ERRATA IN DO CARMO, DIFFERENTIAL GEOMETRY OF CURVES AND SURFACES

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This is a list of errata in do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, 1976 (25th printing). The errata were discovered by Bjorn Poonen and some students in his Math 140 class, Spring 2004: Dmitriy Ivanov, Michael Manapat, Gabriel Pretel, Lauren Tompkins, and Po Yee Wong. Some errata were discovered later by Brent Doyle.

- p. 5, line 9, "using properties 3 and 4": Actually, property 2 also is being used, since property 4 gives linearity in the second variable only.
- p. 5, bottom: The definition of the tangent line is confusing. It is not the line passing through the points $\alpha(t)$ and $\alpha'(t)$ but rather the line through $\alpha(t)$ in the direction of $\alpha'(t)$, namely $\{\, \alpha(t) + \lambda\alpha'(t) : \lambda \in \mathbb{R} \,\}$.
- p. 7, Exercise 3, "On $0B$ mark off the segment $0p = CB$." It would be better to say "On $0B$ mark a point p such that $0p = CB$."
- p. 8, Figure 1-8: The labelling is wrong: the points p and C should lie on the same half-line r through 0 as B.
- p. 8, Figure 1-8: It might be good to add a point labelled Y on the y-axis.
- p. 8, Figure 1-9: The angle labelled t is in the wrong place. (See problem 4 on p. 7.)
- p. 13, line 4: Change "It follows from property 4 that the vector product $u \wedge v \neq 0$ is normal to a plane generated by u and v." to "It follows from property 4 that if u and v are linearly independent, then $u \wedge v$ is normal to the plane spanned by u and v." or better yet, "It follows from properties 3 and 4 that if u and v are linearly independent, then $u \wedge v$ is a nonzero vector normal to the plane spanned by u and v."
- p. 13, bottom: Again change "a plane generated by" to "the plane spanned by"
- p. 13, bottom: Change "a parallelogram" to "the parallelogram" (twice). Also it would be better to speak of "the parallelogram formed by u and v" instead of "the parallelogram generated by u and v".
- p. 14, Exercise 5: change "the equation" to "an equation".
- p. 17, line 1, "Therefore, $\alpha''(s)$ and the curvature remain invariant under a change of orientation.": This is not a direct consequence of the previous statement, so replace "Therefore" by "Similarly".
- p. 18, last sentence: It would be better to change "It follows that" to "Similarly,"
- p. 18, last sentence: change "remains" to "remain".
- p. 21: In the right part of Figure 1-16, the labels e_1 and e_2 should be reversed.
- pp. 47–48: Exercise 6 is about a *convex* curve, but the curve in Figure 1-37 is not convex.
- p. 49, Exercise 12: The ratio M_1/M_2 actually equals 1/2. In the solution on p. 480, the inner integral in the definition of M_1 should go from 0 to 1/2, so $M_1 = \pi$.

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- p. 57, Figure 2-5: The angle ϕ should be φ to be consistent with the text.
- p. 59, proof of Proposition 2: change "axis" to "axes".
- p. 59, proof of Proposition 2: change "in $\mathbb{R}^3$ where F takes its values" to "in the image of F".
- p. 61, before Example 3: The notion of "connected" given here is usually called "path connected". (But the definitions are equivalent in the case of regular surfaces, so no harm is done.)
- p. 61, 8th line from bottom: change "by contradiction" to "for sake of obtaining a contradiction".
- p. 70, Proposition 1: There's no need to mention p.
- p. 71, middle, before third display: change "axis" to "axes".
- p. 72, Definition 1: This requires the notion "V is open in S", which is not defined until the appendix to Chapter 5. The definition of that notion should appear earlier in the book.
- p. 72, third line after Definition 1: $p \in \mathbf{x}(V)$ should be $p \in \mathbf{y}(V)$.
- p. 76, Example 4: The definition of regular curve does not force C to be connected, so the assumption that C does not meet the z-axis needs to be replaced by the assumption that C is contained in the open right half of the xz-plane.
- p. 76, Example 4, first display: It is confusing to include the last inequality $f(v) > 0$ here, since it is not part of the parametrization, but rather a hypothesis on f.
- p. 81: Exercise 10 is poorly stated. It is not clear whether p and q are in C. As defined, regular curves are not supposed to have "endpoints". But if p and q are not part of C, and C is contained in the open right half-plane, then we get a regular surface of revolution automatically.
- p. 82: Exercise 15b is wrong as stated, even if one assumes $t_0 = h(\tau_0)$. For instance, if C is a circle, then the left and right hand sides could be the lengths of the major and minor arcs connecting two points, respectively. In other words, h might not be defined on the whole interval $[\tau_0, \tau]$, in which case one cannot perform the substitution to transform one integral into the other.
- p. 85, Figure 2-24: change ϕ to φ (twice).
- p. 88, Exercise 3: "Tangent plane" means two different things in the two parts of this problem. In the first part, it is a plane passing through p_0. In the second part, it is a subspace of $\mathbb{R}^3$, that is, a plane passing through the origin.
- p. 94, Example 3: In the parametrization, there is no need to restrict u to the interval $(0, 2\pi)$.
- p. 97, definition of domain: It is not clear whether the boundary is the boundary as a subset of $\mathbb{R}^3$ or the boundary as a subset of S. Either way, we run into trouble.

