CHARACTERIZING INTEGERS AMONG RATIONAL NUMBERS WITH A UNIVERSAL-EXISTENTIAL FORMULA

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ABSTRACT. We prove that $\mathbb{Z}$ in definable in $\mathbb{Q}$ by a formula with 2 universal quantifiers followed by 7 existential quantifiers. It follows that there is no algorithm for deciding, given an algebraic family of $\mathbb{Q}$-morphisms, whether there exists one that is surjective on rational points. We also give a formula, again with universal quantifiers followed by existential quantifiers, that in any number field defines the ring of integers.

1. INTRODUCTION

1.1. **Background.** D. Hilbert, in the 10th of his famous list of 23 problems, asked for an algorithm for deciding the solvability of any multivariable polynomial equation in integers. Thanks to the work of M. Davis, H. Putnam, J. Robinson [DPR61], and Y. Matijasevič [Mat70], we know that no such algorithm exists. In other words, the positive existential theory of the integer ring $\mathbb{Z}$ is undecidable.

It is not known whether there exists an algorithm for the analogous problem with $\mathbb{Z}$ replaced by the field $\mathbb{Q}$ of rational numbers. But Robinson showed that the full first-order theory of $\mathbb{Q}$ is undecidable: she reduced the problem to the corresponding known result for $\mathbb{Z}$ by showing that $\mathbb{Z}$ could be defined in $\mathbb{Q}$ by a first-order formula [Rob49, Theorem 3.1]. If there were a positive existential formula defining $\mathbb{Z}$ in $\mathbb{Q}$, then an easy reduction from $\mathbb{Q}$ to $\mathbb{Z}$ would show that Hilbert's 10th problem over $\mathbb{Q}$ would have a negative answer.

G. Cornelissen and K. Zahidi [CZ06] ask:

(1) What is the smallest part of the first-order theory of $\mathbb{Q}$ that can be proved undecidable?
(2) How complicated must a formula defining $\mathbb{Z}$ in $\mathbb{Q}$ be?

To make these questions precise, they define the *positive arithmetical hierarchy* as follows: $\Sigma_0^+ = \Pi_0^+$ is the set of atomic formulas (which, in the language of rings, are polynomial equations), and for $n \in \mathbb{Z}_{\geq 0}$, inductively define Σ_{n+1}^+ as the set of formulas consisting of any number of existential quantifiers followed by a formula in Π_n^+, and Π_{n+1}^+ as the set of formulas consisting of any number of universal quantifiers followed by a formula in Σ_n^+. Thus, for instance, positive existential formulas are equivalent to those in Σ_1^+, and the formula

$$(\forall x_1 \forall x_2 \exists y \forall z_1 \forall z_2)\ x_1^2 w + x_2^3 y - x_2 z_1 = x_1 z_2 + w^7$$

is a Π_3^+-formula with one free variable, w.

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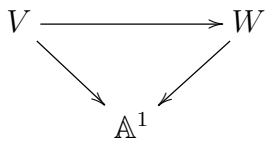
As remarked in [CZ06], Robinson's definition of $\mathbb{Z}$ in $\mathbb{Q}$ uses a Π_4^+-formula, and it follows that the Σ_5^+-theory of $\mathbb{Q}$ is undecidable. Theorems 4.2 and 5.3 of [CZ06] show that a conjecture about elliptic curves implies that $\mathbb{Z}$ is definable in $\mathbb{Q}$ by a Π_2^+-formula, and that the Π_2^+-theory of $\mathbb{Q}$ is undecidable, even if one allows only formulas with a single universal quantifier.

1.2. **Our results.** We prove unconditionally that $\mathbb{Z}$ is definable in $\mathbb{Q}$ by a Π_2^+-formula. Combining this with the negative answer to Hilbert's tenth problem over $\mathbb{Z}$ shows that the Σ_3^+-theory of $\mathbb{Q}$ is undecidable. Our proof uses not elliptic curves, but quaternion algebras.

