Abstract. Many multivariate statistics are expressed as functions of the hypergeometric function of a
matrix argument, or more generally, as series of Jack functions.

This work is a collection of formulas, identities, and algorithms useful for the computations of these
statistics in practice. Numerical examples are presented.

1. Definitions

1.1. Partitions and hook lengths. For an integer \(k \geq 0\) we say that \(\kappa = (\kappa_1, \kappa_2, \ldots)\) is a partition of \(k\)
(denoted \(\kappa \vdash k\)) if \(\kappa_1 \geq \kappa_2 \geq \cdots \geq 0\) are integers such that \(\kappa_1 + \kappa_2 + \cdots = k\). The quantity \(|\kappa| = k\) is also
called the size \(\kappa\).

We introduce a partial ordering of partitions and say that \(\mu \subseteq \lambda\) if \(\mu_i \leq \lambda_i\) for all \(i = 1, 2, \ldots\). Then
\(\lambda/\mu\) is called a skew shape and consists of those boxes in the Young diagram of \(\lambda\) that do not belong to \(\mu\).

Clearly, \(|\lambda/\mu| = |\lambda| - |\mu|\).

The skew shape \(\kappa/\mu\) is a horizontal strip when \(\kappa_1 \geq \mu_1 \geq \kappa_2 \geq \mu_2 \geq \cdots\) [30, p. 339].

The upper and lower hook lengths at a point \((i, j)\) in a partition \(\kappa\) (i.e., \(i \leq \kappa'_j, j \leq \kappa_i\)) are
\[h^*_{\kappa}(i, j) \equiv \kappa'_j - i + \alpha(\kappa_i - j + 1);\]
\[h^*_{\kappa}(i, j) \equiv \kappa'_j - i + 1 + \alpha(\kappa_i - j).\]

The products of the upper and lower hook lengths are denoted, respectively, as
\[H^*_{\kappa} = \prod_{(i, j) \in \kappa} h^*_{\kappa}(i, j)\]
and the product of the two plays an important role in what follows, thus we define
\[(1.1) \quad j_{\kappa} = H^*_{\kappa} H^*_{\kappa}.

1.2. Pochhammer symbol and multivariate Gamma function. For a partition \(\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)\)
and \(\alpha > 0\), the Generalized Pochhammer symbol is defined as
\[(1.2) \quad (a)_{\kappa}^{(\alpha)} \equiv \prod_{(i, j) \in \kappa} \left(a - \frac{i - 1}{\alpha} + j - 1\right) = \prod_{i=1}^{n} \prod_{j=1}^{\kappa_i} \left(a - \frac{i - 1}{\alpha} + j - 1\right) = \prod_{i=1}^{n} \left(a - \frac{i - 1}{\alpha}\right)^{\kappa_i},\]
where \((a)_k = a(a + 1) \cdots (a + k - 1)\) in the rising factorial.

For a partition \(\kappa = (k)\) in only one part \((a)_k^{(\alpha)} = (a)_k\) is independent of \(\alpha\).

The multivariate Gamma function of parameter \(\alpha\) is defined as
\[(1.3) \quad \Gamma_m^{(\alpha)}(c) = \pi^{\frac{m(m-1)}{2\alpha}} \prod_{i=1}^{m} \Gamma\left(c - \frac{i - 1}{\alpha}\right) \quad \text{for } \Re(c) > \frac{m - 1}{\alpha}.\]
1.3. Jack function and hypergeometric function of a matrix argument.

**Definition 1.1** (Jack function). The Jack function $J^{(\alpha)}_\kappa(X) = J^{(\alpha)}_\kappa(x_1, x_2, \ldots, x_m)$ is a symmetric, homogeneous polynomial of degree $|\kappa|$ in the eigenvalues $x_1, x_2, \ldots, x_m$ of $X$. It is defined recursively [29, Proposition 4.2]:

\[
J^{(\alpha)}_\kappa(x_1, x_2, \ldots, x_n) = 0, \text{ if } \kappa_{n+1} > 0;
\]

\[
J^{(\alpha)}_\kappa(x_1) = x_1^\alpha (1 + \alpha) \cdots (1 + (k-1)\alpha);
\]

\[
J^{(\alpha)}_\kappa(x_1, x_2, \ldots, x_n) = \sum_{\mu \subseteq \kappa} J^{(\alpha)}_\mu(x_1, x_2, \ldots, x_{n-1}) x_1^{(\kappa/\mu)_x} \kappa_{\mu}, \quad n \geq 2,
\]

where the summation in (1.4) is over all partitions $\mu \subseteq \kappa$ such that $\kappa/\mu$ is a horizontal strip, and

\[
\beta_{\kappa\mu} = \frac{\prod_{(i,j) \in \kappa} B^{(\alpha)}_{\kappa\mu}(i,j)}{\prod_{(i,j) \in \mu} B^{(\alpha)}_{\mu}(i,j)}, \quad \text{where} \quad B^{(\alpha)}_{\mu}(i,j) = \begin{cases} h^{(\alpha)}(i,j), & \text{if } \kappa_j = \mu_j; \\ k^{(\alpha)}(i,j), & \text{otherwise}. \end{cases}
\]

