor (\deg f - 1)/2 \rfloor, 2)$ (Theorem 8.4).

2. Discriminant

Fix $n \geq 1$ and $d \geq 2$. Let $x^i$ range over the degree $d$ monomials in $\mathbb{Z}[x_0, \ldots, x_n]$, and let $a_i$ be independent indeterminates, so that $F := \sum_i a_i x^i$ is the generic degree $d$ homogeneous polynomial in $x_0, \ldots, x_n$. Then the affine space $\mathbb{A}^N := \text{Spec} \, \mathbb{Z}[\{a_i\}]$ may be viewed as a moduli space for hypersurfaces (one could also remove the origin, or projectivize as in [Sai12, §2.4]). Let $\mathcal{H} \subset \mathbb{P}^n \times \mathbb{A}^N$ be the closed subscheme defined by $F = 0$, so the projection $\phi : \mathcal{H} \to \mathbb{A}^N$ is the universal hypersurface. Let $\mathcal{H}_{\text{sing}}$ be the relative singular subscheme, the closed subscheme...
defined by \( F = \partial F/\partial x_0 = \cdots = \partial F/\partial x_n = 0 \). More precisely, \( \mathcal{H}_{\text{sing}} \) is the locus of points where \( \phi \) is not smooth of relative dimension \( n - 1 \).

The other projection \( \mathcal{H}_{\text{sing}} \to \mathbb{P}^n \) is a rank \( N - n - 1 \) vector bundle since the equations \( F = \partial F/\partial x_0 = \cdots = \partial F/\partial x_n = 0 \) are linear in the \( a_i \) and independent above each point of \( \mathbb{P}^n \) except for the Euler relation \( d \cdot F = \sum x_i(\partial F/\partial x_i) \). Thus \( \mathcal{H}_{\text{sing}} \) is integral and smooth of relative dimension \( N - 1 \) over \( \mathbb{Z} \). Its scheme-theoretic image under the proper morphism \( \phi \) is a closed subscheme \( D \subset \mathbb{A}^N \), the locus parametrizing singular hypersurfaces. In fact, \( D \subset \mathbb{A}^N \) is a divisor and the restriction \( \mathcal{H}_{\text{sing}} \to D \) of \( \phi \) is birational (cf. [Sai12, §2.9]); this is a Bertini-type statement saying essentially that among hypersurfaces singular at a point, most have singular subscheme consisting of just that point. Thus \( D \subset \mathbb{A}^N \) is the zero locus of some polynomial \( \Delta \in \mathbb{Z}[\{a_i\}] \) determined up to a unit, i.e., up to sign; \( \Delta \) is called the discriminant. (See [GKZ08, Dem12, Sai12] for other descriptions of \( \Delta \).) By definition, if the \( a_i \) are specialized to elements of a field \( k \), the resulting hypersurface in \( \mathbb{P}^n_k \) is singular (not smooth of dimension \( n - 1 \)) if and only if \( \Delta \) specializes to 0 in \( k \).

3. Quadratic Forms

**Proposition 3.1.** Suppose that \( d = 2 \). Let \( \text{Det} = \det(\partial^2 F/\partial x_i\partial x_j) \in \mathbb{Z}[\{a_i\}] \). If \( n \) is odd, then \( \Delta = \pm \text{Det} \). If \( n \) is even, then \( \Delta = \pm \text{Det}/2 \).

*Proof.* This is well known, except perhaps the power of 2, which can be determined by evaluating \( \text{Det} \) for a quadratic form defining a smooth quadric over \( \mathbb{Z} \), since \( \Delta = \pm 1 \) for such a form. Use \( x_0x_1 + \cdots + x_{n-1}x_n \) if \( n \) is odd, and \( x_0x_1 + \cdots + x_{n-2}x_{n-1} + x_n^2 \) if \( n \) is even. \( \square \)

A symmetric bilinear space over \( R \) is a pair \((M, \beta)\) where \( M \) is a finite-rank projective module \( R \) (hence free since \( R \) is a discrete valuation ring) and \( \beta: M \times M \to R \) is a symmetric \( R \)-bilinear pairing.

