Accurate Eigenvalues of the Laplacian

Plamen Koev
Massachusetts Institute of Technology
Joint work with James Demmel (Berkeley) and Steve Vavasis (Cornell & Waterloo)
The most common way to compute eigenvalues of the weighted Laplacian is via finite element discretization.

This yields a symmetric generalized eigenvalue problem of the form $Kx = \lambda Mx$.

We argue that this system has special structure allowing high relative precision calculation of all eigenvalues including the smallest ones.
Consider a moving two-dimensional membrane defined by bounded set $\Omega \subset \mathbb{R}^2$ whose boundaries are clamped.

Assume the stiffness varies over the membrane and is given by a coefficient field $c$. Assume the displacement is small and all motion is elastic.

The governing equation is a two-dimensional wave equation: $u_{tt} = \nabla \cdot (c \nabla u)$ on $\Omega$ and $u = 0$ on $\partial \Omega$. 
A standing wave solution to this problem has the form
\[ u(x, t) = e^{i\lambda t}u_0(x). \]
Substituting this formula into the PDE yields the continuum eigenvalue problem
\[ \nabla \cdot (c \nabla u_0) + \lambda^2 u_0 = 0. \]
Finite element discretization (piecewise linear)

• Assume $\mathcal{T}$ is a finite element mesh for the domain $\Omega$, that is, a simplicial subdivision into $r$ triangles.
• Let $w_1, \ldots, w_n$ be the mesh nodes not on $\partial \Omega$.
• Let $V_h$ denote the set of piecewise linear continuous functions $u$ on this triangulation satisfying $u|_{\partial \Omega} = 0$.

Note: $\dim(V_h) = n$. 
**Discrete linear equations**

- Obtain weak form of PDE: multiply PDE by a test function \( q \) satisfying \( q|_{\partial \Omega} = 0 \); integrate by parts:
  \[
  \int_{\Omega} \nabla q \cdot c \nabla u = \int_{\Omega} \lambda qu
  \]

- Discrete FE equations: find eigenpairs
  \( u \in V_h, \lambda \in \mathbb{R} \) such that the weak form holds for all \( q \in V_h \).
• Functions $q \in V_h$ are in 1-1 correspondence with vectors $q \in \mathbb{R}^n$ according to $q_i = q(w_i)$ for $i = 1 : n$ (homeomorphism of vector spaces).

• Let $u \in \mathbb{R}^n$ be the vector corresponding to FE solution $u$. Then $u$ satisfies: for all $q \in \mathbb{R}^n$,

$$q^T K u = q^T \lambda M u$$

for matrix $K$ called the stiffness matrix, and matrix $M$, called the mass matrix.

Equivalent to $K u = \lambda M u$. 

Closed-form expressions for entries of $K$ are obtained by considering $u$ of the form $[0; 0; \cdots ; 0; 1; 0; \cdots 0]$ and similarly for $q$ and evaluating the weak form for the corresponding $u$ and $q$. Expressions also available for $M$.

Matrix $K$ so determined is $n \times n$ symmetric positive definite. Matrix $M$ is $n \times n$ is symmetric and strongly positive definite.
A test case

- Consider the unit square domain with a border of width $\eta$ that surrounds an inner square of width $1 - 2\eta$.

- Assign a very high stiffness $s$ to the border and a constant stiffness of 1 to the inner square.

- It can be proved using classical minimax arguments that as $s \to \infty$, the smaller eigenpairs of this domain tend to eigenpairs of the geometry of the subsquare.
Experiments with this test case

• In exact arithmetic, one expects that as $s$ gets larger, the smallest eigenvalue of this border problem converges to the smallest eigenvalue of the inner domain.

• In the presence of roundoff, convergence is noted up to a certain threshold value $s^*$; after this point, the solution diverges because roundoff error prevents accurate computation of the small eigenvalue.
Why is roundoff error a problem

- Roundoff error corrupts the solution to the problem described above because as $s \to \infty$, we have that $\|\mathbf{K}\| \to \infty$ proportionally.

- Thus, the largest entries of $\mathbf{K}$ as well as the larger eigenvalues increase without bound. Under this circumstance, conventional eigenvalue algorithms cannot recover the smaller eigenvalues accurately.
Our Approach

- Solving $K u = \lambda M u$, where $K = A^T D J^T J DA$
- $A$ is DSTU, $D$ is diagonal, $J$ is well conditioned; assume $M$ is diagonal
- Equivalent to SVD of $J D A M^{1/2}$
- DSTU problem solved by Demmel; also Pelaez–Moro
- One more time for SVD of $J U \Sigma V^T$