8.3.2) Suppose the triangles were in the same plane \( \Pi' \). Then \( \Pi' \) would contain \( A_1, C_1, A_2, C_2 \). The plane \( \Pi \) is a plane containing these points, and it is the only one because the lines \( A_1C_1 \) and \( A_2C_2 \) are distinct in the hypotheses of Desargues’ theorem. Thus \( \Pi' = \Pi \), and this plane contains \( B_1 \) and \( D_1 \), so it contains \( P \), which lies on the line \( B_1D_1 \). This contradicts the assumption that \( P \) is not on \( \Pi \).

Points \( A_1, A_2, O \) are collinear, points \( C_1, C_2, O \) are collinear, and points \( D_1, D_2, O \) are collinear. This is what it means for triangles \( A_1C_1D_1 \) and \( A_2C_2D_2 \) to be in perspective from \( O \).

8.3.3) Since the projection from \( P \) maps \( A_1, D_1, A_2, D_2 \) to \( A_1, B_1, A_2, B_2 \), respectively, the intersection of \( A_1D_1 \) and \( A_2D_2 \) is projected to the intersection of \( A_1B_1 \) and \( A_2B_2 \). The other two statements are similar.

The first three intersection points lie on a line (by the non-planar Desargues’ theorem), so their projections from \( P \) lie on the projection of this line.

8.7.1) Let \( f(x, y) = 0 \) and \( g(x, y) = 0 \) be the two degree \( n \) curves, and let \( c(x, y) = 0 \) be the degree \( m \) curve. Let \( P \) be a point on \( c(x, y) = 0 \) not equal to one of the \( nm \) intersection points of \( f = 0 \) and \( g = 0 \) (this is possible since \( c(x, y) = 0 \) is not degenerate). Choose \( \alpha, \beta \in \mathbb{R} \) not both zero such that \( \alpha f(P) + \beta g(P) = 0 \). Then the curve \( \alpha f(x, y) + \beta g(x, y) = 0 \) of degree \( \leq n \) intersects \( c(x, y) = 0 \) in at least \( nm + 1 \) points. This contradicts Bézout’s Theorem unless \( \alpha f(x, y) + \beta g(x, y) \) and \( c(x, y) \) have a common factor. Since \( c(x, y) \) is irreducible, we get

\[
\alpha f(x, y) + \beta g(x, y) = c(x, y)p(x, y)
\]

for some polynomial \( p(x, y) = 0 \) of degree at most \( n - m \). Of the \( n^2 \) points where \( f = g = 0 \), only \( nm \) lie on \( c(x, y) = 0 \) (if there were more, then it would contradict Bézout’s Theorem applied to \( f = c = 0 \) or \( g = c = 0 \), since \( c = 0 \) cannot be a factor of both \( f \) and \( g \)). The other \( n^2 - nm = n(n - m) \) points where \( f = g = 0 \) satisfy \( \alpha f(x, y) + \beta g(x, y) \) but \( c(x, y) \neq 0 \), so by the factorization above, they must satisfy \( p(x, y) = 0 \).

9.4.1)(first part) It suffices to prove

\[
\sin x = 2^n \cos \frac{x}{2^n} \cos \frac{x}{4} \cdots \cos \frac{x}{2^n} \sin \frac{x}{2^n}
\]

since then we can divide by \( 2^n \sin(x/2^n) \). We prove the formula by induction on \( n \). The case \( n = 1 \) is the given identity \( \sin x = 2 \cos \frac{x}{2} \sin \frac{x}{2} \). Suppose the formula is true for \( n \). Then we apply the identity \( \sin y = 2 \cos \frac{y}{2} \sin \frac{y}{2} \) with \( y = x/2^n \) and substitute to get the result for \( n + 1 \).

9.4.1)(second part) By definition of the derivative (or by L’Hôpital’s Rule), \( \lim_{t \to 0} \frac{\sin t}{t} = \cos 0 = 1 \), so \( \lim_{n \to \infty} \frac{\sin t_n}{t_n} = 1 \) for any sequence \( t_n \) tending to 0. Apply this to \( t_n = x/2^n \) to
get \( \lim_{n \to \infty} \frac{\sin(x/2^n)}{x/2^n} = 1 \). Take reciprocals and divide by \( x \) to get

\[
\lim_{n \to \infty} \frac{1}{2^n \sin(x/2^n)} = \frac{1}{x}.
\]

for any \( x \neq 0 \). Plugging this into the first part of 9.4.1 gives the desired formula, since the infinite product on the right hand side also is defined as a limit.