**Proposition 3.2.** Let \( R \) be a discrete valuation ring.

(a) Each symmetric bilinear space over \( R \) is an orthogonal direct sum of spaces of rank 1 and 2.

(b) Every quadratic form \( f(x_0, \ldots, x_n) \) over \( R \) is equivalent to one of the form

\[
\sum_{i=1}^{I} (a_ix_i^2 + b_i x_iz_i) + \sum_{j=1}^{J} d_j z_j^2
\]

with \( 2I + J = n + 1 \) and \( a_i, b_i, c_i, d_j \in R \).

(c) Let \( f \) be as in (b). Let \( H = \text{Proj} \ R[x_0, \ldots, x_n]/(f) \). Then \( \nu(\Delta(f)) \geq \dim(H_k)_{\text{sing}} + 1 \).

*Proof.*

(a) (We paraphrase an argument of Jean-Pierre Tignol adapted from the proof of [Ver19, Proposition 4.10].) Let \((M, \beta)\) be a nonzero symmetric bilinear space. We may assume that \( \beta \neq 0 \). By dividing \( \beta \) by a nonzero element of \( R \), we may assume that \( \beta(M, M) \not\subset \mathfrak{m} \). We claim that there exists a free \( R \)-module \( N \) of rank 1 or 2 with a homomorphism \( N \to M \) such that \( \beta \) induces a regular pairing on \( N \) (i.e., the composition \( N \to M \to M^\vee \to N^\vee \) is an isomorphism); then \( N \to M \) is injective, and \( M \) is the orthogonal direct sum of \( N \) and \( N^\perp := \ker(M \to N^\vee) \), so we are done by induction on \( \text{rank}(M) \).
If there exists \( e \in M \) with \( \beta(e, e) \in R^* \) a unit, then let \( N = Re \). Otherwise, choose \( c, d \in M \) with \( \beta(c, d) \in R^* \) and let \( N = Rc \oplus Rd \); the induced pairing is regular since its matrix is invertible, being congruent mod \( \pi \) to \( \begin{pmatrix} 0 & \beta(c, d) \\ \beta(c, d) & 0 \end{pmatrix} \).

(b) Decomposing a quadratic space is equivalent to decomposing the associated symmetric bilinear space, even if \( \text{char } k = 2 \).

(c) First suppose \( \text{char } k \neq 2 \). Then \( f \) is equivalent to \( \sum a_i x_i^2 \) for some \( a_i \in R \), and

\[
\dim (H_k)^{\text{sing}} = \# \{ i : v(a_i) \geq 1 \} - 1 \leq v(\text{Det}(f)) - 1 = v(\Delta(f)) - 1,
\]

by Proposition 3.1.

Now suppose \( \text{char } k = 2 \). Let \( I_0 = \# \{ i : v(b_i) = 0 \} \) and \( I_1 = \# \{ i : v(b_i) \geq 1 \} \). Let \( J_0 = \# \{ j : v(d_j) = 0 \} \) and \( J_1 = \# \{ j : v(d_j) \geq 1 \} \). If \( n \) is odd, let \( J' := J \). If \( n \) is even, then \( J \) is odd, so let \( J' := J - 1 \). In both cases \( J' \geq 0 \). The common zero locus in \( \mathbb{P}_k^n \) of the polynomials \( \partial f / \partial x_i \) and \( \partial f / \partial y_i \), for \( i \in I_0 \) is of dimension \( n - 2I_0 \), and including the condition \( f = 0 \) drops the dimension by 1 more if \( J_0 \geq 1 \). Thus \( \dim (H_k)^{\text{sing}} \leq n - 2I_0 \), with strict inequality if \( J_0 \geq 1 \). On the other hand, \( v(4a_i c_i - b_i^2) \geq 2 \) whenever \( v(b_i) \geq 1 \), and \( v(2d_j) \geq v(2) + v(d_j) \) for all \( j \), so Proposition 3.1 implies

\[
v(\Delta(f)) \geq 2I_1 + J'v(2) + J_1
\]

\[
= (n - 2I_0) + J'v(2) - J_0 + 1
\]

\[
\geq \dim (H_k)^{\text{sing}} + J'v(2) - J_0 + 1.
\]