  If it is the boundary in $\mathbb{R}^3$, then a region in S need not be contained in S! For instance, if we slice an infinite cylinder in two (with a circular cross section), S could be one of the open halves, so its boundary is the circle, and its closure is a region in S!

  If it is the boundary in S, then just below Figure 2-28 the assumption that R is bounded does not imply that $Q = \mathbf{x}^{-1}(R)$ is bounded, and the integral defining the area could diverge.

Perhaps one should require in the definition of domain that its closure in S be compact?

- p. 99, Exercise 2: Add a comma between φ and θ.
- p. 100, Exercise 8: change "quadratic" to "fundamental".
- p. 109, Exercise 1: One must assume that V_1 and V_2 are connected.
- p. 118, definition of "neighborhood": It is more common to define a neighborhood of p to be any set (open or not) that contains an open set containing p.
- p. 120, sentence after the second display beginning "In other words,": This sentence is incorrect and should be deleted.
- pp. 121–122, proof of Proposition 1: The notation $S_\delta(p)$ should be replaced by $B_\delta(p)$, which was used earlier to denote balls. (This appears in three places.)
- p. 122, end of Example 4: $\mathbb{R}^n$ should be $\mathbb{R}^m$.
- p. 123, definition of "continuous in A": It is more common to define this to mean that for all $a \in A$ and all $\epsilon > 0$, there exists $\delta > 0$ such that the conditions $x \in A$ and $|x - a| < \delta$ imply $|F(x) - F(a) < \epsilon$. This definition agrees with continuity with respect to the subspace topology on A, whereas do Carmo's definition does not. (To see that the definitions do not agree, consider $A = \mathbb{R} - \{1/n : n \geq 1\}$, and define $F \colon A \to \mathbb{R}$ so that $F(x) = 1/n$ if $x \in (1/(n+1), 1/n)$ for some integer $n \geq 1$, and $F(x) = 0$ otherwise. This should be continuous on A, but is not by do Carmo's definition.)
- p. 124, Proposition 5: The statement should begin "Let $f \colon [a, b] \to \mathbb{R}$".
- p. 124, Proposition 6 (Heine-Borel): The set I has not been defined. Also, the I_α need to be open as subsets in $[a, b]$, not open intervals in $\mathbb{R}$ that are contained in $[a, b]$. Since this appendix has not defined the notion of one set being open *in another*, it would be best to restate the result as follows:

  > Let $[a, b]$ be a closed interval, and let I_α, $\alpha \in A$, be a collection of open intervals such that $[a, b] \subseteq \bigcup_\alpha I_\alpha$. Then it is possible to choose a finite number $I_{k_1}, \ldots, I_{k_n}$ of I_α such that $[a, b] \subseteq \bigcup_{i=1}^{n} I_{k_i}$.
- p. 125, line -2: the word "performed" should perhaps be changed to "taken".
- p. 127, Definition 1, "we associate a linear map": It is not proved to be linear until the next Proposition 7. So perhaps it is better to call it just a "map" at this point.
- p. 132, line -3: The second left hand side should have $(0, 1)$ instead of $(1, 0)$.
- p. 136, Definition 1: Delete the $|$ after the semicolon in the display.
- p. 146, 4 lines from the bottom, "any of the sides of the tangent plane": it would better to replace "any" with "either".
- p. 147, just after Definition 8: It would make more sense to refer to Example 5 instead of "the method of Example 6".
- p. 153, second line from the bottom: "parametrization" should be "parametrizations".
- p. 154, line 4, "all functions to appear below denote their values at the point p": Strictly speaking, some of the functions are functions of t and should be evaluated at $t = 0$.
- p. 154, near the bottom: "coefficientes" should be "coefficients".