**Remark 1.2.** There are several normalizations of the Jack function: $C^{(\alpha)}_\kappa$, $J^{(\alpha)}_\kappa$, $P^{(\alpha)}_\kappa$, and $Q^{(\alpha)}_\kappa$, related as in (2.28). The normalization $J^{(\alpha)}_\kappa(X)$ is such that for $|\kappa| = n$ the coefficient of $x_1 x_2 \cdots x_n$ in $J^{(\alpha)}_\kappa(X)$ is $n!$ [29, Theorem 1.1]. $C^{(2)}_\kappa(X)$ is the zonal polynomial and $P^{(1)}_\kappa(X) = Q^{(1)}_\kappa(X) = s_\lambda(X)$ is the Schur function.

**Definition 1.3** (Hypergeometric function of a matrix argument). Let $p \geq 0$ and $q \geq 0$ be integers, and let $X$ be an $m \times m$ complex symmetric matrix with eigenvalues $x_1, x_2, \ldots, x_m$. The hypergeometric function of a matrix argument $X$ and parameter $\alpha > 0$ is defined as

\[
pFq(a_1, \ldots, a_p; b_1, \ldots, b_q; X) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(a_1)^{(k)} \cdots (a_p)^{(k)}}{(b_1)^{(k)} \cdots (b_q)^{(k)}} C^{(\alpha)}_k(X).
\]

The hypergeometric function of two matrix arguments $X$ and $Y$, and parameter $\alpha > 0$ is defined as

\[
pFq(a_1, \ldots, a_p; b_1, \ldots, b_q; X, Y) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(a_1)^{(k)} \cdots (a_p)^{(k)}}{(b_1)^{(k)} \cdots (b_q)^{(k)}} \frac{C^{(\alpha)}_k(X) C^{(\alpha)}_k(Y)}{C^{(\alpha)}_k(I)}.
\]

1.4. Random matrix ensembles—Wishart, Laguerre, and Jacobi. An $m \times m$ matrix is distributed as $\beta$-Wishart with $n$ ($n \geq m$) degrees of freedom and covariance matrix $\Sigma$ (denoted $A \sim W^{(\alpha)}_m(n, \Sigma)$) if the joint density function of its eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ is

\[
\frac{(\det \Sigma)^{\frac{n}{2}} \prod_{i=1}^{m} \lambda_i^{\frac{n-m+1}{2}}}{\mathcal{K}^{(\alpha)}_m(\frac{\alpha}{2})} \prod_{j<k} |\lambda_k - \lambda_j|^\frac{\alpha}{2} \cdot qF_0^{(\alpha)}(-\frac{1}{\alpha} L, \Sigma^{-1}),
\]

where $\alpha = 2/\beta$, $L = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$, and

\[
\mathcal{K}^{(\alpha)}_m(a) = \frac{m! a^{m+1}}{\Gamma^{(\alpha)}_m(a) \Gamma^{(\alpha)}_m(\frac{m}{\alpha})}. \frac{\Gamma^{(\alpha)}_m(a) \Gamma^{(\alpha)}_m(\frac{m}{\alpha})}{\Gamma^{(\alpha)}_m(\frac{\alpha}{2})}. \frac{\Gamma^{(\alpha)}_m(a) \Gamma^{(\alpha)}_m(\frac{m}{\alpha})}{\Gamma^{(\alpha)}_m(\frac{\alpha}{2})}.
\]

For $\beta > 0$, $a > \frac{m-1}{\alpha}$, and $\alpha = \frac{2}{\beta}$, a matrix $A$ is said to be $\beta$–Laguerre distributed (denoted $A \sim \mathcal{L}^{(\alpha)}_m(a)$) if $A \sim W^{(\alpha)}_m(\alpha a, I)$. The joint density of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$ of $A$ is then [10]:

\[
\frac{1}{\mathcal{K}^{(\alpha)}_m(a)} \prod_{i=1}^{m} \lambda_i^{\frac{n-m+1}{2}} e^{-\frac{\lambda_i}{\alpha}} \prod_{j<k} |\lambda_k - \lambda_j|^\frac{2}{\alpha}.
\]
For \( \beta > 0, \alpha = \frac{2}{\beta} \), and \( a, b > \frac{m-1}{\alpha} \), a matrix \( C \) is said to be \( \beta \)-Jacobi distributed (denoted \( C \sim J_{m}^{(\alpha)}(a, b) \)) if its joint eigenvalue density is