If \( J_0 \geq 1 \), then the inequality above is strict and \( J'v(2) \geq (J_0 - 1)v(2) \geq J_0 - 1 \), so \( v(\Delta(f)) \geq \dim (H_k)^{\text{sing}} + 1 \). If \( J_0 = 0 \), then instead use \( J'v(2) \geq 0 \) to again get \( v(\Delta(f)) \geq \dim (H_k)^{\text{sing}} + 1 \). \( \square \)

4. Nondegenerate double points

**Definition 4.1** ([SGA 7] VI.6). Let \( k \) be a field. Let \( X \) be a finite-type \( k \)-scheme. A \( k \)-point \( Q \in X \) is called a **nondegenerate double point** (or **nondegenerate quadratic point**) if there exist \( n \geq 1 \) and \( f \in k[[x_1, \ldots, x_n]] \) such that there is an isomorphism of complete \( k \)-algebras \( \widehat{\mathcal{O}}_{X, Q} \cong k[[x_1, \ldots, x_n]]/(f) \) and an equality of ideals \( (\partial f / \partial x_1, \ldots, \partial f / \partial x_n) = (x_1, \ldots, x_n) \).

**Remark 4.2.** The ideal equality is equivalent to saying that \( Q \) is an isolated reduced point of the singular subscheme \( X_{\text{sing}} \).

**Remark 4.3.** Suppose that \( n \) and \( f \) exist. Then \( f \) can be taken to be a quadratic form [SGA 7] VI.6.1]. If, moreover, \( k \) is algebraically closed, then

- if \( \text{char } k \neq 2 \), then one can take \( f := x_1^2 + \cdots + x_n^2 \);
- if \( \text{char } k = 2 \), then \( n \) must be even and one can take \( f := x_1 x_2 + x_3 x_4 + \cdots + x_{n-1} x_n \).

**Remark 4.4** ([SGA 7] Definition VI.6.6]). There is also notion of **ordinary double point**, which is the same except that when \( \text{char } k = 2 \) and \( n \) is odd, since nondegeneracy is impossible one allows singularities analytically equivalent over an algebraic closure to the singularity defined by the “least degenerate” quadratic form \( f := x_1 x_2 + \cdots + x_{n-2} x_{n-1} + x_n^2 \).
5. COMMUTATIVE ALGEBRA

A ring extension \( R' \supset R \) is called a weakly unramified extension if \( R' \) too is a discrete valuation ring and \( \pi \) is also a uniformizer of \( R' \).

**Lemma 5.1.** For any field extension \( k' \supset k \), there exists a weakly unramified extension \( R' \supset R \) with residue field \( k' \) (i.e., isomorphic to \( k' \) as \( k \)-algebra).

**Proof.** If \( k'/k \) is generated by one algebraic element, say a zero of a monic irreducible polynomial \( \tilde{f} \in k[x] \), then we may take \( R' := R[x]/(f) \) for any monic \( f \in R[x] \) reducing to \( \tilde{f} \) [Ser79, I.§6, Proposition 15]. If \( k'/k \) is generated by one transcendental element \( t \), then we may take the localization \( R' := R[t]_{(\pi)} \) of the (regular) polynomial ring \( R[t] \) at the codimension \( 1 \) prime \((\pi)\); the residue field of \( R' \) is \( \text{Frac}(R[t]_{(\pi)}) = k(t) \). The general case follows from Zorn’s lemma, using direct limits.

**Lemma 5.2.** Let \( A \) be a noetherian local domain. Let \( \hat{A} \) be its completion. Let \( B \) be the integral closure of \( A \). Then \[
\#\{\text{minimal primes of } \hat{A}\} \geq \{\text{maximal ideals of } B\}.
\]

**Proof.** Combine [SP Tag 0C24] and [SP Tag 0C28(1)].

6. HYPERSURFACES WITH SEVERAL SINGULARITIES

Let notation be as in Theorem [I.1]. We use subscripts to denote base change: e.g., \( D_A := D \times_{\text{Spec} \mathbb{Z}} \text{Spec} A \) for any ring \( A \). Restricting \( \phi_R \) yields a proper morphism \( \varphi : (\mathcal{H}_R)_{\text{sing}} \to D_R \).