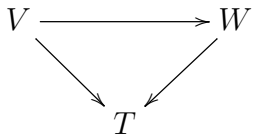
These results may be restated in geometric terms. By $\mathbb{Q}$-*variety* we mean a separated scheme of finite type over $\mathbb{Q}$. Given a $\mathbb{Q}$-morphism $\pi\colon V \to T$ and $t \in T(\mathbb{Q})$, let $V_t = \pi^{-1}(t)$ be the fiber. Then

(a) There exists a diagram of $\mathbb{Q}$-varieties



such that $\mathbb{Z}$ equals the set of $t \in \mathbb{Q} = \mathbb{A}^1(\mathbb{Q})$ such that $V_t(\mathbb{Q}) \to W_t(\mathbb{Q})$ is surjective. (In fact, we may take $W = \mathbb{A}^3$, and take its map to $\mathbb{A}^1$ to be a coordinate projection.)

(b) There is no algorithm that takes as input a diagram of $\mathbb{Q}$-varieties



and decides whether or not there exists $t \in T(\mathbb{Q})$ such that $V_t(\mathbb{Q}) \to W_t(\mathbb{Q})$ is surjective.

In the final section of the paper, we generalize to number fields: we find a Π_2^+-formula that in every number field k defines its ring of integers $\mathcal{O}_k$.

2. Quaternion algebras

We use a quaternion algebra argument similar to that in the proof of [Eis05, Theorem 3.1]. Let $\mathcal{P} = \{2, 3, 5, \ldots\}$ be the set of prime numbers. Given $a, b \in \mathbb{Q}^\times$, let $H_{a,b}$ be the quaternion algebra over $\mathbb{Q}$ generated by i and j satisfying $i^2 = a$, $j^2 = b$, and $ij = -ji$. Let $\Delta_{a,b}$ be the set of $p \in \mathcal{P}$ that ramify in $H_{a,b}$. Let $S_{a,b}$ be the set of reduced traces of elements of $H_{a,b}$ of reduced norm 1. For $p \in \mathcal{P}$, define $S_{a,b}(\mathbb{Q}_p)$ similarly for $H_{a,b} \otimes \mathbb{Q}_p$. For any prime power q, let U_q be the set of $s \in \mathbb{F}_q$ such that $x^2 - sx + 1$ is irreducible in $\mathbb{F}_q[x]$. Let $\mathbb{Z}_p$ be the ring of p-adic integers, and let $\mathrm{red}_p\colon \mathbb{Z}_p \to \mathbb{F}_p$ be the reduction map.

Lemma 2.1.

(i) *If* $p \notin \Delta_{a,b}$, *then* $S_{a,b}(\mathbb{Q}_p) = \mathbb{Q}_p$.
(ii) *If* $p \in \Delta_{a,b}$, *then* $\mathrm{red}_p^{-1}(U_p) \subseteq S_{a,b}(\mathbb{Q}_p) \subseteq \mathbb{Z}_p$.

Proof. We have $s \in S_{a,b}(\mathbb{Q}_p)$ if and only if $x^2 - sx + 1$ is the reduced characteristic polynomial of an element of $H_{a,b} \otimes \mathbb{Q}_p$.

(i) If $p \notin \Delta_{a,b}$, then $H_{a,b} \otimes \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)$, and any monic quadratic polynomial is a characteristic polynomial.

(ii) Now suppose that $p \in \Delta_{a,b}$. Then $H_{a,b} \otimes \mathbb{Q}_p$ is the ramified quaternion algebra over $\mathbb{Q}_p$, and $x^2 - sx + 1$ is a reduced characteristic polynomial if and only if it is a power of a monic irreducible polynomial in $\mathbb{Q}_p[x]$. If $\mathrm{red}_p(s) \in U_p$, then $x^2 - sx + 1$ is irreducible over $\mathbb{Q}_p$. If $s \in \mathbb{Q}_p - \mathbb{Z}_p$, then $x^2 - sx + 1$ is a product of two distinct factors, by the theory of Newton polygons. □

Lemma 2.2. *If $a, b \in \mathbb{Q}^\times$ and either $a > 0$ or $b > 0$, then $S_{a,b} = \mathbb{Q} \cap \bigcap_p S_{a,b}(\mathbb{Q}_p)$.*

Proof. This is a special case of the Hasse principle for rational numbers represented by quadratic forms: see [Ser73, p. 43, Corollary 1], for example. □

Lemma 2.3. *For any prime power q, the set U_q is nonempty. If $q > 11$ then $U_q + U_q = \mathbb{F}_q$.*