\[
\frac{1}{S_m^{(\alpha)}(a, b)} \prod_{i=1}^{m} \lambda_i^{a-\frac{m-1}{\alpha}-1} (1 - \lambda_i)^{b-\frac{m-1}{\alpha}-1} \prod_{j<k} |\lambda_j - \lambda_k|^\frac{2}{\alpha},
\]

where \( S_m^{(\alpha)}(a, b) \) is the value of the Selberg Integral [27]:

\[
S_m^{(\alpha)}(a, b) = \frac{m! \cdot \Gamma_m^{(\alpha)}(\frac{m}{2})}{\pi^{\frac{m(m-1)}{2}}} \cdot \frac{\Gamma_m^{(\alpha)}(a)}{\Gamma_m^{(\alpha)}(a+b)}.
\]

### 1.5. Empirical models for the classical random matrix ensembles.

The notation is convenient to use uniform notation for real, complex, and quaternion normal distributions: \( N^{(\alpha)}(\cdot, \cdot) \) for \( \alpha = 2, 1, \frac{1}{2} \), respectively.

**Definition 1.4** (Real, complex, and quaternion Wishart ensembles). Let \( \Sigma \) be an \( m \times m \) symmetric semi-definite matrix and let \( Z \sim N^{(\alpha)}(0, I_m \otimes \Sigma) \) by an \( n \times m \) real, complex, or quaternion Gaussian random matrix for \( \alpha = 2, 1, \frac{1}{2} \), respectively. Then the matrix \( A = ZDZ \) is said to have a real, complex, or quaternion \( m \times m \) central Wishart distribution with \( n \) degrees of freedom and covariance matrix \( \Sigma \). The notation is \( A \sim W_m^{(\alpha)}(n, \Sigma) \) for \( \alpha = 2, 1, \frac{1}{2} \), respectively [23, Definition 3.1.3, p. 82], [26].

For \( \alpha = 2, 1, \frac{1}{2} \), the density of \( A \sim W_m^{(\alpha)}(n, \Sigma) \) is [26, Definition 3.1], [23, (18), p. 62]:

\[
\frac{\det(\alpha \Sigma)^{-\frac{n}{2}}}{\Gamma_m^{(\alpha)}(\frac{n}{2})} \cdot e^{\text{tr} \left(-\frac{1}{2} \Sigma^{-1}A\right)} \cdot \left(\text{det} \ A\right)^{-\frac{n-m+1}{2}},
\]

and the eigenvalues of \( A \) are distributed as (1.8) [23, Theorem 9.4.1, p. 388], [26, (3.2)].

**Definition 1.5** (Real, complex, and quaternion Jacobi ensembles). Let the matrices \( A \sim W_m^{(\alpha)}(n_1, \Sigma) \) and \( B \sim W_m^{(\alpha)}(n_2, \Sigma) \) be independent real, complex, or quaternion Wishart matrices (where \( \alpha = 2, 1, \frac{1}{2} \), respectively). Then \( C = A(A + B)^{-1} \) is called a real, complex, or quaternion Jacobi matrix, respectively.

**Remark 1.6.** The eigenvalues of \( C \) are unaffected by \( \Sigma \), which can thus be assumed to equal \( I \) without loss of generality.

**Theorem 1.7.** For the matrix \( C \) from Definition 1.5, we have \( C \sim J_{m}^{(\alpha)}(\frac{n_1}{\alpha}, \frac{n_2}{\alpha}) \) for \( \alpha = 2, 1, \frac{1}{2} \).

**Proof.** Theorem 3.3.4 in Muirhead [23] for \( \alpha = 2,1, \frac{1}{2} \).

### 1.6. Empirical models for the classical random matrix ensembles.

The complex norms \( a \sim N^{(1)}(0, 1) \) are generated as \( a = \frac{1}{\sqrt{2}}(a_1 + ia_2) \), where \( a_1, a_2 \sim N^{(2)}(0, 1) \), and the quaternion normals are generated as \( a = \frac{1}{2}(a_1 + ia_2 + ja_3 + ka_4) \), where \( a_1, a_2, a_3, a_4 \sim N^{(2)}(0, 1) \).

A Wishart matrix is empirically generated as \( \Sigma^{1/2} \cdot ZD \cdot Z \cdot \Sigma^{1/2} \), where \( Z \) is \( n \times m \) and \( z_{ij} \) are i.i.d. \( N^{(\alpha)}(0, 1) \).