**Proposition 6.1.** The proper morphism \( \varphi : (\mathcal{H}_R)_{\text{sing}} \to D_R \) is birational.

**Proof.** This follows from [Sai12, Proposition 2.12] applied over \( \text{Frac}(R) \).

**Theorem 6.2.** If the space \( (H_k)_{\text{sing}} \) consists of \( r \) closed points, then \( v(\Delta(f)) \geq r \).

**Proof.** Using Lemma 5.1 we may reduce to the case in which \( k \) is algebraically closed.

Let \( P \in D_R(k) \) correspond to \( H_k \), so \( \varphi^{-1}(P) = (H_k)_{\text{sing}} \). Since \( R \) is regular, the local ring \( \mathcal{O}_{(H_k)_{\text{sing}}} \) is regular, and hence factorial [AB59, Theorem 5].

Let \( D' := \{d \in D_R : \dim_{\varphi^{-1}(d)} = 0\} \), so \( P \in D' \). By [EGA IV$_3$, Corollaire 13.1.5], \( D' \) is open in \( D_R \). By Proposition 6.1 \( \varphi^{-1}(D') \to D' \) is birational. It is also quasi-finite and proper, hence finite by Zariski’s main theorem [EGA III], Corollaire 4.4.11]. Moreover, \( (H_R)_{\text{sing}} \) is smooth over a discrete valuation ring, hence normal. The previous three sentences imply that \( \varphi^{-1}(D') \to D' \) is the normalization of \( D' \).

Take \( A := \mathcal{O}_{D',P} = \mathcal{O}_{D,P} = \mathcal{O}_{(H_k)_{\text{sing}},P}/(\Delta) \), and define \( \hat{A} \) and \( B \) as in Lemma 5.2. Then the maximal ideals of \( B \) correspond to the points of \( \varphi^{-1}(D') \) above \( P \), which are the \( r \) points of \( (H_k)_{\text{sing}} \). Lemma 5.2 implies that \( \hat{A} \) has at least \( r \) minimal primes. Their inverse images in \( \mathcal{O}_{(H_k)_{\text{sing}},P} \) correspond to prime factors of \( \Delta \) in this factorial ring, so \( \Delta = p_1 \cdots p_rq \), for some \( p_1, \ldots, p_r, q \in \mathcal{O}_{(H_k)_{\text{sing}},P} \) with each \( p_i \) vanishing at \( P \). Evaluating both sides at (the coefficient tuple of) \( f \) shows that \( v(\Delta(f)) \geq 1 + \cdots + 1 + 0 = r \).
7. Valuations of polynomial values

**Lemma 7.1.** Suppose that $k$ is infinite, and $\ell \geq n$. Let $\rho: \mathbb{A}^\ell_k \to \mathbb{A}^n_k$ be a projection. Let $V \subset \mathbb{A}^\ell_k$ be a closed subscheme. Then $\{a \in k^n : \rho^{-1}(a)(k) \subseteq V(k)\}$ is the set of $k$-points of a closed subscheme $Z \subseteq \mathbb{A}^n_k$.

**Proof.** Since $k$ is infinite, $\rho^{-1}(a)(k) \subseteq V(k)$ is equivalent to $\rho^{-1}(a) \subseteq V$, which fails if and only if $a \in \rho(\mathbb{A}^\ell_k - V)$. Since $\rho$ is flat, $\rho$ is open, so $\rho(\mathbb{A}^\ell_k - V)$ is open; let $Z$ be its complement. \hfill \Box

For $b \in R$, let $\bar{b}$ be its image in $k$. Likewise, given $b \in R^n$, define $\bar{b} \in k^n$.

**Proposition 7.2.** Let $\delta \in R[x_1, \ldots, x_n]$ and $m \in \mathbb{Z}_{\geq 0}$. If $k$ is infinite and perfect, then

$$\{a \in k^n : v(\delta(b)) \geq m \text{ for all } b \in R^n \text{ with } \bar{b} = a\}$$

is the set of $k$-points of a closed subscheme of $\mathbb{A}^n_k$.

