REAL REPRESENTATIONS

BJORN POONEN

The goal of these notes is to explain the classification of real representations of a finite group. Throughout, $G$ is a finite group, $W$ is a $\mathbb{R}$-vector space or $\mathbb{R}G$-module, and $V$ is a $\mathbb{C}$-vector space or $\mathbb{C}G$-module (except in Section 2 where $V$ is over any field). Vector spaces and representations are assumed to be finite-dimensional.

1. Vector spaces over $\mathbb{R}$ and $\mathbb{C}$

1.1. Constructions. To get from $\mathbb{R}^n$ to $\mathbb{C}^n$, we can tensor with $\mathbb{C}$. In a more coordinate-free manner, if $W$ is an $\mathbb{R}$-vector space, then its complexification $W_\mathbb{C} := W \otimes_\mathbb{R} \mathbb{C}$ is a $\mathbb{C}$-vector space. We can view $W$ as an $\mathbb{R}$-subspace of $W_\mathbb{C}$ by identifying each $w \in W$ with $w \otimes 1 \in W_\mathbb{C}$. Then an $\mathbb{R}$-basis of $W$ is also a $\mathbb{C}$-basis of $W_\mathbb{C}$. In particular, $W_\mathbb{C}$ has the same dimension as $W$ (but is a vector space over a different field).

Conversely, we can view $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ if we forget how to multiply by complex scalars that are not real. In a more coordinate-free manner, if $V$ is a $\mathbb{C}$-vector space, then its restriction of scalars is the $\mathbb{R}$-vector space $\mathbb{R}V$ with the same underlying abelian group but with only scalar multiplication by real numbers. If $v_1, \ldots, v_n$ is a $\mathbb{C}$-basis of $V$, then $v_1, iv_1, \ldots, v_n, iv_n$ is an $\mathbb{R}$-basis of $\mathbb{R}V$. In particular, $\dim (\mathbb{R}V) = 2 \dim V$.

Also, if $V$ is a $\mathbb{C}$-vector space, then the complex conjugate vector space $\bar{V}$ has the same underlying group but a new scalar multiplication $\cdot$ defined by $\lambda \cdot v := \bar{\lambda}v$, where $\bar{\lambda}v$ is defined using the original scalar multiplication.

Complexification and restriction of scalars are not inverse constructions. Instead:

**Proposition 1.1** (Complexification and restriction of scalars).

(a) If $V$ is a $\mathbb{C}$-vector space, then the map

\[(\mathbb{R}V)_\mathbb{C} \longrightarrow V \oplus \bar{V}
\[v \otimes c \longmapsto (cv, \bar{cv})\]

is an isomorphism of $\mathbb{C}$-vector spaces.

(b) If $W$ is an $\mathbb{R}$-vector space, then

\[\mathbb{R}(W_\mathbb{C}) \cong W \oplus W.\]

*Proof.*

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(a) The map is $\mathbb{C}$-linear, by definition of the scalar multiplication on $\mathbb{V}$. It sends $x \otimes 1 + y \otimes i$ to $(x + iy, x - iy)$, and one can recover $x, y \in V$ uniquely from $(x + iy, x - iy)$, so the map is an isomorphism.

(b) We have $R(W \otimes \mathbb{C}) = W \otimes (\mathbb{R} \oplus i\mathbb{R}) = W \oplus iW \cong W \oplus W$. $\square$

1.2. Linear maps between complexifications. Tensoring $M_{m,n}(\mathbb{R})$ with $\mathbb{C}$ yields $M_{m,n}(\mathbb{C})$. The coordinate-free version of this is:

**Proposition 1.2.** If $W$ and $X$ are $\mathbb{R}$-vector spaces, then

$$\text{Hom}_{\mathbb{R}}(W, X) \otimes \mathbb{C} \cong \text{Hom}_{\mathbb{C}}(W_{\mathbb{C}}, X_{\mathbb{C}}).$$

**Corollary 1.3.** If $W$ is an $\mathbb{R}$-vector space, then

$$\text{End}_{\mathbb{R}}(W) \otimes \mathbb{C} \cong \text{End}_{\mathbb{C}}(W_{\mathbb{C}}).$$

1.3. Descent theory. Let $V$ and $X$ be $\mathbb{C}$-vector spaces. A homomorphism $J : V \to X$ of abelian groups is called $\mathbb{C}$-antilinear if $J(\lambda v) = \overline{\lambda} J(v)$ for all $\lambda \in \mathbb{C}$ and $v \in V$; to give such a $J$ is equivalent to giving a $\mathbb{C}$-linear map $V \to X$.