**Definition 1.8** (\( \beta \)-Laguerre matrix model). For \( \beta > 0 \), let \( \alpha = \frac{2}{\beta} \) and define the \( m \times m \) matrix \( L \) as \( L \equiv \frac{1}{\beta}BB^T \), where

\[
B = \begin{bmatrix}
\chi_{2a} & \chi_{\beta(m-1)} & \chi_{2a-\beta} & \cdots & \chi_{2a-(m-1)} \\
\chi_{\beta(m-1)} & \chi_{2a} & \chi_{\beta} & \cdots & \chi_{2a-(m-1)} \\
\chi_{2a-\beta} & \chi_{\beta} & \chi_{2a} & \cdots & \chi_{2a-(m-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\chi_{2a-(m-1)} & \chi_{2a-(m-1)} & \chi_{2a-(m-1)} & \cdots & \chi_{2a} \\
\end{bmatrix}, \quad a > \frac{\beta}{2}(m-1),
\]

\( \chi \) denotes the characteristic function of normal distribution.

\( \text{The notation } ZD \text{ stands for the quaternion conjugate transpose of the matrix } Z \text{ and reduces to the Hermitian transpose } Z^H \text{ when } Z \text{ is complex and the transpose } Z^T \text{ when } Z \text{ is real.} \)
Then $L \sim \mathcal{L}_m^{(\alpha)}(a)$ [8, 28].

**Remark 1.9.** The above definition differs from the one in [8] in that we have the additional factor of $\frac{1}{\beta}$. This way $W_m^{(\alpha)}(n, I)$ and $\mathcal{L}_m^{(\alpha)}(\frac{m}{n})$ have the same eigenvalue distributions for $\alpha = 2, 1, \frac{1}{2}$.

**Proposition 1.10** (Sutton’s $\beta$-Jacobi matrix model). Let $C \equiv Z^T Z$, where

$$
Z \equiv \begin{bmatrix}
    c_m & -s_m c'_{m-1} & \cdots \\
    & c_{m-1} & -s_{m-1} c'_{m-2} & \cdots \\
    & & \ddots & \ddots \\
    & & & c_1 & -s_1 c'_{0}
\end{bmatrix},
$$

with

$$
c_k \sim \sqrt{\text{Beta}(a - \frac{m-k}{\alpha}, b - \frac{m-k}{\alpha})}, \quad s_k = \sqrt{1 - c_k^2};
$$

$$
c'_k \sim \sqrt{\text{Beta}(\frac{k}{\alpha}, a + b - \frac{2m-k-1}{\alpha})}, \quad s'_k = \sqrt{1 - c'_k^2}.
$$

Then $C \sim J_m^{(\alpha)}(a, b)$ for any $\alpha > 0$ [31, Section 5].

1.7. **Connection between $\beta$–Laguerre and $\beta$–Jacobi.** If $\lambda_i$ are the eigenvalues of a $\beta$–Jacobi matrix distributed as $J_m^{(\alpha)}(a, b)$, then $\bar{\lambda}_i \equiv _{b \rightarrow \infty} \frac{\lambda_i}{1 - \lambda_i}$ are jointly distributed as (1.10), the eigenvalues of a Laguerre matrix distributed as $L_m^{(\alpha)}(a)$ as we now prove.

**Proposition 1.11.** Let $L_m^{(\alpha)}(a; \Lambda)$ be the joint eigenvalue density of the $\beta$–Laguerre ensemble (1.10) and $J_m^{(\alpha)}(a, b; \Lambda)$ be the joint eigenvalue density of the $\beta$–Jacobi ensemble (1.11). Then

$$
\lim_{b \rightarrow \infty} \frac{(ab)^m}{(ab)^m} \cdot J_m^{(\alpha)}(a, b; \Lambda(abI + \Lambda)^{-1}) = L_m^{(\alpha)}(a; \Lambda).
$$

**Proof.** Directly from (1.11):

$$
\frac{J_m^{(\alpha)}(a, b; \Lambda(abI + \Lambda)^{-1})}{(ab)^m} = \frac{1}{(ab)^ma} \cdot S_m^{(\alpha)}(a, b) \prod_{i=1}^{m} \lambda_i^{-a - \frac{m-1}{\alpha} - 1} (1 + \frac{1}{ab} \lambda_i)^{2-a-b} \prod_{j<k} |\lambda_j - \lambda_k|^{\frac{-2}{\alpha}}
$$

$$
= \frac{\pi \, \Gamma^{(m-1)}(\frac{1}{\alpha}) \Gamma^{(m)}(a+b)}{\alpha^{m-1} a! \, \prod_{i=1}^{m} \Gamma^{(m)}(\frac{m}{n}) \prod_{i=1}^{m} \Gamma^{(m)}(\frac{m}{n}) \prod_{i=1}^{m} \lambda_i^{-a - \frac{m-1}{\alpha} - 1} (1 + \frac{1}{ab} \lambda_i)^{2-a-b} \prod_{j<k} |\lambda_j - \lambda_k|^{\frac{-2}{\alpha}}.
$$