Proof. We have $U_q = \mathrm{Tr}\,(\{\beta \in \mathbb{F}_{q^2} - \mathbb{F}_q : N(\beta) = 1\})$, where Tr and N are the trace and norm for $\mathbb{F}_{q^2}/\mathbb{F}_q$. Since $\mathbb{F}_{q^2}$ contains $q+1$ norm-1 elements, $U_q \neq \emptyset$. Also, $-U_q = U_q$, so $0 \in U_q + U_q$. Given $a \in \mathbb{F}_q^\times$ with $q > 11$, we hope to prove $a \in U_q + U_q$.

Suppose that q is odd. Write $\mathbb{F}_{q^2} = \mathbb{F}_q(\sqrt{c})$ with $c \in \mathbb{F}_q^\times - \mathbb{F}_q^{\times 2}$. Then $U_q = \{2x : x, y \in \mathbb{F}_q \text{ and } x^2 - cy^2 = 1\}$. So $a \in U_q + U_q$ if and only if there exist $x_1, y_1, x_2, y_2 \in \mathbb{F}_q$ satisfying

$$x_1^2 - cy_1^2 = 1, \quad x_2^2 - cy_2^2 = 1, \quad 2x_1 + 2x_2 = a, \quad y_1, y_2 \neq 0.$$

These conditions define a smooth curve X in $\mathbb{A}^4$. Eliminating x_2 shows that the projective closure $\overline{X}$ of X is a geometrically integral intersection of two quadrics in $\mathbb{P}^3$, with function field $\mathbb{F}_q(x_1)(\sqrt{c(1-x_1^2)}, \sqrt{c(1-(a/2-x_1)^2)})$. So X is of genus 1 with at most 12 punctures (the intersections of $\overline{X}$ with three hyperplanes: $y_1 = 0$, $y_2 = 0$, and the one at infinity).

If instead q is even, $\mathbb{F}_{q^2} = \mathbb{F}_q(\gamma)$ where $\gamma^2 + \gamma + c = 0$ for some $c \in \mathbb{F}_q$, and we seek an $\mathbb{F}_q$-point on the curve X defined by

$$x_1^2 + x_1y_1 + cy_1^2 = 1, \quad x_2^2 + x_2y_2 + cy_2^2 = 1, \quad y_1 + y_2 = a, \quad y_1, y_2 \neq 0.$$

The geometric properties of X are the same as in the odd q case.

For any $q \geq 23$, the Hasse bound yields

$$\#X(\mathbb{F}_q) \geq (q + 1 - 2\sqrt{q}) - 12 > 0,$$

so $a \in U_q + U_q$. If $11 < q < 23$ we check $U_q + U_q = \mathbb{F}_q$ by exhaustion. □

Remark 2.4. A further calculation shows that the only prime power q less than or equal to 11 for which $U_q + U_q = \mathbb{F}_q$ holds is 9.

Let $N = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310$. Let $T_{a,b}$ be the set of rational numbers of the form $s + s' + n$ where $s, s' \in S_{a,b}$ and $n \in \{0, 1, 2, \ldots, N-1\}$.

Lemma 2.5. *If $a, b \in \mathbb{Q}^\times$ and either $a > 0$ or $b > 0$, then $T_{a,b} = \bigcap_{p \in \Delta_{a,b}} \mathbb{Z}_{(p)}$.*

Proof. Let $T'_{a,b}$ be the right-hand side. Lemmas 2.1 and 2.2 imply $S_{a,b} \subseteq T'_{a,b}$, so $T_{a,b} \subseteq T'_{a,b}$.