To recover $\mathbb{R}^n$ from its complexification $\mathbb{C}^n$ one takes the vectors fixed by coordinate-wise complex conjugation. More generally, given a $\mathbb{C}$-vector space $V$, finding a $\mathbb{R}$-vector space $W$ such that $W_{\mathbb{C}} \cong V$ is equivalent to finding a “complex conjugation” on $V$; more precisely:

**Proposition 1.4.** There is an equivalence of categories

\[ \{\mathbb{R}\text{-vector spaces}\} \leftrightarrow \{\mathbb{C}\text{-vector spaces equipped with $\mathbb{C}$-antilinear $J : V \to V$ such that $J^2 = 1$}\} \]

\[ V^J := \{v \in V : Jv = v\} \leftrightarrow (V, J). \]

**Sketch of proof.** The only tricky part is to show that given $(V, J)$, the map $V^J \otimes \mathbb{C} \to V$ sending $v \otimes c$ to $cv$ is an isomorphism. For this, one can write down the inverse: map $v \in V$ to $\frac{1}{2}(v + Jv) \otimes 1 + \frac{1}{2}(v - Jv) \otimes i \in V^J \otimes \mathbb{C}$. $\square$

**Remark 1.5.** More generally, given any Galois extension of fields $L/k$, an action of $\text{Gal}(L/k)$ on an $L$-vector space $V$ is called semilinear if scalar multiplication is compatible with the actions of $\text{Gal}(L/k)$ on $L$ and $V$, that is, if $g(\ell v) = (\ell g)(gv)$ for all $g \in \text{Gal}(L/k)$, $\ell \in L$ and $v \in V$. Then the category of $k$-vector spaces is equivalent to the category of $L$-vector spaces equipped with a semilinear $\text{Gal}(L/k)$-action. This is called descent, since it specifies what extra structure is needed on an $L$-vector space to make it “descend” to a $k$-vector space.

1.4. Representations. All the constructions and propositions above are natural. In particular, if $G$ acts on $W$, then it acts on any of the spaces constructed from $W$, and likewise for $V$. In particular,
If $W$ is an $\mathbb{R}G$-module, then $W_C$ is a $\mathbb{C}G$-module, and the matrix of $g \in G$ acting on $W$ with respect to a basis is the same as the matrix of $g$ acting on $W_C$, so $\chi_{W_C} = \chi_W$.

If $V$ is a $\mathbb{C}G$-module, then $\overline{V}$ is another $\mathbb{C}G$-module, and $\chi_{\overline{V}} = \overline{\chi_V}$.

If $V$ is a $\mathbb{C}G$-module, then $\mathbb{R}V$ is an $\mathbb{R}G$-module. Taking the characters of both sides in Proposition 1.1 shows that $\chi_{\mathbb{R}V} = \chi_V + \overline{\chi_V}$.

A $\mathbb{C}$-representation $V$ of $G$ is said to be realizable over $\mathbb{R}$ if $V \simeq W_C$ for some $\mathbb{R}$-representation $W$ of $G$. This implies that $\chi_V$ is real-valued, but we will see that the converse can fail.

2. **Pairings**

2.1. **Bilinear forms.** Let $V$ be a (finite-dimensional) vector space over any field $k$. A function $B : V \times V \to k$ is bi-additive if it is an additive homomorphism in each argument when the other is fixed; that is, $B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$ for all $v_1, v_2, w \in V$, and $B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2)$ for all $v, w_1, w_2 \in V$. The left kernel of $B$ is $\{v \in V : B(v, w) = 0 \text{ for all } w \in V\}$, and the right kernel is defined similarly.

A function $B : V \times V \to k$ is a bilinear form (or bilinear pairing) if it is $k$-linear in each argument; that is, $B$ is bi-additive and $B(\lambda v, w) = \lambda B(v, w)$ and $B(v, \lambda w) = \lambda B(v, w)$ for all $\lambda \in k$ and $v, w \in V$. We have

$$\{\text{bilinear forms on } V\} \simeq \text{Hom}(V \otimes V, k) \simeq (V \otimes V)^* \simeq V^* \otimes V^* \simeq \text{Hom}(V, V^*).$$

(here Hom is $\text{Hom}_k$, and $\otimes$ is $\otimes_k$).

Let $B$ be a bilinear form.