9.4.2) Taking \( x = \pi/2 \), we get \( \frac{\sin x}{x} = \frac{1}{\pi/2} = \frac{2}{\pi} \) on the left, and

\[
\cos \frac{\pi}{4} \cos \frac{\pi}{8} \cos \frac{\pi}{16} \cdots
\]

on the right. We have \( \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \) and the successive cosines can be gotten by iterating the half-angle formula \( \cos \frac{t}{2} = \sqrt{\frac{1}{2} (1 + \cos t)} \):

\[
\begin{align*}
\cos \frac{\pi}{8} &= \sqrt{\frac{1}{2} (1 + \frac{\sqrt{2}}{2})} \\
\cos \frac{\pi}{16} &= \sqrt{\frac{1}{2} \left(1 + \sqrt{\frac{1}{2} \left(1 + \frac{\sqrt{2}}{2}\right)}\right)} \\
&\quad \vdots
\end{align*}
\]

9.5.3) Taking \( a = -t^2 \) and \( p = -1/2 \) in the binomial theorem on page 158, we find that the \( t^{2n} \) term is

\[
\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{1}{2}-(n-1)\right)}{n!}(-t^2)^n
\]

so the coefficient of \( t^{2n} \) is

\[
\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\cdots\left(\frac{2n-1}{2}\right)}{n!} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}
\]

(the minus signs cancel) This gives the expansion

\[
\frac{1}{\sqrt{1-t^2}} = 1 + \frac{1}{2} t^2 + \frac{1}{2 \cdot 4} t^4 + \frac{1}{2 \cdot 4 \cdot 6} t^6 + \cdots.
\]

9.5.3) Just integrate term by term.

1a) If \( f \in P_d \), then \( \deg(f(x) - f(x-1)) < d \), since the \( x^d \) terms in the expansions of \( f(x) \) and \( f(x-1) \), if any, cancel. Thus \( \phi \) is well-defined.

If \( f, g \in P_d \), then \( \phi(f+g) = (f+g)(x)-(f+g)(x-1) = f(x)+g(x)-(f(x-1)+g(x-1)) = (f(x)-f(x-1))+(g(x)-g(x-1)) = \phi(f)+\phi(g) \). If \( f \in P_d \) and \( \lambda \in \mathbb{R} \), then \( \phi(\lambda f) \) is \( (\lambda f)(x)-(\lambda f)(x-1) = \lambda f(x)-\lambda f(x-1) = \lambda (f(x)-f(x-1)) = \lambda \phi(f) \). Thus \( \phi \) is linear.

1b) The kernel of \( \phi \) consists of polynomials \( f(x) \) (of degree \( \leq d \)) such that \( f(x)-f(x-1) = 0 \). For such a polynomial \( f(x) = f(x-1) \), so \( f(0) = f(-1) = f(-2) = \cdots \). In particular, \( f(x) - f(0) \) is a polynomial taking the value 0 at 0, -1, -2, \cdots, but a nonzero polynomial could have only finitely many zeros, so \( f(x) - f(0) \) is the zero polynomial. Thus \( f(x) = f(0) \) as polynomials in \( x \), so \( f(x) \) is a constant. Conversely, any constant polynomial is obviously in the kernel.
1c) We have \( \dim P_d = d + 1 \), because \( 1, x, x^2, \ldots, x^d \) is a basis. We have \( \dim \ker \phi + \dim \im \phi = \dim P_d \). This gives \( 1 + \dim \im \phi = d + 1 \), so \( \dim \im \phi = d = \dim P_{d-1} \). Thus \( \im \phi = P_{d-1} \). So \( \phi \) is surjective.

1d) Choose \( d \) such that \( g \in P_{d-1} \). By 1c, there exists \( f \in P_d \) with \( \phi(f) = g \), and we may adjust \( f \) by a constant in order to assume \( f(0) = 0 \). Now

\[
g(1) + g(2) + \cdots + g(n) = (f(1) - f(0)) + (f(2) - f(1)) + \cdots (f(n) - f(n-1)) \\
= -f(0) + f(1) - f(1) + f(2) - \cdots + f(n-1) + f(n) \\
= -f(0) + f(n) \quad \text{(everything else cancels)} \\
= f(n).
\]