The Complexity of Accurate Computations with Structured Matrices

Plamen Koev
Department of Mathematics
M.I.T.

Joint work with James Demmel

Goals

- **Accurate and Efficient** Linear Algebra
  - $\det(A)$
  - Any minor
  - $A^{-1}$
  - Decompositions: LDU, QR, Bidiagonal
  - Solution to $Ax = b$
  - SVD
  - Eigenvalues

- **With Structured Matrices**
  - Vandermonde
    * Polynomial / Generalized / Totally Positive
  - Cauchy
  - Totally Positive (All minors $> 0$)
  - $M$-matrices
  - Green Matrices ...
Goals

- **Efficient** means $O(n^3)$
  - Usual arithmetic model $\text{fl}(a \odot b) = (a \odot b)(1 + \delta), \ |\delta| \leq \epsilon$
- **Accurate** means correct sign and leading digits
  - $\det(A)$
  - minors
  - componentwise $A^{-1}$, LDU, solution to $Ax = b$
  - Eigenvalues, SVD:
    $$|\hat{\lambda}_i - \lambda_i| \leq O(\epsilon)|\lambda_i|, \quad |\hat{\sigma}_i - \sigma_i| \leq O(\epsilon)\sigma_i,$$
- **Contrast**: Traditional algorithms
  $$|\hat{\lambda}_i - \lambda_i| \leq O(\epsilon)\frac{||A||}{|y^*x|}, \quad |\hat{\sigma}_i - \sigma_i| \leq O(\epsilon)\sigma_1$$
Example: $100 \times 100$ Hilbert Matrix $H = 1/(i + j - 1)$

- Eigenvalues range from 1 to $10^{-150}$
- **Old Algorithm**, **New algorithm**, both in 16 digits

- Exploits Cauchy Structure
Cost of Accuracy in TM (1)

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>det$(A)$</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Cauchy</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Term Orth. Poly.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

GENP/PP/CP = Gaussian Elimination with No/Partial/Complete Pivoting
SVD = Singular Value Decomposition
NENP = Neville Elimination (bidiagonal factorization) with No Pivoting
Cost of Accuracy in TM (2)

TP = Totally Positive (all minors nonnegative)

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$C_{ij} = 1/(x_i + y_j)$</td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$x_i \nearrow, y_j \nearrow, x_1 + y_1 &gt; 0$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$V_{ij} = x_i^{j-1}, x_i$ distinct</td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td>$0 &lt; x_i \nearrow$</td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td>if some $x_i$ coincide, differentiate rows of $V$</td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td>$0 &lt; x_i \nearrow$</td>
</tr>
<tr>
<td>Vandermonde 3 Term Orth. Poly.</td>
<td>$V_{ij} = P_j(x_i), P_j$ orthogonal polynomial from 3-term recurrence</td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td>$G_{ij} = x_i^{\lambda_i+j-1}, \lambda_j$ nonnegative increasing integer sequence</td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td>$0 &lt; x_i \nearrow$</td>
</tr>
</tbody>
</table>
How do we lose relative accuracy in floating point?

- **MODEL:** $\text{fl}(a \otimes b) = (a \otimes b)(1 + \delta)$, no under/overflow

- **ACCURATE:**
  - Products, Quotients, Sums of positive numbers
    - Proof: $1 + \delta$ factors can be factored-out
  - $x_i \pm x_j$, where $x_i$ and $x_j$ are initial data (so exact)
How do we lose relative accuracy in floating point?

- **MODEL:** $fl(a \otimes b) = (a \otimes b)(1 + \delta)$, no under/overflow

- **ACCURATE:**
  - Products, Quotients, Sums of positive numbers
    - Proof: $1 + \delta$ factors can be factored out
  - $x_i \pm x_j$, where $x_i$ and $x_j$ are initial data (so exact)

- **POSSIBLE LOSS OF ACCURACY:**
  - **Subtractive Cancellation** when subtracting approximate results:
    
    \[
    \begin{array}{c}
    .12345xxx \\
    - .12345yyy \\
    \hline
    .00000zzz
    \end{array}
    \]
How do we lose relative accuracy in floating point?