- Call $B$ symmetric if $B(v, w) = B(w, v)$ for all $v, w \in V$.
- Call $B$ skew-symmetric if $B(v, w) = -B(w, v)$ for all $v, w \in V$.
- Call $B$ alternating if $B(v, v) = 0$ for all $v \in V$.

If $\text{char } k \neq 2$, then alternating and skew-symmetric are equivalent. (If $\text{char } k = 2$, then alternating is the stronger and better-behaved condition.) The map sending $B$ to the pairing $(x, y) \mapsto B(y, x)$ is a linear automorphism of order 2 of the space of bilinear forms, so if $\text{char } k \neq 2$, it decomposes the space into $+1$ and $-1$ eigenspaces:

$$\{\text{bilinear forms}\} = \{\text{symmetric bilinear forms}\} \oplus \{\text{skew-symmetric bilinear forms}\},$$

which is the same as the decomposition

$$(V \otimes V)^* \simeq (\text{Sym}^2 V)^* \oplus (\wedge^2 V)^*.$$

2.2. **Sesquilinear and hermitian forms.** Now let $V$ be a $\mathbb{C}$-vector space.

- A sesquilinear form (or sesquilinear pairing) is a bi-additive pairing $(\cdot, \cdot)$ that is $\mathbb{C}$-linear in the first variable and $\mathbb{C}$-antilinear in the second variable; that is $(\lambda v, w) = \lambda(v, w)$
and \((v, \lambda w) = \bar{\lambda}(v, w)\) for all \(\lambda \in \mathbb{C}\) and \(v, w \in V\). (The prefix “sesqui” means \(1\frac{1}{2}\): the form is only \(\mathbb{R}\)-linear in the second argument.)

- A **hermitian form** (or **hermitian pairing**) is a bi-additive pairing \((\ , \ )\) such that \((\lambda v, w) = \lambda(v, w)\) and \((w, v) = \overline{(v, w)}\) for all \(\lambda \in \mathbb{C}\) and \(v, w \in V\).

A hermitian pairing is sesquilinear. We have

\[
\{\text{sesquilinear forms on } V\} \simeq \text{Hom}(V \otimes \overline{V}, \mathbb{C}) \simeq (V \otimes V)^* \simeq V^* \otimes \overline{V}^* \simeq \text{Hom}(V, V^*).
\]

### 2.3. Nondegenerate and positive definite forms

A bilinear form (or sesquilinear form) is called **nondegenerate** if its left kernel is 0, or equivalently its right kernel is 0, or equivalently the associated homomorphism \(V \to V^*\) (respectively, \(\overline{V} \to V^*\)) is an isomorphism.

Suppose that \((\ , \ )\) is either a bilinear form on an \(\mathbb{R}\)-vector space or a hermitian form on a \(\mathbb{C}\)-vector space. Then \((v, v) \in \mathbb{R}\) for all \(v\). Call \((\ , \ )\) **positive definite** if \((v, v) > 0\) for all nonzero \(v \in V\). Positive definite forms are automatically nondegenerate.

### 3. Characters of symmetric and alternating squares

Let \(V\) be an \(n\)-dimensional \(\mathbb{C}\)-representation of \(G\). If \(g \in G\) acts on \(V\) with eigenvalues \(\lambda_1, \ldots, \lambda_n\) (listed with multiplicity), then the eigenvalues of \(g\) acting on associated vector spaces are as follows:

<table>
<thead>
<tr>
<th>Representation</th>
<th>Dimension</th>
<th>Eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td>(V)</td>
<td>(n)</td>
<td>(\lambda_1, \ldots, \lambda_n)</td>
</tr>
<tr>
<td>(\overline{V})</td>
<td>(n)</td>
<td>(\overline{\lambda_1}, \ldots, \overline{\lambda_n})</td>
</tr>
<tr>
<td>(V^*)</td>
<td>(n)</td>
<td>(\overline{\lambda_1}, \ldots, \overline{\lambda_n})</td>
</tr>
<tr>
<td>(V \otimes V)</td>
<td>(n^2)</td>
<td>(\lambda_i \lambda_j) for all ((i, j))</td>
</tr>
<tr>
<td>(\text{Sym}^2 V)</td>
<td>(n(n+1)/2)</td>
<td>(\lambda_i \lambda_j) for (i \leq j)</td>
</tr>
<tr>
<td>(\wedge^2 V)</td>
<td>(n(n-1)/2)</td>
<td>(\lambda_i \lambda_j) for (i &lt; j)</td>
</tr>
</tbody>
</table>

These are obvious if \(V\) has a basis of eigenvectors (i.e., \(\rho(g)\) is diagonalizable). In general, we have the Jordan decomposition \(\rho(g) = d + n\), where \(d\) is diagonalizable and \(n\) is nilpotent, and \(dn = nd\); then \(d\) and \(n\) induce commuting diagonalizable endomorphisms and nilpotent endomorphisms of each of the other representations, so the eigenvalues of \(g\) are the same as the eigenvalues of \(d\) on each of them.