Now suppose $t \in T'_{a,b}$. Choose $n \in \{0, 1, 2, \ldots, N-1\}$ such that $\mathrm{red}_p(t-n) \in U_p + U_p$ for all $p \leq 11$. For each $p > 11$, Lemma 2.3 yields $\mathrm{red}_p(t-n) \in U_p + U_p$. So we may choose $s \in \mathbb{Z}$ such that $\mathrm{red}_p(s), \mathrm{red}_p(t-n-s) \in U_p$ for all $p \in \Delta_{a,b}$. Now $s, t-n-s \in S_{a,b}$ by Lemmas 2.1 and 2.2. So $t \in T_{a,b}$. □

Remark 2.6. It follows that the set of $(a,b,c) \in \mathbb{Q}^\times \times \mathbb{Q}^\times \times \mathbb{Q}$ such that at least one of a and b is positive and such that c is integral at all primes ramifying in $H_{a,b}$ is diophantine over $\mathbb{Q}$. This adds to the toolbox that might someday be useful for a negative answer to Hilbert's Tenth Problem over $\mathbb{Q}$. Given a prime p, it is possible to choose $a,b,a',b' \in \mathbb{Q}_{>0}$ with $\Delta_{a,b} \cap \Delta_{a',b'} = \{p\}$, so that $T_{a,b} + T_{a',b'} = \mathbb{Z}_{(p)}$; thus we also quickly recover the well-known fact that $\mathbb{Z}_{(p)}$ is diophantine over $\mathbb{Q}$.

Lemma 2.7. *We have* $\bigcap_{a,b\in\mathbb{Q}_{>0}} T_{a,b} = \mathbb{Z}$.

Proof. By Lemma 2.5, it suffices to show that for each $p \in \mathcal{P}$ there exist $a,b \in \mathbb{Q}_{>0}$ such that $H_{a,b}$ is ramified at p. If $p = 2$, take $a = b = 7$. If $p > 2$, take $a = p$ and choose $b \in \mathbb{Z}_{>0}$ with $\mathrm{red}_p(b) \in \mathbb{F}_p^\times - \mathbb{F}_p^{\times 2}$. □

3. Definition of $\mathbb{Z}$

Theorem 3.1. *The set* $\mathbb{Z}$ *equals the set of* $t \in \mathbb{Q}$ *for which the following* Π_2^+*-formula is true over* $\mathbb{Q}$*:*

$$\begin{aligned}(\forall a,b)(\exists a_1,a_2,a_3,a_4,b_1,b_2,b_3,b_4,x_1,x_2,x_3,x_4,y_1,y_2,y_3,y_4,n)\\ (a+a_1^2+a_2^2+a_3^2+a_4^2)(b+b_1^2+b_2^2+b_3^2+b_4^2)\\ \cdot\big[(x_1^2-ax_2^2-bx_3^2+abx_4^2-1)^2+(y_1^2-ay_2^2-by_3^2+aby_4^2-1)^2\\ +n^2(n-1)^2\cdots(n-2309)^2+(2x_1+2y_1+n-t)^2\big]=0.\end{aligned}$$

Proof. The set of a for which there exist $a_1,\ldots,a_4$ such that $a + a_1^2 + a_2^2 + a_3^2 + a_4^2 = 0$ are those satisfying $a \le 0$. Thus removing this factor and the corresponding factor for b is equivalent to restricting the domain of a,b to $\mathbb{Q}_{>0}$. Now the theorem follows directly from Lemma 2.7. □

4. Reducing the number of quantifiers

The formula in Theorem 3.1 contains 2 universal quantifiers followed by 17 existential quantifiers. We do not see how to reduce the number of universal quantifiers. But we can reduce the number of existential quantifiers:

Theorem 4.1. *It is possible to define* $\mathbb{Z}$ *in* $\mathbb{Q}$ *with a* Π_2^+*-formula with* 2 *universal quantifiers followed by* 7 *existential quantifiers.*

Proof of Theorem 4.1. Starting with the formula in Theorem 3.1, we may replace each a_i and b_i by the corresponding x_i, and hence reuse the variables x_i (we thank Pace Nielsen for this idea, which greatly simplified our original proof). Next we solve $2x_1 + 2y_1 + n - t = 0$ for y_1 to eliminate y_1, and we clear denominators. Finally, the quantifier for n is unnecessary because n takes on only finitely many values. The resulting formula is

$$\begin{aligned}(\forall a,b)(\exists x_1,x_2,x_3,x_4,y_2,y_3,y_4)\\ (a+x_1^2+x_2^2+x_3^2+x_4^2)(b+x_1^2+x_2^2+x_3^2+x_4^2)\\ \cdot\left[\big(x_1^2-ax_2^2-bx_3^2+abx_4^2-1\big)^2+\prod_{n=0}^{2309}\big((n-t-2x_1)^2-4ay_2^2-4by_3^2+4aby_4^2-4\big)^2\right]=0.\end{aligned}$$

□

Remark 4.2. Here we show that if $f(t), g(t) \in \mathbb{Q}(t)$ are rational functions, then the intersection of $T_{f(a),g(a)}$ over all $a \in \mathbb{Q}$ such that $f(a)$ and $g(a)$ are nonzero and not both negative is always much larger than $\mathbb{Z}$; this foils one possible approach to defining $\mathbb{Z}$ in $\mathbb{Q}$ using just one universal quantifier.