By taking the limit as $a_2 \rightarrow \infty$ we get (1.10). \qed
2. Identities involving \( pF_q^{(\alpha)}(a_{1:p}; b_{1:q}; X, Y) \)

Let \( X \) and \( Y \) be \( m \times m \) Hermitian matrices with eigenvalues \( x_1, \ldots, x_m \) and \( y_1, \ldots, y_m \), respectively, and let \( I \) be the \( m \times m \) identity matrix. Also define the \((m-1) \times (m-1)\) matrices \( \tilde{X} \equiv \text{diag}(x_2 - x_1, x_3 - x_1, \ldots, x_m - x_1) \) and \( \bar{Y} \equiv \text{diag}(y_2 - y_1, y_3 - y_1, \ldots, y_m - y_1) \).

2.1. Pure identities.

(2.1) \[ \text{tr}(X) = e^{\text{tr}(X)}; \]

(2.2) \[ \text{tr}(XY - I) = e^{\text{tr}(X + Y - 2)} - 1; \]

(2.3) \[ \text{det}(A-X) = (X < I); \]

(2.4) \[ \text{det}(I - X) = 1; \]

(2.5) \[ \text{det}(I - X) = \text{det}(I - X); \]

(2.6) \[ \text{det}(I - X) = \text{det}(I - X); \]

(2.7) \[ \text{det}(I - X) = \text{det}(I - X); \]

(2.8) \[ \text{det}(I - X) = \text{det}(I - X); \]

(2.9) \[ \text{det}(I - X) = \text{det}(I - X); \]

(2.10) \[ \text{det}(I - X) = \text{det}(I - X); \]

(2.11) \[ \text{det}(I - X) = \text{det}(I - X); \]

(2.12) \[ \text{det}(I - X) = \text{det}(I - X); \]

(2.13) \[ \text{det}(I - X) = \text{det}(I - X); \]

(2.14) \[ \text{det}(I - X) = \text{det}(I - X); \]

(2.15) \[ \text{det}(I - X) = \text{det}(I - X); \]

(2.16) \[ \text{det}(I - X) = \text{det}(I - X); \]

(2.17) \[ \text{det}(I - X) = \text{det}(I - X); \]

The references for the above identities are:

(2.1) [11, (13.3), p. 593] (also [26, p. 444] and [23, p. 262] for \( \alpha = 1 \) and \( \alpha = 2 \), respectively);

(2.2) [2, section 6], also [21] for \( \alpha = 2 \). Reproven in Theorem 5.1;

(2.3) [11, p. 593, (13.4)] (also [26, p. 444] and [23, p. 262] for \( \alpha = 1 \) and \( \alpha = 2 \), respectively);

(2.4) [21, (6.1)]; (both conjectures used appear to have been proven in literature);

(2.4) [21, (6.10)];

(2.5) [11, p. 596, (13.16)] (also [23, (6), p. 265] for the case \( \alpha = 2 \));

(2.7) New result, see Theorem 5.3. For \( m = 3, \alpha = 2 \) also proven by Bingham [3, Lemma 2.1];
This is a result of Raymond Kan, see Theorem 5.3;
(2.9) [11, (13.14), p. 594] (also [4, p. 24, (51)] for the case $\alpha = 2$);
(2.10) [13];
(2.11) [11, Proposition 13.1.6, p. 595]; see also [23, p. 265, (7)] for the case $\alpha = 2$;
(2.12) [11, Proposition 13.1.7, p. 596]. The condition $a$ or $b \in \mathbb{Z}_{\leq 0}$ implies that the series expansion for $2F_1^{(\alpha)}$ terminates.

(2.13) New result, see Theorem 5.2. See also [12, (5.13)] for $\alpha$;
(2.14) [25, (33), p. 281];
(2.15) [12, Theorem 4.2] (there is a typo in [12, (4.8)]: $\beta_n^{-1}$ should be $\beta_n$), see also [25, (34), p. 281];
(2.16) New result, see Theorem 5.4;
(2.17) [11, (13.5)]. Trivially implied by $\lim_{x \to \infty} (x)_\alpha^n \cdot x^{-|\alpha|} = 1$.

### 2. Integral identities

Let $X$ be an $m \times m$ real matrix and $dH$ be the invariant (normalized) measure on the orthogonal group $O(m)$. The identities (2.20) and (2.21) have complex equivalents, see, e.g., James [14]. The identities (2.18) and (2.19) are valid only for $\alpha \in \{\frac{1}{2}, 1, 2\}$. In those cases the matrix $Y$ is $m \times m$ real symmetric, complex Hermitian, or symplectic self-dual for $\alpha = 2, 1, \frac{1}{2}$, respectively.