### 4. Classification of division algebras over \(\mathbb{R}\)

**Lemma 4.1.** The only finite-dimensional field extensions of \(\mathbb{R}\) are \(\mathbb{R}\) and \(\mathbb{C}\).

**Proof.** The fundamental theorem of algebra states that \(\mathbb{C}\) is algebraically closed, so every finite extension of \(\mathbb{R}\) embeds in \(\mathbb{C}\). Since \([\mathbb{C} : \mathbb{R}] = 2\), there is no room for other fields in between.

\qed
**Theorem 4.2** (Frobenius 1877). *The only finite-dimensional (associative) division algebras over \(\mathbb{R}\) are \(\mathbb{R}\), \(\mathbb{C}\), and \(\mathbb{H}\).*

**Proof.** Let \(D\) be a finite-dimensional (associative) division algebras over \(\mathbb{R}\) not equal to \(\mathbb{R}\) or \(\mathbb{C}\). For any \(d \in D - \mathbb{R}\), the \(\mathbb{R}\)-subalgebra \(\mathbb{R}[d] \subseteq D\) generated by \(d\) is a commutative domain of finite dimension over a field, so it is a field extension of finite degree over \(\mathbb{R}\), hence a copy of \(\mathbb{C}\). Fix one such copy, and let \(i\) be a \(\sqrt{-1}\) in it. View \(D\) as a left \(\mathbb{C}\)-vector space. Conjugation by \(i\) on \(D\) (the map \(x \mapsto ix^{-1}\)) is a \(\mathbb{C}\)-linear automorphism of \(D\), and it is of order 2 since conjugation by \(i^2 = -1\) is the identity, so it decomposes \(D\) into \(+1\) and \(-1\) eigenspaces \(D^+\) and \(D^-\). Explicitly,

\[
D^+ = \{ x : ix^{-1} = x \} = \{ x \text{ that commute with } i \} \supseteq \mathbb{C}
\]

\[
D^- = \{ x : ix^{-1} = -x \}.
\]

If \(x \in D^+\), then \(\mathbb{C}[x]\) is commutative, hence a finite field extension of \(\mathbb{C}\), but \(\mathbb{C}\) is algebraically closed, so \(\mathbb{C}[x] = \mathbb{C}\), so \(x \in \mathbb{C}\). Thus \(D^+ = \mathbb{C}\).

Since \(D \neq \mathbb{C}\), we have \(D^- \neq 0\). Choose \(j \in D^-\) such that \(j \neq 0\). Right multiplication by \(j\) defines a \(\mathbb{C}\)-linear map \(D^+ \to D^-\) (if \(d \in D^+\), then \(i(dj)^{-1} = (idi^{-1})(ij^{-1}) = d(-j) = -dj\), so \(dj \in D^-\)), and it is injective since \(D\) is a division algebra. Thus \(\dim_{\mathbb{C}} D^- \leq \dim_{\mathbb{C}} D^+ = 1\). Hence \(D^- = \mathbb{C}j\). Since \(\mathbb{R}[j]\) is another copy of \(\mathbb{C}\), we have \(j^2 \in \mathbb{R} + \mathbb{R}j\). On the other hand \(j^2 \in D^+ = \mathbb{C}\). Thus \(j^2 \in (\mathbb{R} + \mathbb{R}j) \cap \mathbb{C}\), which is \(\mathbb{R}\), since \(\mathbb{R} + \mathbb{R}j\) and \(\mathbb{C}\) are different 2-dimensional subspaces in \(D\). Also, \(j^2 \neq 0\).

If \(j^2 > 0\), then \(j^2 = r^2\) for some \(r \in \mathbb{R}\), so \((j+r)(j-r) = 0\), so \(j = \pm r \in \mathbb{R}\), a contradiction since \(D^- \cap \mathbb{R} = 0\).

