16.4.1) By definition,
\[ \varphi(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda} \left( \frac{1}{(z - \lambda)^2} - \frac{1}{\lambda^2} \right), \]
where \( \Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \). For any constant \( \lambda \), the derivative of \( \frac{1}{(z - \lambda)^2} \) is \( -2/(z - \lambda)^3 \), so differentiating termwise gives
\[ \varphi'(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z - \lambda)^3}. \]
The numbers \(-m\omega_1 - n\omega_2 \) with \( m, n \in \mathbb{Z} \) give each element of \( \Lambda \) exactly one, so
\[ \varphi'(z) = -2 \sum_{(m,n) \in \mathbb{Z}^2} \frac{1}{(z + m\omega_1 + n\omega_2)^3}. \]
Then
\[ \varphi'(z + \omega_1) = -2 \sum_{(m,n) \in \mathbb{Z}^2} \frac{1}{(z + (m+1)\omega_1 + n\omega_2)^3}. \]
A shift of variable \( m' = m + 1 \) shows that this equals \( \varphi'(z) \). A similar proof shows that \( \varphi'(z + \omega_2) = \varphi'(z) \).

16.4.2) The derivative of \( \varphi(z + \omega_1) - \varphi(z) \) is \( \varphi'(z + \omega_1) - \varphi'(z) = 0 \), so \( \varphi(z + \omega_1) - \varphi(z) \) is a constant. The same is true of \( \varphi(z + \omega_2) - \varphi(z) \).

16.4.3) Substitute \( z = -\omega_1/2 \) in the first equation of 16.4.2, and \( z = -\omega_2/2 \) in the second equation of 16.4.2.

16.4.4) We have
\[ \varphi(-z) = \frac{1}{(-z)^2} + \sum_{\lambda \in \Lambda} \left( \frac{1}{(-z - \lambda)^2} - \frac{1}{\lambda^2} \right), \]
\[ = \frac{1}{z^2} + \sum_{\lambda \in \Lambda} \left( \frac{1}{(z - (-\lambda))^2} - \frac{1}{\lambda^2} \right), \]
\[ = \varphi(z), \]
since as \( \lambda \) ranges over elements of \( \Lambda \), the value of \(-\lambda \) takes each value in \( \Lambda \) exactly once (since \( \Lambda \) is an additive subgroup of \( \mathbb{C} \)).

Substituting \( \omega_1/2 \) into \( \varphi(z) = \varphi(-z) \) and applying 16.4.3 shows that \( c = 0 \). Similarly \( d = 0 \). Thus \( \varphi \) is doubly periodic.

1) The real and imaginary parts are \( u(x, y) = e^x \cos y \) and \( v(x, y) = e^x \sin y \); these functions are continuously differentiable functions \( \mathbb{R}^2 \to \mathbb{R} \). Also
\[ \frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}. \]
\[ \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}, \]

so the Cauchy-Riemann equations are satisfied. Thus \( e^z \) is differentiable in the complex sense.

2) (b) (Below we will identify complex numbers with points \((a, b)\) in \( \mathbb{R}^2 \) frequently.)

The segment from 0 to 2 is mapped to the segment from 0 = (0, 0) to 4 = (4, 0).

The segment from 0 to \( i \) is mapped to the segment from 0 = (0, 0) to \(-1 = (-1, 0)\) (since \((ti)^2 = -t^2\) for \( t \in [0, 1] \)).

The segment from 2 to 2 + \( i \) is mapped to \( \{(2+ti)^2 : t \in [0, 1]\} = \{(4-t^2)+4ti : t \in [0, 1]\} \), which is the part of the parabola \( 16x + y^2 = 64 \) between (4, 0) and (3, 4).

The segment from \( i \) to 2 + \( i \) is mapped to \( \{(t+i)^2 : t \in [0, 2]\} = \{(t^2-1)+2ti : t \in [0, 1]\} \), which is the part of the parabola \( y^2 - 4x = 4 \) between (-1, 0) and (3, 4).

(c) The derivative of \((4-t^2, 4t)\) at \( t = 1 \) is \((-2, 4)\), so the tangent line to the parabola \( 16x + y^2 = 64 \) at (3, 4) has slope \( 4/(-2) = -2 \) and is given by \( (y-4) = -2(x-3) \), or equivalently \( y = -2x + 10 \).

The derivative of \((t^2-1, 2t)\) at \( t = 2 \) is \((4, 2)\), so the tangent line to the parabola \( y^2 - 4x = 4 \) at (3, 4) has slope \( 2/4 = 1/2 \) and is given by \( (y-4) = (1/2)(x-3) \), or equivalently \( y = (1/2)x + (5/2) \).

The slopes are negative reciprocals of each other, so the two tangent lines form a right angle.

(d) It opens it out into a 180° angle.