• **MODEL:** \( \text{fl}(a \otimes b) = (a \otimes b)(1 + \delta) \), no under/overflow

• **ACCURATE:**
  - Products, Quotients, Sums of positive numbers
    Proof: \( 1 + \delta \) factors can be factored out
  - \( x_i \pm x_j \), where \( x_i \) and \( x_j \) are initial data (so exact)

• **POSSIBLE LOSS OF ACCURACY:**
  - Subtractive Cancellation when subtracting approximate results:
    \[
    \begin{array}{c}
    \text{.12345xxx} \\
    - \text{.12345yyy} \\
    \hline
    \text{.00000zzz}
    \end{array}
    \]
    \[
    .12345(1 + \delta_1) - .12345(1 + \delta_2) = .12345(\delta_1 - \delta_2) - \text{BAD}
    \]
    \[
    .12345(1 + \delta_1) + .12345(1 + \delta_2) = .23490(1 + (\delta_1 + \delta_2)/2) - \text{OK}
    \]
How do we lose relative accuracy in floating point?

• **MODEL:** \( \text{fl}(a \otimes b) = (a \otimes b)(1 + \delta) \), no under/overflow

• **ACCURATE:**
  
  - Products, Quotients, Sums of positive numbers
    
    Proof: \( 1 + \delta \) factors can be factored out
  
  - \( x_i \pm x_j \), where \( x_i \) and \( x_j \) are initial data (so exact)

• **POSSIBLE LOSS OF ACCURACY:**
  
  - **Subtractive Cancellation** when subtracting approximate results:
    
    \[
    \begin{array}{c}
    .12345xxx \\
    - .12345yyy \\
    \hline
    .00000zzz
    \end{array}
    \]
    
    \[
    .12345(1 + \delta_1) - .12345(1 + \delta_2) = .12345(\delta_1 - \delta_2) \text{ – BAD}
    \]
    
    \[
    .12345(1 + \delta_1) + .12345(1 + \delta_2) = .23490(1 + (\delta_1 + \delta_2)/2) \text{ – OK}
    \]

• **TRICK:** Only compute:
  
  - \( 1/(1 + xy) \); \( 1/x \); \( xy \); \( 1/(x - y) \); \( x, y > 0 \)
  
  - never subtract approximate quantities
Central Role of Minors

• Being able to compute det(A) accurately and efficiently is necessary for
  – \( A = LU \) with pivoting
  – \( A = QR \)
  – Eigenvalues \( \lambda_i \) of \( A \) ...
    * Proof: \( \det(A) = \pm \prod_i U_{ii} = \pm \prod_i R_{ii} = \prod \lambda_i = \cdots \)

• Being able to compute all minors of \( A \) is sufficient for
  – \( A^{-1} \)
    * Proof: Cramer’s rule, only need \( n^2 + 1 \) minors
  – \( A = LU \) or \( A = LDU \) with pivoting
    * Proof: Each entry of \( L, D, U \) a quotient of minors; \( O(n^3) \) needed
  – Singular values of \( A \) (SVD)
    * Proof: \( A = LDU \) with complete pivoting, then SVD of \( LDU \)
  – Eigenvalues of Totally Positive matrices (yesterday’s talk)

• Other methods for getting accurate \( \lambda_i, \sigma_i \) developed
Accurate det(A), minors

- **det(A):**
  - Vandermonde \( V = x_i^{j-1} \)
    
    \[
    \det(V) = \prod_{i>j} (x_i - x_j)
    \]
  - Cauchy \( C = 1/(x_i + y_j) \)
    
    \[
    \det(C) = \frac{\prod_{i<j}(x_i - x_j)(y_i - y_j)}{\prod_{i,j}(x_i + y_j)}
    \]
  - TP Generalized Vandermonde \( G = x_i^{a_j}, x_i > 0, \) increasing
    
    \[
    \det(G) = \det(V) \cdot s_\lambda(x_1, x_2, \ldots, x_n)
    \]
    
    where \( s_\lambda(x_1, x_2, \ldots, x_n) = \sum x^\mu \) – sum of positives

- **Minors:** Cauchy – OK, Vandermonde – OK, but not efficient

- **LDU:** Each entry of \( L, D, U \) – quotient of minors of \( A \)
Accurate Solution to $Ax = b$