Thus \(j^2 < 0\). Scale \(j\) to assume that \(j^2 = -1\). Then \(D = \mathbb{C} + \mathbb{C}j = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}ij\) with \(i^2 = -1\), \(j^2 = -1\), and \(ij = -ji\), so \(D \simeq \mathbb{H}\). \(\square\)

If \(D\) is an \(\mathbb{R}\)-algebra, then \(D \otimes_{\mathbb{R}} \mathbb{C}\) is a \(\mathbb{C}\)-algebra.

**Proposition 4.3.** We have

\[
\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}
\]

\[
\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}
\]

\[
\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq \text{M}_2(\mathbb{C}).
\]

**Proof.** The first isomorphism is a special case of the general isomorphism \(A \otimes_A B \simeq B\).

The map \(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \times \mathbb{C}\) sending \(a \otimes b\) to \((ab, \hat{ab})\) is an isomorphism by Proposition [1.1] and it respects multiplication.

There is a \(\mathbb{C}\)-algebra homomorphism \(\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to \text{M}_2(\mathbb{C})\) sending \(h \otimes 1\) for each \(h \in \mathbb{H}\) to the linear endomorphism \(x \mapsto hx\) of the 2-dimensional right \(\mathbb{C}\)-vector space \(\mathbb{H}\) with basis 1, \(j\).
Explicitly, we have
\[
1 \otimes 1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
\[
i \otimes 1 \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\]
\[
j \otimes 1 \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]
\[
ij \otimes 1 \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.
\]

For example, to get the image of \(i \otimes 1\), observe that
\[
i 1 = 1 \cdot 1 + j \cdot 0
\]
\[
ij = 1 \cdot 0 + j \cdot (-i).
\]

The four matrices on the right are linearly independent over \(\mathbb{C}\), so \(\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to M_2(\mathbb{C})\) is an isomorphism of 4-dimensional \(\mathbb{C}\)-algebras.  

5. Real and complex representations

Let \(G\) be a finite group. Let \(W\) be an irreducible \(\mathbb{R}\)-representation of \(G\). Let \(V\) be one irreducible \(\mathbb{C}\)-subrepresentation of \(W_C\). The following table gives facts about this situation.

<table>
<thead>
<tr>
<th>(D)</th>
<th>(\text{End}_G(W_C))</th>
<th>(W_C)</th>
<th>(\mathbb{R}V)</th>
<th>(\dim_{\mathbb{R}} W)</th>
<th>(\dim_{\mathbb{C}} V)</th>
<th>(V) realiz. over (\mathbb{R})?</th>
<th>(V \simeq \overline{V}) real-valued?</th>
<th>(V \simeq V^*) (\exists\ G\text{-inv. } B)?</th>
<th>(\text{FS}(V))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\mathbb{R})</td>
<td>(\mathbb{C})</td>
<td>(V)</td>
<td>(W \oplus W)</td>
<td>(n)</td>
<td>(n)</td>
<td>YES</td>
<td>YES</td>
<td>YES (symmetric)</td>
<td>1</td>
</tr>
<tr>
<td>(\mathbb{C})</td>
<td>(\mathbb{C} \times \mathbb{C})</td>
<td>(V \oplus \overline{V})</td>
<td>(W)</td>
<td>(2n)</td>
<td>(n)</td>
<td>NO</td>
<td>NO</td>
<td>NO</td>
<td>0</td>
</tr>
<tr>
<td>(\mathbb{H})</td>
<td>(\text{M}_2(\mathbb{C}))</td>
<td>(V \oplus V)</td>
<td>(W)</td>
<td>(4n)</td>
<td>(2n)</td>
<td>NO</td>
<td>YES</td>
<td>YES (skew-sym.)</td>
<td>-1</td>
</tr>
</tbody>
</table>

