7.2.1) Substitution shows that the coefficient of $x'y'$ is as stated, and can be written as $(b, c - a) \cdot (\cos 2\theta, \sin 2\theta)$. This can be made 0 by choosing $\theta$ so that $2\theta$ equals $\pi/2$ plus the argument of $(b, c - a)$, so that the vectors are orthogonal.

7.2.2) The substitution of 7.2.1 transforms the equation into one of the same shape, and only the $ax^2 + bxy + cy^2$ part contributes to the coefficient of $x'y'$, so we get an equation as stated, with the $x'y'$ coefficient equal to 0.

7.2.3) Dividing by $a'$, we may assume that $a' = 1$. The substitution $x' = x'' + f$ preserves the shape of the equation in 7.2.2, and the new coefficient of $x''$ is $2f + c'$, which can be made 0 by choosing $f = -c'/2$. If the the coefficient of $y'$ is 0, we are left with $x'^2 + e = 0$, which is either empty (if $e > 0$), a double line (if $e = 0$), or a pair of parallel lines (if $e < 0$). The double line is called that because its equation factors as a product of two terms, each of which defines the same line. It arises as the intersection of the cone with a tangent plane.

7.2.4) The substitutions $x' = x'' + f$ and $y' = y'' + g$ for suitable $f$ and $g$ allow one to eliminate the $x'$ and $y'$ terms, leaving only an equation of the form $ax^2 + cy^2 = e$, with $a, c \neq 0$. (I have dropped the primes, for simplicity).

If $e = 0$, then either $a$ and $c$ have the same sign, in which case the only solution is $(0, 0)$, or $a$ and $c$ have opposite signs, in which case the equation factors into the equations of a pair of lines with opposite slopes.

Finally, suppose $e \neq 0$. Then we may divide by $e$ to assume $e = 1$. If $a, c < 0$, there are no solutions. If $a > 0 > c$, we write $a = 1/A^2$ and $c = -1/B^2$ to get $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$, the equation of a hyperbola. The case $a < 0 < c$ is similar. If $a, c > 0$, we write $a = 1/A^2$ and $c = 1/B^2$ to get $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$, the equation of an ellipse.

7.4.2) Given any rational solution to $y^2 = x^2(x + 1)$ with $x \neq 0$, the slope $t = y/x$ of the line through $(x, y)$ and $(0, 0)$ is a rational number. Conversely, given $t \in \mathbb{Q}$, the intersection of the line $y = tx$ with the curve is obtained by solving $y = tx$, $y^2 = x^2(x + 1)$, which gives

$$t^2x^2 = x^2(x + 1),$$

or equivalently,

$$(t^2 - x - 1)x^2 = 0,$$

which has the solutions $x = 0$ and $x = t^2 - 1$. The latter gives $y = tx = t^3 - t$. So we get a solution $(x, y) = (t^2 - 1, t^3 - t)$, which has $x \neq 0$ as long as $t \neq \pm 1$.

In other words, the rational points on the curve other than $(0, 0)$ (which is the unique point on the curve with $x = 0$) are exactly the points of the form $(t^2 - 1, t^3 - t)$ for $t \in \mathbb{Q} - \{\pm 1\}$. Thus the set of rational points on the curve is

$$\{(t^2 - 1, t^3 - t) : t \in \mathbb{Q} - \{\pm 1\}\} \cup \{(0, 0)\}.$$
1) Starting with $x_0 = \sqrt{13}$, define 
\[ a_0 = \lfloor x_0 \rfloor = 3 \]
\[ x_1 = \frac{1}{x_0 - a_0} = \frac{3 + \sqrt{13}}{4} \]
\[ a_1 = \lfloor x_1 \rfloor = 1 \]
\[ x_2 = \frac{1}{x_1 - a_1} = \frac{1 + \sqrt{13}}{3} \]
\[ a_2 = \lfloor x_2 \rfloor = 1 \]
\[ x_3 = \frac{1}{x_2 - a_2} = \frac{2 + \sqrt{13}}{3} \]
\[ a_3 = \lfloor x_3 \rfloor = 1 \]
\[ x_4 = \frac{1}{x_3 - a_3} = \frac{1 + \sqrt{13}}{4} \]
\[ a_4 = \lfloor x_4 \rfloor = 1 \]
\[ x_5 = \frac{1}{x_4 - a_4} = 3 + \sqrt{13} \]

so 
\[ \sqrt{13} = 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{3 + \sqrt{13}}}}} \]

We compute 
\[ 3 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = \frac{18}{5}, \]

and 
\[ 18^2 - 13 \cdot 5^2 = -1, \]

so we compute 
\[ (18 + 5\sqrt{13})^2 = 649 + 180\sqrt{13}. \]

Then $(649, 180)$ is the smallest positive solution to $x^2 - 13y^2 = 1$.

2) Let $f(x, y) = 0$ be a conic section. We will show that its projective closure is equivalent to the projective closure of $x^2 + y^2 = 1$. By earlier exercises on this homework, by means of a rotation and translation, we may assume that $f$ is one of the following: $x^2/a^2 + y^2/b^2 - 1$, $x^2/a^2 - y^2/b^2 - 1$, $y - Ax^2$. By replacing $x$ and $y$ by a scalar multiple, and multiplying $f$ by a nonzero constant, we may reduce to the cases where $f$ is one of the following: $x^2 + y^2 - 1$, $x^2 - y^2 - 1$, $y - x^2$. The first one is already the circle, and its projective closure is given in homogeneous coordinates by $X^2 + Y^2 - Z^2 = 0$. The second one has projective closure
given by $X^2 - Y^2 - Z^2 = 0$, and the substitution $(X, Y, Z) \mapsto (Z, Y, X)$ transforms this to $Z^2 - Y^2 - X^2 = 0$, which is the same curve as $X^2 + Y^2 - Z^2 = 0$. Finally, the third one has projective closure given by $YZ - X^2 = 0$, and the substitution $(X, Y, Z) \mapsto (X, Y + Z, Y - Z)$ transforms this to $Y^2 - Z^2 - X^2 = 0$, and permuting the coordinates and multiplying by $-1$ yields $X^2 + Y^2 - Z^2 = 0$.

Remark: The result of this problem can also be seen geometrically: if one views the two conic sections as being planar sections of the same cone, then projection from the vertex of the cone maps one conic section (in its plane) onto the other (in its plane).