18.782: INTRODUCTION TO ARITHMETIC GEOMETRY, FALL 2009

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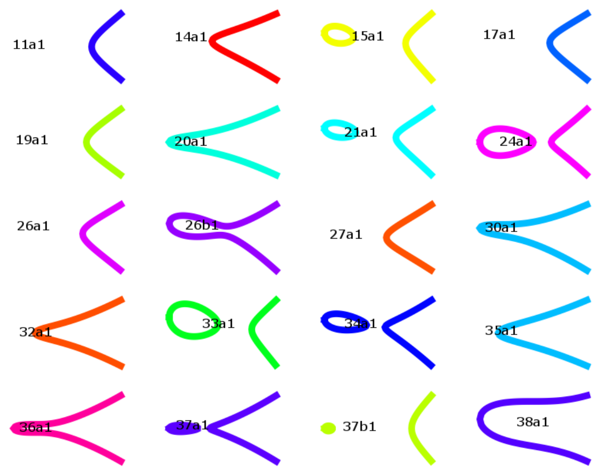

The class meets Tuesdays and Thursdays, 9:30-11:00am. The prerequisite is 18.702.

1. Tentative list of topics

This course will be an introduction to some of the key ideas of arithmetic geometry, which is the subject that applies methods from algebra and geometry to answer questions like "What are all the rational number solutions to the equation $y^2 = x^3 + 4$?"

1.1. p-adic numbers.

- Absolute values on fields
- The p-adic absolute value on $\mathbb{Q}$
- Ostrowski's theorem characterizing all absolute values on $\mathbb{Q}$
- Construction of $\mathbb{Q}_p$ and $\mathbb{Z}_p$ via Cauchy sequences
- Construction of $\mathbb{Z}_p$ as an inverse limit
- p-adic expansions of elements of $\mathbb{Q}_p$
- Hensel's lemma (the p-adic analogue of Newton's method)
- The structure of the multiplicative group $\mathbb{Q}_p^\times$

1.2. Further field theory.

- Algebraic closures
- Finite fields
- Galois groups of infinite algebraic extensions

The figure, showing various elliptic curves with their Cremona labels, was created by William Stein using Sage.

1.3. **Basic algebraic geometry (omitting many proofs).**

- Affine varieties
- Projective varieties
- Smooth varieties
- Morphisms and rational maps
- Divisors on a curve
- Linear equivalence of divisors and the Picard group
- Statement of the Riemann-Roch theorem

1.4. **Plane conics.**

- Genus 0 curves with a rational point
- The anticanonical embedding of an arbitrary genus 0 curve
- Number of points of a plane conic over a finite field
- Testing whether a plane conic over $\mathbb{Q}$ has $\mathbb{Q}_p$-points for all p
- The local-global principle for plane conics over $\mathbb{Q}$

1.5. **Elliptic curves.**

- Genus 1 curves with a rational point
- Weierstrass equations
- The group law, and the connection with the Picard group
- Elliptic curves over finite fields: statement of the Weil conjectures in this setting
- Elliptic curves over $\mathbb{Q}$: torsion subgroup, height functions, the Mordell-Weil theorem (in the case of rational 2-torsion)

2. Grading

Grades will be based on weekly problem sets, an in-class midterm on Thursday, November 5, and a final exam in December.

References

There will be no required textbook for the course. But [Kob84, Chapter I], [Ser73, Chapters I and II (and a little of IV)], [Ful89], and [Sil92] are recommended references for most of the material above.

The book [ST92] is an alternative to [Sil92]; it is more elementary, but this sometimes comes at the cost of replacing conceptual understanding with tedious calculations, so I recommend [Sil92] if you have the prerequisites for it (or can deal with not having them).

[Ful89] William Fulton, *Algebraic curves*, Advanced Book Classics, Addison-Wesley Publishing Company Advanced Book Program, Redwood City, CA, 1989. An introduction to algebraic geometry; Notes written with the collaboration of Richard Weiss; Reprint of 1969 original. MR **1042981 (90k:**14023) ↑2

[Kob84] Neal Koblitz, *p-adic numbers, p-adic analysis, and zeta-functions*, 2nd ed., Graduate Texts in Mathematics, vol. 58, Springer-Verlag, New York, 1984.MR754003 (86c:11086) ↑2

[Ser73] J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York, 1973. Translated from the French; Graduate Texts in Mathematics, No. 7. MR 0344216 (49 #8956) ↑2

[Sil92] Joseph H. Silverman, *The arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 106, Springer-Verlag, New York, 1992. Corrected reprint of the 1986 original.MR 95m:11054 ↑2

[ST92] Joseph H. Silverman and John Tate, *Rational points on elliptic curves*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992.MR 93g:11003 ↑2

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