Define

$$F_m^{(\alpha)}(a, c) = \frac{\Gamma(m)^{\alpha}}{\Gamma_m^{(\alpha)}(a)^{\alpha}(c - a)}.$$ 

Then

(2.18) $\int_0^1 e^{\text{tr}(XY)}(\det Y)^{a - \frac{m - 1}{\alpha} - 1} \det(I - Y)^{c - a - \frac{m - 1}{\alpha} - 1} dY$,

where $\Re(a) > \frac{m - 1}{\alpha}$, $\Re(c - a) > \frac{m - 1}{\alpha}$;

(2.19) $\int_0^1 e^{\text{tr}(XY)^{-1}}(\det Y)^{-b}(\det(I - Y)^{a - \frac{m - 1}{\alpha} - 1} \det(I - Y)^{c - a - \frac{m - 1}{\alpha} - 1} dY$,

where $\|X\| < 1$, $\Re(a) > \frac{m - 1}{\alpha}$, $\Re(c - a) > \frac{m - 1}{\alpha}$;

(2.20) $\int_0^1 e^{\text{tr}(XY)^{-a}} dH$;

(2.21) $\int_0^1 e^{\text{tr}(XY)^{-a}} dH$;

(2.22) $\int_0^1 e^{\text{tr}(XY)^{-a}} dH$;

(2.23) $\int_0^1 e^{\text{tr}(XY)^{-a}} dH$;

(2.24) $\int_0^1 e^{\text{tr}(XY)^{-a}} dH$;

(2.25) $\int_0^1 e^{\text{tr}(XY)^{-a}} dH$;

where $p = (m - 1)\beta + 1$, $c_m^{(\alpha)}(a) = \pi^{-\frac{(m-1)}{2\alpha}} m! \prod_{i=1}^{m} \Gamma\left(\frac{a + 1}{\alpha}\right)$, $c_m^{(\beta)} = c_m^{(2/\beta)}$, and $\Delta(X) = \prod_{i<j}(x_i - x_j)$.

The references for the above identities are:

(2.18) [23, p. 264, (4)] for $\alpha = 2$; [9] for $\alpha \in \{\frac{1}{2}, 1\}$;

(2.19) [23, p. 264, (5)] for $\alpha = 2$; [9] for $\alpha \in \{\frac{1}{2}, 1\}$;

(2.20) [24]:
(2.21) $U^{(\alpha)}(n)$ stands for the orthogonal group of size $n$ for $\alpha = 2$ and the unitary group of size $n$ for $\alpha = 1$ \cite{14, 27, p. 479, and 91, p. 488}, direct consequence of (2.20) via a limit, perhaps also due to Herz (1955) \cite{24};

(2.22) \cite{23, (1), p. 390};

(2.23) \cite{21, (6.20)};

(2.24) \cite{21, (6.21)}.

(2.25) \cite{21, Conjecture C, section 8}. Proven in \cite{2, section 6}, but the scaling constant in Macdonald is the correct one.

2.3. Identities involving the Jack function.

(2.26) \[ \sum_{\kappa \vdash k} C^{(\alpha)}_{\kappa}(X) = (\text{tr } X)^k; \]

(2.27) \[ J_n(x I_m) = (x \alpha)^{\kappa}(\frac{m}{\alpha})^{(\alpha)}_\kappa \]

(2.28) \[ J^{(\alpha)}_\kappa(X) = \frac{j_\kappa}{\alpha^{\kappa}|\kappa|!} C^{(\alpha)}_{\kappa}(X) = H^*_\kappa Q^{(\alpha)}_\kappa(X) = H^*_\kappa P^{(\alpha)}_\kappa(X); \]

(2.29) \[ J^{(\alpha)}_\kappa(X) = \text{det}(X) \cdot J^{(\alpha)}_{\kappa_1, \kappa_2-1, \ldots, \kappa_m-1}(X) \prod_{i=1}^m (m-i+1+\alpha(\kappa_i-1)), \quad (\kappa_m > 0) \]

(2.30) \[ \frac{C^{(\alpha)}_{\kappa}(I + X)}{C^{(\alpha)}_{\kappa}(I)} = \sum_{\sigma \leq \kappa} \left( \begin{array}{c} \kappa \\ \sigma \end{array} \right) \frac{C^{(\alpha)}_{\sigma}(X)}{C^{(\alpha)}_{\sigma}(I)}; \]

(2.31) \[ P^{(\alpha)}_{\hat{\mu}}(X) = \text{det}(X)^N P^{(\alpha)}_{\hat{\mu}}(X^{-1}), \quad \text{where } \mu_1 \leq N \text{ and } \hat{\mu}_i = N - \mu_{m+1-i}, \ i = 1, 2, \ldots, m. \]

For partitions $\kappa = (k)$ in only one part,

(2.32) \[ C^{(\alpha)}_k(I) = (\frac{m}{\alpha})_k \left( \frac{1}{\alpha} \right)_k^{-1}; \]