- Classical Example: Björck-Pereyra Methods for Vandermonde systems

$$
\begin{bmatrix}
1 & x_1 & \ldots & x_1^{n-1} \\
1 & x_2 & \ldots & x_2^{n-1} \\
1 & x_3 & \ldots & x_3^{n-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_n & \ldots & x_n^{n-1}
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3 \\
\vdots \\
y_n
\end{bmatrix}
=
\begin{bmatrix}
+ \\
- \\
+ \\
\vdots \\
-
\end{bmatrix}
$$

and $0 < x_1 < \ldots < x_n$.

- In $O(n^2)$ time
- With small forward error: $|y_i - \hat{y}_i| \leq O(\epsilon)|y_i|$
- With small backward error:
  - If $\tilde{V}\hat{y} = b$ then $|V_{ij} - \tilde{V}_{ij}| \leq O(\epsilon)|V_{ij}|$. 
Accuracy of the Björck-Pereyra Method for Vandermondes

\[
\begin{bmatrix}
1 & x_1 & x_1^2 & x_1^3 \\
1 & x_2 & x_2^2 & x_2^3 \\
1 & x_3 & x_3^2 & x_3^3 \\
1 & x_4 & x_4^2 & x_4^3
\end{bmatrix}^{-1} = \begin{bmatrix}
1 & -x_1 \\
1 & -x_2 \\
1 & -x_3
\end{bmatrix} \times
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} \times \begin{bmatrix}
1 & -1 \\
1 & -1 \\
1 & -1
\end{bmatrix} \times \begin{bmatrix}
1 & + & + & + & + \\
1 & + & - & + & - \\
1 & - & - & - & +
\end{bmatrix}
\]

Accuracy ... OK

Other TP matrices? ... Yes. Cauchy
### The Björck-Pereyra Method for Cauchy Matrices

- **Cauchy** matrices
  
  \[ C_{ij} = \frac{1}{x_i - y_j} \]

- **TP**, if \( x_1 > \ldots > x_n > y_1 > \ldots > y_n \)
  
  \[
  \begin{bmatrix}
  \frac{1}{x_1 - y_1} & \frac{1}{x_1 - y_2} & \frac{1}{x_1 - y_3} \\
  \frac{1}{x_2 - y_1} & \frac{1}{x_2 - y_2} & \frac{1}{x_2 - y_3} \\
  \frac{1}{x_3 - y_1} & \frac{1}{x_3 - y_2} & \frac{1}{x_3 - y_3}
  \end{bmatrix}^{-1} =
  \begin{bmatrix}
  1 & -(x_1 - y_1) & 0 \\
  0 & \frac{1}{y_1 - y_2} & \frac{-(x_1 - y_2)}{y_1 - y_2} \\
  0 & 0 & \frac{1}{y_2 - y_3} \\
  \end{bmatrix}
  \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
  \end{bmatrix}
  \begin{bmatrix}
  1 & 0 & 0 \\
  -(x_1 - y_1) & \frac{x_2 - y_1}{y_1 - y_2} & \frac{-(x_2 - y_1)}{y_1 - y_2} \\
  0 & \frac{x_3 - y_1}{x_2 - x_1} & \frac{x_3 - x_1}{x_2 - x_1}
  \end{bmatrix}
  \times
  \begin{bmatrix}
  1 & 0 & 0 \\
  -(x_1 - y_1) & \frac{x_2 - y_1}{y_1 - y_2} & \frac{-(x_2 - y_1)}{y_1 - y_2} \\
  0 & \frac{x_3 - y_1}{x_2 - x_1} & \frac{x_3 - x_1}{x_2 - x_1}
  \end{bmatrix}
  \]

(Vadim Olshevsky, 1995)

- **Unifying Characteristic?**
The Connection with Minors

- Which TP matrices permit *accurate* bidiagonal decomposition?