The columns are as follows:
- First, \(D := \text{End}_G W\). By Schur’s lemma, \(D\) is a division algebra over \(\mathbb{R}\), so \(D\) is \(\mathbb{R}\), \(\mathbb{C}\), or \(\mathbb{H}\). Accordingly, \(V\) is said to be of real type, complex type, or quaternionic type. Let \(n\) be the dimension of \(W\) as a right \(D\)-vector space.
- We have \(\text{End}_G(W_C) \simeq (\text{End}_G W) \otimes_{\mathbb{R}} \mathbb{C} = D \otimes_{\mathbb{R}} \mathbb{C}\) by taking \(G\)-invariants in Corollary 1.3.
- The \(W_C\) column gives the decomposition of \(W_C\) into irreducible \(\mathbb{C}\)-representations.
- The \(\mathbb{R}V\) column gives the decomposition of \(\mathbb{R}V\) into irreducible \(\mathbb{R}\)-representations.
- The \(\dim_{\mathbb{R}} W\) column gives \(\dim_{\mathbb{R}} W = [D : \mathbb{R}] \dim_D W = [D : \mathbb{R}] n\).
• The \( \dim_{\mathbb{C}} V \) column entries follow from the \( W_{\mathbb{C}} \) column and the column giving \( \dim_{\mathbb{R}} W = \dim_{\mathbb{C}} (W_{\mathbb{C}}) \).

• Is \( V \) realizable over \( \mathbb{R} \)? That is, is \( V \cong X_{\mathbb{C}} \) for some \( \mathbb{R} \)-representation \( X \) of \( G \)?

• Is \( V \cong V^* \) as a \( \mathbb{C} \)-representation of \( G \)? Equivalently, is \( \chi_V = \bar{\chi}_V \)? That is, is it true that \( \chi_V (g) \in \mathbb{R} \) for all \( g \in G \)?

• Is \( V \cong V^* \) as a \( \mathbb{C} \)-representation of \( G \)? Since

\[
\text{Hom}(V, V^*) \cong \{ \text{bilinear forms on } V \}
\]

as a \( \mathbb{C} \)-representation of \( G \), and since isomorphisms correspond to nondegenerate bilinear forms, taking \( G \)-invariants shows that this question is the same as asking whether there exists a nondegenerate \( G \)-invariant bilinear form \( B : V \times V \to \mathbb{C} \). We will show that in the cases where \( B \) exists, \( B \) is either symmetric or skew-symmetric.

• The Frobenius–Schur indicator of a \( \mathbb{C} \)-representation \( V \) of \( G \) is defined by

\[
\text{FS}(V) := \frac{1}{\#G} \sum_{g \in G} \chi_V (g^2).
\]

Proof that the table is correct. Some columns have already been checked above. Let us now verify the rest.

\( W_{\mathbb{C}} \) column: In general, if \( V_1, \ldots, V_r \) are the irreducible \( \mathbb{C} \)-representations of \( G \), and \( X = \bigoplus_{i=1}^r n_i V_i \), then \( \text{End}_{\mathbb{C}} X = \prod_{i=1}^r M_{n_i}(\mathbb{C}) \). Thus the \( \text{End}_{\mathbb{C}} (W_{\mathbb{C}}) \) column implies the \( W_{\mathbb{C}} \) column, except that in the \( \mathbb{C} \) case, we deduce only that \( W_{\mathbb{C}} \cong V \oplus V' \) for some distinct \( \mathbb{C} \)-representations \( V \) and \( V' \). In that case, \( W \) has an action of \( D = \mathbb{C} \), and hence \( W \cong \mathbb{R} \mathcal{W} \) for some \( \mathbb{C} \)-vector space \( \mathcal{W} \); then \( W_{\mathbb{C}} = (\mathbb{R} \mathcal{W})_{\mathbb{C}} \cong \mathcal{W} + \overline{\mathcal{W}} \), but then the Jordan–Hölder theorem implies that \( V, V' \) must be \( \mathcal{W}, \overline{\mathcal{W}} \) in some order, so \( V' \cong \overline{V} \).

\( \chi_V \) real-valued column: In the \( \mathbb{R} \) case, \( \chi_V = \chi_{W_{\mathbb{C}}} = \chi_W \), which is real-valued. In the \( \mathbb{C} \) case, \( V \not\cong \overline{V} \), so \( \chi_V \) is not real-valued. In the \( \mathbb{H} \) case, \( 2\chi_V = \chi_{V \oplus V^*} = \chi_{W_{\mathbb{C}}} = \chi_W \), so \( \chi_V \) is real-valued.

\( \mathbb{R}V \) column: Since \( V \) is a subrepresentation of \( W_{\mathbb{C}} \), the restriction of scalars \( \mathbb{R}V \) is a subrepresentation of \( \mathbb{R}(W_{\mathbb{C}}) \), which is isomorphic to \( W \oplus W \) by Proposition 1.1(b). Thus \( \mathbb{R}V \) is a direct sum of copies of \( W \). If \( D = \mathbb{R} \), then \( V = W_{\mathbb{C}} \), so \( \mathbb{R}V \cong W \oplus W \). If \( D \) is \( \mathbb{C} \) or \( \mathbb{H} \), then \( V \) is half the dimension of \( W_{\mathbb{C}} \), so \( V \cong W \).