(2.33) \[ C^{(\alpha)}_k(X + tI) = C^{(\alpha)}_k(I) \sum_{s=0}^k t^{k-s} \left( \begin{array}{c} k \\ s \end{array} \right) \frac{C^{(\alpha)}_s(X)}{C^{(\alpha)}_s(I)} = \left( \frac{m}{\alpha} \right)_k \sum_{s=0}^k \frac{t^{k-s}}{(\frac{m}{\alpha})_s} \left( \begin{array}{c} k \\ s \end{array} \right) C^{(\alpha)}_s(X). \]

The references for the above identities are:

(2.26) This is the definition of the normalization $C^{(\alpha)}_\kappa(X)$ \cite{29, Theorem 2.3};

(2.27) \cite{29, Theorem 5.4};

(2.28) \cite{17, (16)], \cite{22, (10.22), p. 381}. Then $P^{(1)}_\kappa = Q^{(1)}_\kappa = s_\kappa$, the Schur function \cite{29, Proposition 1.2};

(2.29) \cite{29, Propositions 5.1 and 5.5};

(2.30) \cite{17, (33)}. This also serves as a definition for the generalized binomial coefficient $\binom{\alpha}{\sigma}$;

(2.31) \cite{21, (4.5)]. $\hat{\mu}$ is the complement of $\mu$ in the partition $(N^n)$;

(2.32) Directly from (2.27);

(2.33) Directly from (2.30).

2.4. Multivariate Gamma function identities.

(2.34) \[ \Gamma^{(\alpha)}_m(a) = \int_{A>0} e^{tr A} \det A^{\alpha-\frac{m+1}{\alpha}-1} (dA) \quad \text{for } \alpha = 1, 2, \]

\[ \text{where } A \text{ is Hermitian or symmetric positive definite, respectively; } \]

(2.35) \[ \frac{\Gamma^{(\alpha)}_m(x)}{\Gamma^{(\alpha)}_m(x - \frac{1}{\alpha})} = \frac{\Gamma(x)}{\Gamma(x - \frac{m}{\alpha})}. \]

The references for the above identities are:
Most densities and distributions in this paper are expressed in terms of the hypergeometric functions of the form

\[ f(x) = {}_pF_q^{(a)}(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{k=0}^{\infty} x^k \sum_{\kappa\vdash k} \frac{1}{k!} \cdot \frac{\prod_{j=1}^{p} (a_j^{(a)}) \cdot \prod_{j=1}^{q} (b_j^{(a)})}{\prod_{j=1}^{p} (b_j^{(a)}) \cdot \prod_{j=1}^{q} (b_j^{(a)})} \cdot C^{(a)}(x), \]

where \( F \) is a given matrix and

\[ c_k = \sum_{\kappa\vdash k} \frac{1}{k!} \cdot \frac{\prod_{j=1}^{p} (a_j^{(a)}) \cdot \prod_{j=1}^{q} (b_j^{(a)})}{\prod_{j=1}^{p} (b_j^{(a)}) \cdot \prod_{j=1}^{q} (b_j^{(a)})} \cdot C^{(a)}(x). \]

If \( f(x) \) is to be evaluated at many points \( x \), then it makes perfect sense to evaluate the coefficients \( c_k \) only once. Then \( f(x) \) can be evaluated at as many points as necessary at a negligible cost using Horner’s method.

For example, in order to evaluate \( s = {}_pF_q^{(a)}(a_1, \ldots, a_p; b_1, \ldots, b_q; X) \) we use the algorithms of [19]. We first choose a truncation limit \( M \) (the summation in (3.1) will be only for \( k \leq M \) ) and call

\[ [s, c] = \text{mhg}(M, a_1, \ldots, b_1, \ldots, x_{1:m}), \]

The array \( c \) returns the marginal sums for partitions of size \( 0, 1, \ldots, M \), so that \( s = \text{sum}(c) \).

When \( X = xI \) is a multiple of the identity, we call

\[ [s, c] = \text{mhg1}([M, K], a_1, \ldots, b_1, m, x). \]

The second output \( c = (c_1, c_2, \ldots, c_{m+1}) \) is such that \( s = c_1 + c_2 x + \cdots + c_{m+1} x^m \).

The optional parameter \( K \) is such that it truncates the summation to be over only those partitions whose parts do not exceed \( K \), i.e., \( k_i \leq K \). This is particularly useful in evaluating the expressions (4.3) and (4.9). The parameter \( K \) defaults to \( M \) if unspecified.

4. DISTRIBUTIONS OF THE EIGENVALUES OF THE RANDOM MATRIX ENSEMBLES

4.1. Wishart. Let \( A \sim W_m^{(a)}(n, \Sigma), \alpha = 2, 1, \frac{1}{2} \), be a real, complex, or quaternion Wishart matrix.