- Each entry is *product of quotients of initial minors*

\[
L_{i+1,i}^{(k)} = - \frac{\det(A(i - k + 2 : i + 1, 1 : k))}{\det(A(i - k + 2 : i, 1 : k - 1))} \cdot \frac{\det(A(i - k + 1 : i - 1, 1 : k - 1))}{\det(A(i - k + 1 : i, 1 : k))}
\]

**INITIAL MINORS**
- Contiguous
- Include first row or column

- Accurate Initial Minors ⇒ Björck-Pereyra Methods

- Initial Minors easier than all minors

- **New results:** TP Generalized Vandermonde, \( x_i > 0 \) increasing

\[
G = \begin{bmatrix}
1 & x_1 & x_1^6 & x_1^8 \\
1 & x_2 & x_2^6 & x_2^8 \\
1 & x_3 & x_3^6 & x_3^8 \\
1 & x_3 & x_3^6 & x_4^8
\end{bmatrix}
\]
The SVD

- Demmel, Kahan 1991: Accurate SVD of bidiagonals
  \[
  B = \begin{bmatrix}
  a_1 & b_1 \\
  a_2 & b_2 \\
  \cdots & \cdots \\
  a_{n-1} & b_{n-1} \\
  a_n 
  \end{bmatrix}
  \]

- Demmel et al. 1997:
  Given an accurate LDU from GECP
  \[ A = LDU \]
  \[ \implies \text{Accurate SVD} \]
  - LDU – computed accurately through minors
  - Works for Cauchy, totally unimodular, Green matrices, unit displacement rank, total signed compound
  - **New result:** Diagonally dominant M-matrices
New results:

- Given $A$, first decompose as $A = B \cdot (\text{WELL-CONDITIONED})$
- Polynomial Vandermonde involving orthonormal polynomials

$$V_P = \begin{bmatrix}
  P_0(x_1) & P_1(x_1) & \ldots & P_{n-1}(x_1) \\
  P_0(x_2) & P_1(x_2) & \ldots & P_{n-1}(x_2) \\
  \ldots \\
  P_0(x_n) & P_1(x_n) & \ldots & P_{n-1}(x_n)
\end{bmatrix} = (\text{CAUCHY}) \cdot (\text{ORTHOGONAL})$$
Accurate Eigenvalues

- Interesting case – unsymmetric
- TP Tridiagonals: If $b_i, d_i, c_i > 0$ then the eigenvalues of

\[
T = \begin{bmatrix}
1 & 1 & & \\
& b_1 & 1 & \\
& & \ddots & \\
& & & b_{n-2} & 1 & \\
& & & & b_{n-1} & 1
\end{bmatrix}
\begin{bmatrix}
d_1 & & & \\
& d_2 & & \\
& & \ddots & \\
& & & d_{n-1}
\end{bmatrix}
\begin{bmatrix}
1 & c_1 & & \\
& 1 & c_2 & \\
& & \ddots & \\
& & & 1 & c_{n-1}
\end{bmatrix}
\]

are the squares of the singular values of

\[
\begin{bmatrix}
\sqrt{d_1} & \sqrt{d_1 b_1 c_1} & & \\
& \sqrt{d_2} & \sqrt{d_2 b_2 c_2} & \\
& & \ddots & \\
& & & \sqrt{d_{n-1}} & \sqrt{d_{n-1} b_{n-1} c_{n-1}} & \\
& & & & \sqrt{d_n}
\end{bmatrix}
\]

- Computed to high relative accuracy
Accurate Eigenvalues

• Thm: If $A$ is TP

$$A = (TP) \cdot T \cdot (TP)^{-1}$$

• If we start with a bidiagonal decomposition (think Björck-Pereyra decomposition)

\[
A = \begin{bmatrix}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & + & 1 & 1 \\
+ & 1 & + & 1 \\
+ & 1 & + & 1 \\
\end{bmatrix}
\]

• New Result: Accurate $A \rightarrow T \rightarrow \lambda_i$
**New approach to the accurate SVD of TP matrices:**
- Applying accurate Givens rotations