- p. 155: It may be preferable to write equation (3) in the form

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

(the transpose of the current version of the equation), so that the matrix (a_{ij}) that appears in it is exactly the matrix of dN_p, instead of its transpose.
- p. 155, middle, just before the formula for a_{11}: Each x in $\{x_u, x_v\}$ should be bold.
- p. 161, line 2 of Example 4: change "changed" to "replaced".
- p. 162, middle, "If a parametrization of a regular surface is such that $F = f = 0$, then the principal curvatures are given by e/E and g/G." A more enlightening explanation of this is given by the equation following equation (3) on page 155.
- p. 214, first display: Some indices are backwards (the ij-th entry of a matrix is traditionally the entry in the i-th row and j-th column, and in fact this convention is followed on page 154). Change this line to the following:

$$\alpha_{ij} = \langle Ae_j, e_i \rangle = \langle e_j, Ae_i \rangle = \langle Ae_i, e_j \rangle = \alpha_{ji};$$

- p. 215, Lemma: The letter "a" of the word "at" should be italicized.
- p. 221, top right of Figure 4-2: $d\phi_p(W)$ should be $d\phi_p(w)$.
- pp. 224–225: The function F is being confused with $F \circ \overline{\mathbf{x}}$ (which appears on page 225). For F to be a function from U to $\mathbb{R}^3$, it should be defined as a function of x and y. For $F \circ \overline{\mathbf{x}}$ to make sense, F should be a function of x, y, z. But F was defined as a function of ρ and θ!
- p. 226, display in Definition 3: Since the subscript p is used on the inner product on the right, it would make sense to use the subscript $\varphi(p)$ on the left.
- p. 226, second to last display: The range for θ should be $0 \le \theta \le \pi$.
- p. 227, sentence after Proposition 2: With the given definition of local conformality, it seems to be very difficult to prove that it is a symmetric relation, unless one accepts the unproven theorem on page 227. Also, the sentence as written misleadingly suggests that transitivity is the only property required for an equivalence relation.
- p. 232, end first paragraph: Italicize the first appearance of "Christoffel symbols" instead of the second.
- p. 234, sentence after Theorema Egregium: Presumably the codomain of φ was supposed to be a possibly different surface $\bar{S}$. In this case, the S near the end of this sentence also should be $\bar{S}$.
- p. 238, sentence before Definition 1: Start this sentence with "The vector field" (to avoid starting it with the mathematical symbol w), and change "for every $p \in U$" to "at every $p \in U$".
- p. 239, middle: "Sec. 4-1" should be "Sec. 4-3".
- p. 239: In (1), the functions a and b depend on the curve α, so it should be explained that the values of a' and b' depend on α only through its tangent vector.
- p. 243, Figure 4-12: ϕ should be φ.
- p. 245, Figure 4-14: ϕ should be φ (4 times).
- p. 245, last line of Definition 8: "for all $t \in I$" should be "at all $t \in I$" (to match the previous part of Definition 8).
- p. 246, Example 3: "there exists exactly one geodesic $C \subset S$ passing through p and tangent to this direction". Strictly speaking, any connected open neighborhood of

p in C will be another geodesic, so strictly speaking it is not quite unique. This sloppiness persists throughout the section. For instance:

  – p. 246, end of Example 3: Great circles are not the only geodesics on the sphere: arcs of great circles (including the empty arc!) are also geodesics.
  – p. 247, last paragraph, first sentence: lines are not the only geodesics of the plane (open intervals in lines are also geodesics)
  – p. 260, Exercise 3: Same comment.

- p. 248, Definition 10: change "contained on" to "contained in".
- p. 248, last sentence of Definition 10: "$[D\alpha'(s)/ds] = k_g$" should follow "algebraic value", not "covariant derivative".
- p. 249, Figure 4-18: change ϕ to φ in two places.
- p. 249, Figure 4-18: The formula for $|k_g|$ can be simplified to $|k_g| = |\operatorname{cotan}\varphi|$. (No need for the 1!)
- p. 249, middle: There is no need to mention k_n; Figure 4-18 shows directly that

$$|k_g| = |k\cos\varphi| = \left|\frac{1}{\sin\varphi}\sin(\pi/2-\varphi)\right| = |\operatorname{cotan}\varphi|.$$

