(1) (5 pts. each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. (Please do not use the abbreviations T and F, since in handwriting they are sometimes indistinguishable.) No explanations are required in this problem.

(a) There exist integers \(x\) and \(y\) satisfying \(709x + 100y = 4\).
   TRUE, since \(\gcd(709, 100) = 1\), which divides 4. (The \(\gcd\) can be computed using the Euclidean algorithm, or by observing that the only prime factors of 100 are 2 and 5, neither of which divides 709.)

(b) Starting from any two distinct points in the plane, it is possible to construct a regular 7-gon using straightedge and compass.
   FALSE, since 7 is not a Fermat prime. (Alternatively, if a regular 7-gon were constructible, it would follow that \(e^{2\pi i/7}\) would be an algebraic number of 2-power degree, whereas in fact it has degree 6, being a zero of the irreducible polynomial \(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1\).)

(c) Starting from two points in the plane 1 unit apart, it is possible to construct a circle of radius \(\sqrt{\sqrt{2} + \sqrt{3}}\) using straightedge and compass.
   TRUE, since \(\sqrt{\sqrt{2} + \sqrt{3}}\) is a constructible number.

(d) The equation \(x^2 - 9y^2 = 1\) has infinitely many integer solutions.
   FALSE. (It’s not Pell’s equation, since 9 is a square.) The solutions have \(x + 3y = x - 3y = \pm 1\). This gives two linear systems, each of which gives one solution.

(2) (30 pts.) Prove that there are infinitely many primes congruent to 2 modulo 3.

Suppose there were only finitely many primes congruent to 2 modulo 3. Let \(p_1, \ldots, p_s\) be all of them. Let \(n = 3p_1 \cdots p_s - 1\). Since \(n \equiv -1 \equiv 2 \pmod{3}\), 3 does not divide \(n\), and not all prime factors of \(n\) are 1 mod 3. So we can pick a prime factor \(q\) of \(n\) that is 2 mod 3. Since \(p_1, \ldots, p_s\) are all primes congruent to 2 mod 3, we have \(q = p_i\) for some \(i\). Then \(q\) divides \(3p_1 \cdots p_s = n + 1\) as well as \(n\), so \(q\) divides 1, which is impossible. This contradiction proves that there are infinitely many primes congruent to 2 modulo 3.

(3) (30 pts.) Either find a parametrization of the rational solutions to \(x^2 + 2y^2 = 7\), or prove that no rational solutions exist.

We will prove that no rational solutions exist. Suppose that \((x, y)\) is a rational solution. Let \(c \in \mathbb{Z}_{>0}\) be the least common denominator for \(x\) and \(y\). Thus \(x = a/c\) and \(y = b/c\) for some \(a, b \in \mathbb{Z}\). Moreover, \(\gcd(a, b, c) = 1\), since otherwise dividing \(a, b, c\) by the \(\gcd\) shows that \(c\) is not the least common denominator.

Substituting \(x = a/c\) and \(y = b/c\) into \(x^2 + 2y^2 = 7\) and multiplying by \(c^2\) gives \(a^2 + 2b^2 = 7c^2\). Thus \(a^2 + 2b^2 \equiv 0 \pmod{7}\).

If \(b \equiv 0 \pmod{7}\), then \(a^2 \equiv -2b^2 \equiv 0 \pmod{7}\), so 7 divides \(a^2\), so 7 divides \(a\), so 7 divides \(a^2\) and \(b^2\), so 7 divides \(a^2 + 2b^2 = 7c^2\), so 7 divides \(c^2\), so 7 divides \(c\), so 7 is a common factor of \(a, b, c\), contradicting \(\gcd(a, b, c) = 1\).
If \( b \not\equiv 0 \pmod{7} \), then (denoting by \( \bar{z} \) the image of \( z \in \mathbb{Z} \) in the field \( \mathbb{F}_7 = \mathbb{Z}/7\mathbb{Z} \)) we get \((\bar{a}/\bar{b})^2 + 2 = 0\) in \( \mathbb{F}_7 \), so \(-2\) is a square in \( \mathbb{F}_7 \). But the squares in \( \mathbb{F}_7 \) are \( 0^2 = 0, \ 1^2 = 1, \ 2^2 = 4, \ 3^2 = 2 \) (and then \( 4^2 = (-3)^2 = 2 \) and so on). So this case gives a contradiction too.

(4) (20 pts.) Show that for any nonzero real number \( a \), the projective closure of the plane curve \( y^2 = ax^3 \) is projectively equivalent to the projective closure of the curve \( y = x^3 \).

The projective closures are given in homogeneous coordinates by \( Y^2Z = aX^3 \) and \( YZ^2 = X^3 \). The invertible linear substitution \((X, Y, Z) \mapsto (\sqrt[3]{a}X, Z, Y)\) transforms the second equation into the first.