**Start with the bidiagonal decomposition of a TP Matrix $A$**

\[
A = \begin{bmatrix}
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & + \\
+ & + & + & +
\end{bmatrix}
= 
\begin{bmatrix}
1 & 1 \\
1 & + & 1 \\
+ & 1 & + & 1 \\
+ & + & + & + & + & + & + & + & + & + & + & + & + & +
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 & + & 1 \\
+ & 1 & + & 1 \\
+ & + & + & + & + & + & + & + & + & + & + & + & + & +
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 & + & 1 \\
+ & 1 & + & 1 \\
+ & + & + & + & + & + & + & + & + & + & + & + & + & +
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 & + & 1 \\
+ & 1 & + & 1 \\
+ & + & + & + & + & + & + & + & + & + & + & + & + & +
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 & + & 1 \\
+ & 1 & + & 1 \\
+ & + & + & + & + & + & + & + & + & + & + & + & + & +
\end{bmatrix}
\]

**Givens Rotation =**
- Subtract a row from next to make a 0
- Add a multiple of row with 0 to previous
- Scale both rows
### Cost of Accuracy in TM (1)

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>det($A$)</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Cauchy</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Term Orth. Poly.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

GENP/PP/CP = Gaussian Elimination with No/Partial/Complete Pivoting  
SVD = Singular Value Decomposition  
NENP = Neville Elimination ( bidiagonal factorization) with No Pivoting
### Cost of Accuracy in TM (3)

**Known results**

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>det$(A)$</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
<td></td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
<td></td>
</tr>
<tr>
<td>Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Term Orth. Poly.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Proof:** Exploit \( \det(C) = \prod_{i<j}(x_j - x_i)(y_j - y_i)/\prod_{ij}(x_i + y_j) \)
### Cost of Accuracy in TM (4)

**Known results + New Results**

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>det($A$)</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vandermonde 3 Term Orth. Poly.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Proof:** Do GECP, apply new SVD algorithm
Cost of Accuracy in TM (5)

Known results

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>det($A$)</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$n^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Term Orth. Poly.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Proof: Björck-Pereyra
Cost of Accuracy in TM (6)
Known results + **New Results**

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>det($A$)</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$n^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Term Orth. Poly.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Proof:** Vandermonde = Cauchy \times DFT
Cost of Accuracy in TM (7)
Known results + New Results

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>det($A$)</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$n^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>exp</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>exp</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n^3$</td>
<td></td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n^3$</td>
<td></td>
</tr>
<tr>
<td>Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Term Orth. Poly.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Proof: Special case of TP Generalized Vandermonde
**Cost of Accuracy in TM (8)**

**Known results + New Results**

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>det($A$)</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>exp</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>exp</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Term Orth. Poly.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Proof:** Can’t add $x + y + z$ accurately
### Cost of Accuracy in TM (9)

#### Known results

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>$\det(A)$</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>exp</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>exp</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td>$n^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vandermonde</td>
<td></td>
<td>$n^3$</td>
<td>$n^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Term Orth. Poly.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Proof:** Higham
Cost of Accuracy in TM (10)
Known results + New Results

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>$\det(A)$</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>exp</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>exp</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td></td>
<td>$n^3$</td>
<td></td>
<td></td>
<td>$n^2$</td>
<td></td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Term Orth. Poly.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Proof: Can’t add $x + y + z$ accurately
### Known results

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>det($A$)</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>exp</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>exp</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td></td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td></td>
<td>$n^3$</td>
<td></td>
<td></td>
<td></td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde 3 Term Orth. Poly.</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n^2$</td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Proof: Higham
## Cost of Accuracy in TM (12)

**Known results + New Results**

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>det$(A)$</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>exp</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>exp</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td></td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde 3 Term Orth. Poly.</td>
<td>$n^2$</td>
<td></td>
<td>$n^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n^3$</td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Proof:** 

\[
\text{Poly}_\text{Vand}(x) = \text{Cauchy}(x,y) \times \text{Poly}_\text{Vand}(y)
\]