- p. 249, last line: change "relatively to any" to "relative to either".
- p. 251, middle, "and observing that $\langle v, \bar{v}\rangle = 0$, $\langle v, v'\rangle = 0$": After these two equations, "$\langle \bar{v}, \bar{v}'\rangle = 0$" should be inserted, since it also is being used.
- p. 253, Proposition 4: change "the angle that $\mathbf{x}_u$ makes with $\alpha'(s)$" to "the angle from $\mathbf{x}_u$ to $\alpha'(s)$".
- p. 255, Proposition 5: The uniqueness statement should be as follows: For each $\epsilon > 0$ for which there exists a parametrized geodesic $\gamma\colon (-\epsilon, \epsilon) \to S$ with $\gamma(0) = p$ and $\gamma'(0) = w$, such a parametrized geodesic is unique. Maybe it would be better to state a result for an arbitrary interval (a, b) with $a < 0 < b$. The parametrized geodesic with minimal a and maximal b (and $\gamma(0) = p$ and $\gamma'(0) = w$) is truly unique, and all others (with $\gamma(0) = p$ and $\gamma'(0) = w$) are restrictions of this one to subintervals of (a, b) containing 0.
- p. 260, Exercise 1b: "nonrectilinear" is not defined. For this exercise (and for the application of this exercise to Exercise 8), I suggest changing this hypothesis to "C does not contain an open segment of a straight line".
- p. 261, Exercise 7(a): The conclusion is false if $\theta = \pi/2$.
- p. 261, Exercise 7(b): It's not clear what "at the points where C meets their axes" is referring to. I suggest changing it to "at the endpoints of the major and minor axes of the ellipse" (and adding the assumption $0 < \theta < \pi/2$ so that this makes sense).
- p. 261, Exercise 10: Since the geodesic curvature could have a sign, it should be made clear that "the curvature of the plane curve" refers to the signed curvature defined on page 21, using the orientation of $T_p(S)$ coming from the orientation of S that is implicitly used in defining the geodesic curvature. Or else the problem should discuss only the absolute value of the geodesic curvature.
- p. 262, Exercise 17: The first conclusion is false: It can happen that for all $\epsilon > 0$, the set $\mathbf{x}(I \times (-\epsilon, \epsilon))$ fails to be a regular surface. (Consider a curve $\alpha\colon (0,1) \to \mathbb{R}^3$ such that $\alpha(s)$ approaches $(0,0,0)$ from the same direction as $s \to 0^+$ or $s \to 1^-$, and such that the part of α near $s = 0$ is contained in a plane, and the part of α near $s = 1$ is contained in a different plane.)

- p. 265, Figure 4-23: Change ϕ to φ (3 times).
- p. 266, paragraph beginning "Assume now...": One should allow the possibility $|\theta_i| = 0$, which is possible even if α fails to be differentiable (C^∞) at t_i.
- p. 266, last sentence: This is false; ϵ' need not exist. For example, consider the piecewise regular curve in $\mathbb{R}^2$ given by

  $$\alpha(t) := \begin{cases} (-t, 0), & \text{if } t \leq 0 \\ (t, e^{-1/t}(2 + \sin(3/t))), & \text{if } t > 0 \end{cases}$$

  The y-coordinate of $\alpha'(t)$ changes sign infinitely often as $t \to 0^+$.