Realizability over \( \mathbb{R} \): In the \( \mathbb{R} \) case, \( V \cong W_{\mathbb{C}} \), so \( V \) is realizable by definition. In the \( \mathbb{C} \) and \( \mathbb{H} \) cases, if \( V \cong X_{\mathbb{C}} \) for some \( \mathbb{R} \)-representation \( X \), then \( W \cong \mathbb{R}V \cong \mathbb{R}(X_{\mathbb{C}}) \cong X \oplus X \) by Proposition 1.1(b), contradicting the irreducibility of \( W \).

Nondegenerate \( G \)-invariant hermitian form: The averaging argument shows that there exists a positive definite \( G \)-invariant hermitian form \( ( , ) \) on \( V \). Fix one; it defines an isomorphism \( \overline{V} \to V^* \). Thus \( V \cong \overline{V} \) if and only if \( V \cong V^* \), so these two columns have the same YES/NO
answers. By Section 2.1 we have isomorphisms
\[
\text{Hom}(V, V^*) \simeq \{\text{symmetric bilinear forms}\} \oplus \{\text{skew-symmetric bilinear forms}\}.
\]

Taking $G$-invariants yields
\[
\text{Hom}_G(V, V^*) \simeq \{G\text{-invariant symm. bilinear forms}\} \oplus \{G\text{-invariant skew-symm. bilinear forms}\}.
\]

Suppose that $V \simeq V^*$. Then $\text{Hom}_G(V, V^*) \simeq \text{End}_G V \simeq \mathbb{C}$ by Schur’s lemma, so there exists a unique nondegenerate $G$-invariant bilinear form $B$ up to a scalar in $\mathbb{C}^\times$, and it is either symmetric or skew-symmetric. Since $B$ is nondegenerate, the $\mathbb{C}$-linear functional $(-, w)$ equals $B(-, Jw)$ for a unique $Jw \in V$. Then $J := V \to V$ is $\mathbb{C}$-antilinear, and it is an isomorphism since $(\ , \ )$ too is nondegenerate. Now $J^2$ is a $\mathbb{C}$-linear automorphism of the representation $V$, so by Schur’s lemma, $J^2$ is multiplication-by-$r$ for some $r \in \mathbb{C}^\times$. Also by Schur’s lemma, every other $\mathbb{C}$-antilinear $G$-equivariant isomorphism is $cJ$ for some $c \in \mathbb{C}$, and replacing $J$ by $cJ$ changes $r$ to $c\bar{c}r$ (Proof: For $v \in V$, if $JJv = rv$, then $cJ(cJ(v)) = c\bar{c}J(J(v)) = c\bar{c}rv$).

- If $B$ is symmetric, then for any choice of nonzero $v \in V$,
  \[
  (Jv, Jv) = B(Jv, J^2v) = B(Jv, rv) = rB(Jv, v) = rB(v, Jv) = r(v, v)
  \]
  but $(\ , \ )$ is positive definite, so $r$ is a positive real number.
- If $B$ is skew-symmetric, the same calculation shows that $r$ is a negative real number.

Finally, the following are equivalent:

- $V$ is realizable over $\mathbb{R}$
- We can choose $c \in \mathbb{C}^\times$ so that $(cJ)^2 = 1$.
- We can choose $c \in \mathbb{C}^\times$ so that $c\bar{c}r = 1$.
- $r$ is positive.
- $B$ is symmetric.

**Frobenius–Schur indicator:** We have
\[
\text{FS}(V) = \frac{1}{\# G} \sum_g \chi_V^*(g^2) = \frac{1}{\# G} \sum_g \left(\chi_{(\text{Sym}^2 V)^*}(g) - \chi_{(\wedge^2 V)^*}(g)\right) \text{ (by the formulas in Section 3)}
\]
\[
= (\mathbb{C}, (\text{Sym}^2 V)^*) - (\mathbb{C}, (\wedge^2 V)^*)
\]
\[
= \dim\{G\text{-invariant symm. bilinear forms}\} - \dim\{G\text{-invariant skew-symm. bilinear forms}\}
\]
\[
= \begin{cases} 1 & \text{if } D = \mathbb{R}; \\ 0 & \text{if } D = \mathbb{C}; \\ -1 & \text{if } D = \mathbb{H}. \end{cases}
\]

\[\square\]
Proposition 5.1. Every irreducible \( \mathbb{C} \)-representation \( V \) of \( G \) occurs in \( W_C \) for a unique irreducible \( \mathbb{R} \)-representation \( W \) of \( G \).