Choose $y$ as roots of Orth Poly $\Rightarrow$ Poly$_\text{Vand}(y) = \text{diagonal} \times \text{orthogonal}$
### Cost of Accuracy in TM (13)

#### New Results

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>det($A$)</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>exp</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>exp</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td></td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td></td>
<td>$n^3$</td>
<td></td>
<td></td>
<td></td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde 3 Term Orth. Poly.</td>
<td>$n^2$</td>
<td></td>
<td>$n^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n^3$</td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Proof:** Can’t add $x + y + z$ accurately
### New Results

<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>$\det(A)$</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>exp</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>exp</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td></td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td></td>
<td>$n^3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vandermonde 3 Term Orth. Poly.</td>
<td>$n^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n^3$</td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td>$\Lambda n^2$</td>
<td>$\Lambda n^3$</td>
<td>exp</td>
<td>$\Lambda n^2$</td>
<td>$\Lambda n^2$</td>
<td>exp</td>
<td>exp</td>
<td>$\Lambda n^2$</td>
</tr>
</tbody>
</table>

- $G_{ij} = x_i^{\lambda_j + j - 1}$, $0 \leq \lambda_i$
- $\Lambda \leq 2(\lambda_1 + 1) \cdot (\lambda_2 + 1)^2 \cdots (\lambda_n + 1)^2 \cdot (#\lambda_i > 1)$
- Exponential speedup over previous best algorithm: $n^{\lambda_1 + \cdots + \lambda_n}$
- Proof: Divide-and-conquer to evaluate Schur polynomials
<table>
<thead>
<tr>
<th>Matrix Type</th>
<th>$\det(A)$</th>
<th>$A^{-1}$</th>
<th>Minor</th>
<th>GENP</th>
<th>GEPP</th>
<th>GECP</th>
<th>SVD</th>
<th>NENP</th>
<th>EVD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^2$</td>
</tr>
<tr>
<td>TP Cauchy</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td>$n^3$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>$n^3$</td>
<td>$n^2$</td>
<td></td>
</tr>
<tr>
<td>TP Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>exp</td>
<td>$n^2$</td>
<td>$n^2$</td>
<td>exp</td>
<td>$n^3$</td>
<td>$n^2$</td>
<td>$n^3$</td>
</tr>
<tr>
<td>Confluent Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>$n^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TP Confluent Vandermonde</td>
<td>$n^2$</td>
<td>$n^3$</td>
<td>$n^3$</td>
<td></td>
<td></td>
<td></td>
<td>$n^3$</td>
<td>$n^2$</td>
<td>$n^3$</td>
</tr>
<tr>
<td>Vandermonde</td>
<td>$n^2$</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 Term Orth. Poly.</td>
<td>$n^2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$n^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generalized Vandermonde</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>TP Generalized Vandermonde</td>
<td>$\Lambda n^2$</td>
<td>$\Lambda n^3$</td>
<td>exp</td>
<td>$\Lambda n^2$</td>
<td>$\Lambda n^2$</td>
<td>exp</td>
<td>$\Lambda n^3$</td>
<td>$\Lambda n^2$</td>
<td>$\Lambda n^3$</td>
</tr>
<tr>
<td>Any TP</td>
<td>$n$</td>
<td>$n^3$</td>
<td>exp</td>
<td>$n^3$</td>
<td>exp</td>
<td>exp</td>
<td>$n^3$</td>
<td>0</td>
<td>$n^3$</td>
</tr>
</tbody>
</table>

- Any TP: assuming NENP already done
Conclusions

- We have identified many classes of structured matrices that permit accurate and efficient matrix computations

- New results:
  - $O(n^3)$ algorithms for the eigenvalues and the SVD of (unsymmetric) TP Matrices to high relative accuracy
  - Björck-Pereyra Methods for Generalized Vandermondes
  - Accurate SVDs of M-matrices
  - Accurate SVDs of Polynomial Vandermonde matrices

- Open problems remain
  - TP eigenvectors, singular vectors
  - Non-TP eigenproblem

- This talk and Matlab software: math.mit.edu/~plamen