Tsen's theorem implies that the quaternion algebra $H_{f(t),g(t)}$ over $\mathbb{Q}(t)$ is split by $\overline{\mathbb{Q}}(t)$ and hence by $k(t)$ for some number field k, and hence by $\mathbb{Q}_p(t)$ for any prime p splitting in k; for such p, we have that $H_{f(a),g(a)} \otimes \mathbb{Q}_p$ is split for all $a \in \mathbb{Q}$ such that $f(a)$ and $g(a)$ are defined and nonzero. Hence any rational number whose denominator is divisible only by primes splitting in k will be in the intersection of the sets $T_{f(a),g(a)}$ mentioned above.

We can also give a new proof of the following result, which was first proved by G. Cornelissen and A. Shlapentokh [CS07].

Theorem 4.3 (Cornelissen and Shlapentokh)**.** *For every $\epsilon > 0$, there is a set R of primes of natural density at least $1 - \epsilon$ such that $\mathbb{Z}$ is definable in $\mathbb{Z}[R^{-1}]$ using a Π_2^+-formula with just* one *universal quantifier (instead of two).*

Proof. Given ϵ, choose a positive integer m such that $2^{-m} < \epsilon$, and let B be the set of the first m primes. Let R be the set of $p \in \mathcal{P}$ that fail to split in $\mathbb{Q}(\sqrt{b})$ for at least one $b \in B$. The density of R equals $1 - 2^{-m} > 1 - \epsilon$.

For fixed $b > 0$, the set $\bigcup_{a \in \mathbb{Z}[R^{-1}]} \Delta_{a,b}$ equals the set of primes that fail to split in $\mathbb{Q}(\sqrt{b})$, so $\bigcap_{a \in \mathbb{Z}[R^{-1}]} T_{a,b} = \mathbb{Z}[S_b^{-1}]$, where S_b is the set of primes that split in $\mathbb{Q}(\sqrt{b})$. Thus

$$\mathbb{Z}[R^{-1}] \cap \bigcap_{b \in B} \bigcap_{a \in \mathbb{Z}[S^{-1}]} T_{a,b} = \mathbb{Z}.$$

The set on the left is definable in $\mathbb{Z}[R^{-1}]$ by a Π_2^+-formula, since positive existential formulas over $\mathbb{Q}$ may be modeled by equivalent positive existential formulas over $\mathbb{Z}[R^{-1}]$. Moreover, only one universal quantifier (for a) is needed, since b ranges over only finitely many variables. □

Remark 4.4. The proof of Theorem 4.3 shows also that for every $\epsilon > 0$, there is a subset $S \subset \mathcal{P}$ of density less than ϵ (namely, $\bigcap_{b \in B} S_b$) such that $\mathbb{Z}[S^{-1}]$ is definable in $\mathbb{Q}$ by a Π_2^+-formula with just one universal quantifier.

5. Defining rings of integers

Theorem 5.1. *There is a Π_2^+-formula that in any number field k defines its ring of integers.*

Proof. Let ℓ be a prime greater than 11 such that $(\ell - 1)/2$ also is prime, e.g., $\ell = 23$. Let ζ_ℓ be a primitive ℓ^{th} root of 1 in an algebraic closure of k. We will break into cases according to which subfields of $\mathbb{Q}(\zeta_\ell)$ are contained in k. Let $\ell^* = (-1)^{(\ell-1)/2}\ell$. Let $f(x) \in \mathbb{Z}[x]$ be the minimal polynomial of $\zeta_\ell + \zeta_\ell^{-1}$ over $\mathbb{Q}$.