Proof. By Proposition 1.1(a), \( V \) occurs in \( (V)_C \), so \( V \) occurs in \( W_C \) for some irreducible \( \mathbb{R} \)-subrepresentation \( W \) of \( V \). If \( W \) is any irreducible \( \mathbb{R} \)-representation such that \( V \) occurs in \( W_C \), then the \( V \) column of the table shows that \( W \) equals the unique irreducible \( \mathbb{R} \)-subrepresentation of \( V \), so \( W \) is uniquely determined by \( V \). \( \square \)

Theorem 5.2 (Frobenius–Schur). We have

\[
\# \{ g \in G : g^2 = 1 \} = \sum_V (\dim V) \text{FS}(V),
\]

where \( V \) ranges over the irreducible \( \mathbb{C} \)-representations of \( G \) up to isomorphism.

Proof. The character of the regular representation \( \mathbb{C}G \) is given by

\[
\chi(g) = \begin{cases} 
\#G, & \text{if } g = 1; \\
0, & \text{if } g \neq 1.
\end{cases}
\]

Thus

\[
\# \{ g \in G : g^2 = 1 \} = \frac{1}{\#G} \sum_g \chi(g^2) = \text{FS}(\mathbb{C}G) = \sum_V (\dim V) \text{FS}(V),
\]

since \( \mathbb{C}G \simeq \bigoplus_V (\dim V)V \). \( \square \)

Remark 5.3. Everything above for finite groups \( G \) holds also for compact groups \( G \). The only changes required are:

- All representations should be given by continuous homomorphisms.
- Averages over \( G \) (such as in the definition of the Frobenius–Schur indicator) should be defined as integrals with respect to normalized Haar measure.

Remark 5.4. Let \( k \) be a field such that \( \text{char } k \nmid \#G \). Let \( X_1, \ldots, X_r \) be the irreducible \( k \)-representations of \( G \). Let \( D_i = \text{End}_G X_i \). Let \( n_i \) be the dimension of \( X_i \) as a right \( D_i \)-vector space. Then

\[
kG \simeq \prod_{i=1}^r \text{End}_{D_i} X_i \\
\simeq \prod_{i=1}^r \text{M}_{n_i}(D_i).
\]
In particular,
\[ \mathbb{R}G \simeq \prod M_{d_i}(\mathbb{R}) \times \prod M_{e_j}(\mathbb{C}) \times \prod M_{f_k}(\mathbb{H}) \]
for some positive integers \(d_i, e_j, f_k\), and tensoring with \( \mathbb{C} \) yields
\[ \mathbb{C}G \simeq \prod M_{d_i}(\mathbb{C}) \times \prod (M_{e_j}(\mathbb{C}) \times M_{e_j}(\mathbb{C})) \times \prod M_{2f_k}(\mathbb{C}). \]

6. Some conclusions to remember

- Every irreducible \( \mathbb{C} \)-representation \( V \) of \( G \) occurs in \( W_{\mathbb{C}} \) for a unique irreducible \( \mathbb{R} \)-representation of \( G \).
- The representation \( V \) is said to be of real, complex, or quaternionic type according to whether \( \text{End}_G W \) is \( \mathbb{R} \), \( \mathbb{C} \), or \( \mathbb{H} \).
- The type can be determined from the character \( \chi_V \) by computing the Frobenius–Schur indicator.
- The representation \( V \) is realizable over \( \mathbb{R} \) if and only if \( V \) is of real type, which happens if and only if there exists a nondegenerate \( G \)-invariant symmetric bilinear form \( B : V \times V \rightarrow \mathbb{C} \).
- The representation \( V \) is of complex type if and only if \( V \not\cong V^* \); in this case, there does not exist any nondegenerate \( G \)-invariant bilinear form \( B : V \times V \rightarrow \mathbb{C} \).
- The representation \( V \) is of quaternionic type if and only if there exists a nondegenerate \( G \)-invariant skew-symmetric bilinear form \( B : V \times V \rightarrow \mathbb{C} \).
- If \( V \) is realizable over \( \mathbb{R} \), then \( \chi_V \) is real-valued. The converse is not true in general (it fails exactly in the quaternionic case).