**Proof.** The $m$th Greenberg functor $\text{Gr}^m$ satisfies $\text{Gr}^m(X)(k) = X(R/m^m)$ for any $R$-scheme $X$; see [Gre61, Gre63, NS08, §2.2; BGA18]. Applying $\text{Gr}_m$ to $\delta: \mathbb{A}^n_R \to \mathbb{A}^1_R$ yields a morphism

$$\text{Gr}^m(\mathbb{A}^n_R) \longrightarrow \text{Gr}^m(\mathbb{A}^1_R);$$

let $V$ be the fiber above $0$. On the other hand, the reduction map $R/m^m \to k$ induces a morphism $\rho: \text{Gr}^m(\mathbb{A}^n_R) \to \text{Gr}^1(\mathbb{A}^n_R)$ that is a projection $\mathbb{A}^m_R \to \mathbb{A}^1_k$ as in Lemma 7.1. For $a \in k^n$,

$$v(\delta(b)) \geq m \text{ for all } b \in R^n \text{ with } \bar{b} = a \iff \rho^{-1}(a)(k) \subseteq V(k),$$

so the result follows from Lemma 7.1. \hfill \Box

8. Hypersurfaces with a positive-dimensional singularity

In Lemma 8.1, Corollary 8.2, and Lemma 8.3 we assume that $n \geq 2$, $r \geq 1$, and $P_1, \ldots, P_r$ are distinct points in $\mathbb{P}^n(k)$. Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}^n}$. For each $P \in \mathbb{P}^n(k)$, let $\mathfrak{m}_P \subset \mathcal{O}$ be the ideal sheaf of $P$.

**Lemma 8.1.** If $d \geq 2r - 1$, then $\mathcal{O}(d) \to \prod_i (\mathcal{O}/\mathfrak{m}_P^2)(d)$ induces a surjection on global sections.

**Proof.** Let $\ell_i$ be a linear form vanishing at $P_i$ but not $P_j$ for any $j \neq i$. Let $h$ be a homogeneous polynomial of degree $d - (2r - 1)$ not vanishing at any $P_i$. For each $s$, as $g$ ranges over linear forms, the image of $g$ in $(\mathcal{O}/\mathfrak{m}_P^2)(1)$ ranges over all its sections, so the images of $gh\prod_{j \neq s} \ell^2_j$ in $\prod_i (\mathcal{O}/\mathfrak{m}_P^2)(d)$ exhaust the $s$th factor of $\prod_i (\mathcal{O}/\mathfrak{m}_P^2)(d)$. \hfill \Box

**Corollary 8.2.** Let $N = \dim_k \Gamma(\mathbb{P}^n, \mathcal{O}(d))$. For $f \in \Gamma(\mathbb{P}^n, \mathcal{O}(d))$, let $H_f := \text{Proj} k[x_0, \ldots, x_n]/(f)$. Then the $f$ for which $(H_f)_{\text{sing}} \supset \{P_1, \ldots, P_r\}$ form a vector space of dimension $N - r(n + 1)$.

**Lemma 8.3.** If $d \geq 3$ and $1 \leq r \leq \max((d - 1)/2, 2)$, then in the locus $\mathbb{A}$ of $f$ for which $(H_f)_{\text{sing}} \supset \{P_1, \ldots, P_r\}$, the open sublocus $U$ for which $(H_f)_{\text{sing}}$ is finite is dense.

**Proof.** Since $\mathbb{A}$ is defined by the vanishing of values of $f$ and its partial derivatives at the $P_i$, it is cut out by linear forms in the coefficients of $f$, so $\mathbb{A}$ is an affine space. Applying [EGA IV, Corollaire 13.1.5] the relative singular subscheme over $\mathbb{A}$ shows that $U$ is open in $\mathbb{A}$, so it remains to show that $U \neq \emptyset$.

First suppose that $r \leq (d - 1)/2$. Let

$$I = \{(f, P_{r+1}) : f \in \mathbb{A}, P_{r+1} \in (H_f)_{\text{sing}} - \{P_1, \ldots, P_r\}\}.$$
Theorem 8.4. Let $I$ at a point $P$ whose singular locus contains $I$.

Proof. We may assume that $k$ is a proper birational morphism, so $\phi$ is an isomorphism. On the other hand, by Remark 4.3,

$$\Delta = \phi^*(\Delta)$$

Proof of Theorem 1.1. Case 1: $r = \dim(A) = \dim(A) - 1$. Therefore $I \to A$ is not dominant, and $U$ contains the complement of its image.