Case 1: k contains a zero of f and a zero of $x^2 - \ell^*$. Then $k \supseteq \mathbb{Q}(\zeta_\ell)$, so the residue field at every prime of k not above ℓ contains a primitive ℓ^{th} root of 1. In particular, every residue field is an $\mathbb{F}_q$ with $q > 11$, so Lemma 2.3 always applies. Also k has no real places. Thus the argument of Section 2 shows that for any $a, b \in k^\times$, the analogously defined $T_{a,b}$ (without the n) equals the set of elements of k that are integral at every prime ramifying in $H_{a,b}$. We can require $a, b \in k^\times$ by adding an equation $abc - 1 = 0$. We may combine equations over

k by observing that if $P_n(z_1, \ldots, z_n)$ is the norm form for a degree-n extension of k, then $P_n(z_1, \ldots, z_n) = 0$ is equivalent to $z_1 = \cdots = z_n = 0$. So $\mathcal{O}_k$ is defined in k by the following formula Φ:

$$(\forall a, b)(\exists c, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)$$
$$P_4(abc - 1, x_1^2 - ax_2^2 - bx_3^2 + abx_4^2 - 1, y_1^2 - ay_2^2 - by_3^2 + aby_4^2 - 1, 2x_1 + 2y_1 - t) = 0.$$

This is not uniform in k, because the norm form must depend on k, but by quantifying over all possible coefficients for the norm form, we can replace Φ by a uniform formula.

Case 2: k contains a zero of f but not a zero of $x^2 - \ell^*$. By using Case 1 and a Π_2^+-analogue of the "Going up and then down" method [Shl07, Lemma 2.1.17] (i.e., modeling a formula over $k' := k[x]/(x^2 - \ell^*)$ by a formula over k, by restriction of scalars), we find a Π_2^+-formula Ψ defining $\mathcal{O}_k$ in k.

Cases 1 and 2: For any number field k containing a zero of f, we use

$$(((\exists u)\ u^2 - \ell^* = 0) \wedge \Phi)\ \vee\ (((\forall v)(\exists w)\ w(v^2 - \ell^*) = 1) \wedge \Psi),$$

which, when written in positive prenex form, is a Π_2^+-formula, by [CZ06, Lemma 1.20.1]. (To replace a conjunction of polynomial equations by a single polynomial equation, we again use a norm form, and we quantify over its coefficients to make it uniform in k.)

The same approach of dividing into two cases lets us generalize to include the case where k does not contain a zero of f: in this case, f is irreducible over k since $\mathbb{Q}(\zeta_\ell + \zeta_\ell^{-1})$ is abelian of prime degree over $\mathbb{Q}$. □

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References

[CS07] Gunther Cornelissen and Alexandra Shlapentokh, *Defining the integers in large subrings of number fields using one universal quantifier*, 2007. in preparation. ↑4

[CZ06] Gunther Cornelissen and Karim Zahidi, *Elliptic divisibility sequences and undecidable problems about rational points* (June 23, 2006). Preprint, to appear in *J. reine angew. Math.* ↑1.1, 1.1, 5

[DPR61] Martin Davis, Hilary Putnam, and Julia Robinson, *The decision problem for exponential diophantine equations*, Ann. of Math. (2) **74** (1961), 425–436. MR0133227 (24 #A3061) ↑1.1

[Eis05] Kirsten Eisenträger, *Integrality at a prime for global fields and the perfect closure of global fields of characteristic $p > 2$*, J. Number Theory **114** (2005), no. 1, 170–181. MR2163911 (2006f:11150) ↑2

[Mat70] Yu. Matiyasevich, *The Diophantineness of enumerable sets*, Dokl. Akad. Nauk SSSR **191** (1970), 279–282 (Russian). MR0258744 (41 #3390) ↑1.1

[Rob49] Julia Robinson, *Definability and decision problems in arithmetic*, J. Symbolic Logic **14** (1949), 98–114. MR0031446 (11,151f) ↑1.1

[Ser73] J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York, 1973. Translated from the French; Graduate Texts in Mathematics, No. 7. MR0344216 (49 #8956) ↑2

[Shl07] Alexandra Shlapentokh, *Hilbert's tenth problem. Diophantine classes and extensions to global fields*, New Mathematical Monographs, vol. 7, Cambridge University Press, Cambridge, 2007. MR2297245 ↑5

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