  Instead one should select $\theta = \pi$ or $\theta = -\pi$ as follows. Using a parametrization, reduce to the case where α is contained in $\mathbb{R}^2$, with $\alpha(t_i) = 0$, and $\alpha'(t_i - 0)$ on the negative x-axis (so $\alpha'(t_i + 0)$ on the positive x-axis). For small enough $\epsilon > 0$, the trace of α restricted to $(t_i - \epsilon, t_i)$ is the graph of a function $f\colon (0, \epsilon') \to \mathbb{R}$ and the trace of α restricted to $(t_i, t_i + \epsilon)$ is the graph of a function $g\colon (0, \epsilon'') \to \mathbb{R}$. Because α has no self-crossings, the Intermediate Value Theorem implies that either $f(x) > g(x)$ for all x for which both are defined, or $f(x) < g(x)$ for all x for which both are defined. In the first case, define $\theta = \pi$; in the second, define $\theta = -\pi$.
- p. 267, 7 lines from the bottom: It is claimed that a proof of the Theorem of Turning Tangents is contained in Section 5-7, but there the theorem is proved only in the case where α is a plane curve.
- p. 268, line 3: The I here (the domain of β) is different from the I on the previous page (the domain of α).
- p. 271, Figure 4-28: Change ϕ to φ.
- p. 276, statement of Jordan curve theorem before application 1: Insert the word "closed" before "piecewise regular curve".
- p. 277, application 3: At the end of the first paragraph of the proof, it is claimed that "$\varphi(\Gamma)$ is the boundary of a simple region of P". This is not necessarily the case, since φ is only a homeomorphism, not a diffeomorphism, and the definition of simple region requires a piecewise C^∞ boundary.
- p. 277, just below Figure 4-33: The interior of $\varphi(\Gamma')$ is *not* homeomorphic to a cylinder as claimed. Instead one should apply the global Gauss-Bonnet theorem to the region R between Γ and Γ' in S.
- p. 279, line 9: Even if $K \neq 0$ on T, it is not necessarily the case that $\iint_T K\, d\sigma$ equals the area of $N(T)$, because the sign could be wrong, and because the area of $N(T)$ could be less than expected if N maps different subregions of T onto the same set (in this case, $\iint_T K\, d\sigma$ counts that part of the area of $N(T)$ with multiplicity). The statement should be "If $K \neq 0$ on T and the restriction $N|_T$ of N to T is injective (or at least these two conditions hold after deleting a subset of measure zero from T), then $\left|\iint_T K\, d\sigma\right|$ equals the area of $N(T)$." Perhaps it would be best to state this for arbitrary regions instead of just triangles, since this is needed to do Exercise 3 on p. 282.
- p. 280, middle of last paragraph: Change "(a ring)" to "(an annulus)".
- p. 282, last sentence before exercises: Maybe it would be nice to mention that the famous theorem regarding the impossibility of combing a hairy sphere has just been proven.

- p. 282, Exercise 1: One must assume in addition that S is connected. (Otherwise a disjoint union of two spheres is a counterexample.)
- p. 282, Exercise 3: One must assume that S has $K > 0$ everywhere and that N is injective, so that the area of $N(A)$ can be computed without worrying about sign or multiplicities.
- p. 282, last line (line 2 of Exercise 5): "$p \notin C$" should be "$p \in C$".
- p. 282, Exercise 5: φ is being used to denote two different things in this problem (the colatitude, and the angle between two vector fields along C).
- p. 425, Definition 1, part 2: There is no need to exclude the case $W = \emptyset$; such an exclusion only clutters the definition. I suggest replacing

  > "For each pair α, β with $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) = W \neq \phi$, we have that"

  with

  > "For each pair α, β, if we set $W = \mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta)$, then".

  Exactly the same comment applies to Definition 1a on p. 438.
- p. 426, top of Figure 5-46: The subscript on the second U should be β, not p.
- p. 426, Remark 1: The statement

  > "This means that any other family satisfying conditions 1 and 2 is already contained in the family $\{U_\alpha, \mathbf{x}_\alpha\}$."

  neglects the possibility of distinct differentiable structures on the same set. It should be replaced by

  > "This means that the family $\{U_\alpha, \mathbf{x}_\alpha\}$ is not properly contained in any other family of coordinate charts satisfying conditions 1 and 2 of Definition 1."
- p. 426, Definition 2: This definition is faulty unless one makes the maximality assumption of Remark 1 on this page. (Without this assumption, it might be impossible to find $\mathbf{x}$ satisfying $\mathbf{x}(U) \subset \mathbf{y}(V)$.)
- p. 427, examples 1 and 2: Change ϕ to $\emptyset$ (once in each example). Also, it would be easier to read if $A \circ \mathbf{x}_\alpha(U_\alpha)$ were replaced by $A(\mathbf{x}_\alpha(U_\alpha))$ (once in each example).
- p. 428, middle of first paragraph: Change

  > "Thus, we must define what is the tangent vector of a curve on an abstract surface."

  to

  > "Thus, we must define what the tangent vector of a curve on an abstract surface is."
- p. 443, Exercise 3: To make sense of the notion of neighborhood of a point in an abstract surface, one needs a notion of open set. I suggest adding the following definition to the text, perhaps after Remark 1 on p. 426: A subset V of an abstract surface S is *open* if and only if $\mathbf{x}_\alpha^{-1}(V)$ is open in U_α (or equivalently, in $\mathbb{R}^2$) for all α.
- p. 444, Exercise 5b: "$d\varphi_p = 0$" should be "$d\varphi_p$ is not an isomorphism".

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