Now suppose instead that $r \leq 2$. Choose a homogeneous degree $d$ form $g(x_3, \ldots, x_n)$ defining a smooth hypersurface in $\mathbb{P}^{n-3}$, let $c_1, \ldots, c_{d-1} \in k$ be distinct (enlarge $k$ if necessary), and let

$$f = x_0 \prod_{i=1}^{d-1} (x_i - c_i x_2) + g.$$ 

At a point $P$ where $f$ and its partial derivatives vanish, $\prod_{i=1}^{d-1} (x_i - c_i x_2) = 0$, so $g = 0$, so $g$ and its derivatives vanish, so $x_3 = \cdots = x_n = 0$; thus $P$ is a singular point of the plane curve $x_0 \prod_{i=1}^{d-1} (x_i - c_i x_2) = 0$, i.e., an intersection point of two components. By a linear change of variable, we may assume that the $P_i$ (of which there are at most two) are among these singular points. Then $f$ gives a $k$-point of $U$. □

Theorem 8.4. Let $H = \text{Proj} R[x_0, \ldots, x_n]/(f)$ for some homogeneous $f$ of degree $d$. If $\dim(H_{\text{sing}}) \geq 1$, then $v(\Delta(f)) \geq \max([(d-1)/2], 2)$.

Proof. We may assume that $n, d \geq 2$. Using Lemma 5.1 we may reduce to the case in which $k$ is algebraically closed. If $d = 2$, then Proposition 3.2(c) implies that $v(\Delta(f)) \geq \dim(H_{\text{sing}}) + 1 \geq 2$.

So assume $d \geq 3$. Let $Z$ be the closed subscheme of Proposition 7.2 for $\delta := \Delta \in R[\{a_i\}]$ and $r := \max([(d-1)/2], 2)$. Choose distinct $P_1, \ldots, P_r \in (H_{\text{sing}})(k)$. If $j \in R[x_0, \ldots, x_n]$ is a degree $d$ homogeneous polynomial, and $J = \text{Proj} R[x_0, \ldots, x_n]/(j)$ is such that $(J_{\text{sing}})^{-1} = \{P_1, \ldots, P_r\}$, then $v(\Delta(j)) \geq r$ by Theorem 6.2, so the corresponding coefficient tuple mod $m$ belongs to $Z(k)$. By Lemma 8.3, any coefficient tuple mod $m$ corresponding to a hypersurface whose singular locus contains $\{P_1, \ldots, P_r\}$ also belongs to $Z(k)$. This applies in particular to the coefficient tuple of $f$ mod $m$, so $v(\Delta(m)) \geq r$ by definition of $Z$. □

9. When the Discriminant Has Valuation 1

Proof of Theorem 1.1. Case 1: $\text{char } k = 2$ and $n$ is odd. By [Sai12, Theorem 4.2], if the sign of $\Delta$ is chosen appropriately, then $\Delta = A^2 + 4B$ for some polynomials $A, B$, so $v(\Delta(f)) \neq 1$. On the other hand, by Remark 4.3, $H_k$ cannot have a nondegenerate double point. Thus (i) and (ii) both fail.

Case 2: $\text{char } k \neq 2$ or $n$ is even. The hypersurface $H \to \text{Spec } R$ is the pullback of $H_R \to \mathbb{A}^N_R$ by some $R$-morphism $\nu: \text{Spec } R \to \mathbb{A}^N_R$. Let $P = \nu(\text{Spec } k) \in \mathbb{A}^N(k)$.

(i)⇒(ii): Suppose that $\nu(\Delta(f)) = 1$. By Theorem 8.4, $(H_k)_{\text{sing}}$ is finite. The surjection $R[\{a_i\}] \to R$ sending the $a_i$ to the corresponding coefficients $a_1$ of $f$ maps $\Delta$ to $\Delta(f)$, so the $a_i - a_1$ and $\Delta$ are local parameters for $\mathbb{A}^N_R$ at $P$. Thus $D_R = \text{Spec } R[\{a_i\}]/(\Delta)$ is regular at $P$, so $D_R$ is normal at $P$. Let $U$ be the largest normal open subscheme of $D_R$ such that $\varphi^{-1}U \to U$ has finite fibers. The fiber above $P$ is $(H_k)_{\text{sing}}$, so $P \in U$. By Proposition 6.1, $\varphi$ is a proper birational morphism, so $\varphi^{-1}U \to U$ has finite fibers by Zariski’s main theorem [EGA III, Corollaire 4.4.9]. In particular, the fiber $(H_k)_{\text{sing}}$ consists of a single reduced $k$-point $Q$. By Remark 1.2, $Q$ is a nondegenerate double point of $H_k$.

Choose an $\mathbb{A}^N_R \subset \mathbb{P}^n_R$ containing $Q$; let $f_0$ be the corresponding dehomogenization of $f$. The point $(H_k)_{\text{sing}}$ is cut out in $\mathbb{A}^N_R$ by $f_0$ and its partial derivatives; these $n + 1$ functions are
therefore local parameters for $\mathbb{P}^n_R$ at $Q$, so the local ring $\mathcal{O}_{H,Q} = \mathcal{O}_{\mathbb{P}^n_R,Q}/(f_0)$ is regular too. On the other hand, $H - \{Q\}$ is smooth over $\text{Spec } R$. Thus $H$ is regular everywhere.

(ii)$\Rightarrow$(i): Now suppose that $H$ is regular and $(H_k)_{\text{sing}}$ consists of a nondegenerate double point $Q \in H(k)$. Hence the underlying space of $H_{\text{sing}}$ is $\{Q\}$.

Since the tangent space of $(H_k)_{\text{sing}}$ at $Q$ is 0, the projection $(H_k)_{\text{sing}} \to \mathbb{A}^n_k$ induces an injection between the tangent spaces at $Q$ and $P$. Since $Q$ is the only point in $(H_k)_{\text{sing}}$ above $P$, this implies that $(H_k)_{\text{sing}} \to D_R$ is étale at $Q$. Pulling back $(H_k)_{\text{sing}} \to D_R \hookrightarrow \mathbb{A}^n_R$ by $\iota$ shows that $H_{\text{sing}} \to \text{Spec}(R/(\Delta(f)))$ is étale. These are connected 0-dimensional schemes with the same residue field, so $H_{\text{sing}} \simeq \text{Spec}(R/(\Delta(f)))$.

Let $f_0$ be as above, so $f_0$ and its partial derivatives lie in the maximal ideal $m_{\mathbb{P}^n_R,Q} \subset \mathcal{O}_{\mathbb{P}^n_R,Q}/(f_0)$. The partial derivatives are independent in $m_{\mathbb{P}^n_R,Q}/m_{\mathbb{P}^n_R,Q}^2$ since they form a basis for $m_{\mathbb{P}^n_k,Q}/m_{\mathbb{P}^n_k,Q}^2$, since $Q$ is a nondegenerate double point. On the other hand, the image of $f_0$ in $m_{\mathbb{P}^n_R,Q}/m_{\mathbb{P}^n_R,Q}^2$ is nonzero (since $\mathcal{O}_{H,Q} = \mathcal{O}_{\mathbb{P}^n_R,Q}/(f_0)$ is regular) and in fact independent of the partial derivatives (since it maps to 0 in $m_{\mathbb{P}^n_k,Q}/m_{\mathbb{P}^n_k,Q}^2$). Thus $f_0$ and its partial derivatives form a basis of $m_{\mathbb{P}^n_R,Q}/m_{\mathbb{P}^n_R,Q}^2$, so by Nakayama’s lemma, they generate $m_{\mathbb{P}^n_R,Q}$, so $H_{\text{sing}} \simeq \text{Spec } k$.

The conclusions of the two previous paragraphs imply $v(\Delta(f)) = 1$. \hfill $\Box$

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**References**


Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA

Email address: poonen@math.mit.edu
URL: http://math.mit.edu/~poonen/

Mathematisches Institut, Universität Bayreuth, 95440 Bayreuth, Germany

Email address: Michael.Stoll@uni-bayreuth.de
URL: http://www.mathe2.uni-bayreuth.de/stoll/