4.1.1. Extreme eigenvalues. For ease of notation, define the common expressions: \( t \equiv \frac{n-m+1}{\alpha} - 1 \),

\[ G_m^{(a)}(a) = \frac{\Gamma_m^{(a)}(m-1/\alpha + 1)}{\Gamma_m^{(a)}(a + m-1/\alpha + 1)}, \quad \text{and} \quad F_m^{(a)}(a) = \frac{\Gamma_m^{(a)}(m-1/\alpha + 1)}{\Gamma_m^{(a)}(a)}. \]

Then

\[ P(\lambda_{\max}(A) < x) = G_m^{(a)}(\frac{m}{\alpha}) \cdot \det(\frac{x}{\alpha} \Sigma^{-1})^{-\frac{m}{\alpha}} F_1^{(a)}(\frac{m}{\alpha} + 1; -\frac{x}{\alpha} \Sigma^{-1}) \]

\[ = G_m^{(a)}(\frac{m}{\alpha}) \cdot \det(\frac{x}{\alpha} \Sigma^{-1})^{-\frac{m}{\alpha}} e^{\text{tr}(-\frac{x}{\alpha} \Sigma^{-1})} F_1^{(a)}(\frac{m}{\alpha} + 1; -\frac{x}{\alpha} \Sigma^{-1} + 1; \frac{x}{\alpha} \Sigma^{-1}); \]

\[ P(\lambda_{\min}(A) < x) = 1 - e^{\text{tr}(\frac{x}{\alpha} \Sigma^{-1})} \sum_{\kappa \subseteq (m')} \frac{1}{|\kappa|!} \cdot C_\kappa^{(a)} \cdot (\frac{x}{\alpha} \Sigma^{-1}), \quad (t \in \mathbb{Z}_{\geq 0}), \]

\[ = 1 - F_m^{(a)}(a) \cdot \det(\frac{x}{\alpha} \Sigma^{-1})^{-t} e^{-\text{tr}(\frac{x}{\alpha} \Sigma^{-1})} \sum_{\kappa \subseteq (m')} \frac{1}{|\kappa|!} \cdot C_\kappa^{(a)} \cdot (\frac{x}{\alpha} \Sigma^{-1})^{-t}, \quad (t \in \mathbb{Z}_{\geq 0}). \]
For the density we differentiate (4.2) and (4.3) with respect to $x$ to obtain:

$$
dens_{\lambda_{\text{max}}}(A) = G_m(\frac{\alpha}{\nu}) \cdot x^{\frac{mn-1}{\nu}} \det(\alpha \Sigma)^{-\frac{1}{\nu}} e^{-\frac{1}{\nu} \text{tr}(-\frac{\alpha}{\nu} \Sigma^{-1})} \sum_{k=0}^{\infty} x^k \left( k + \frac{mn}{\alpha} \right) - \text{tr} \left( \frac{\alpha}{\nu} \Sigma^{-1} \right) c_k;
$$

$$
dens_{\lambda_{\text{min}}}(A) = e^{\text{tr}(-\frac{\alpha}{\nu} \Sigma^{-1})} \left( \text{tr} \left( \frac{1}{\nu} \Sigma^{-1} \right) + \sum_{k=1}^{m} x^{k-1} \left( \text{tr} \left( \frac{\alpha}{\nu} \Sigma^{-1} \right) - k \right) \sum_{\kappa\geq k, \kappa \leq \nu} \frac{C_{\kappa}^{(\nu)} \left( \frac{1}{\nu} \Sigma^{-1} \right)}{k!} \right), \quad (t \in \mathbb{Z}_{\geq 0}),
$$

where

$$
c_k \equiv \sum_{\kappa\geq k} \frac{(m-1)^{(\nu)}_{\kappa}}{(\alpha + m-1 + 1)_{\kappa}} \cdot \frac{C_{\kappa}^{(\nu)} \left( \frac{1}{\nu} \Sigma^{-1} \right)}{k!}.
$$

The expression (4.2) appears to be more numerically stable than (4.1) and is obtained using (2.6).

The expressions (4.1) and (4.3) come from [23, Theorems 9.7.1 and 9.7.4] for $\alpha = 2$ and from [26, Corollaries 3.3 and 3.5] for $\alpha = 1$; (4.4) is a new result (see (2.16) and Theorem 5.4).

4.1.2. **Trace**.

$$
dens_{\text{tr}_A}(x) = \det(\frac{1}{\nu} \Sigma^{-1}) \pi e^{-\frac{1}{\nu} \sum_{k=0}^{\infty} \frac{1}{\Gamma(m+1)} \cdot \frac{1}{\nu^2} \left( \frac{1}{\alpha} \right) \sum_{\kappa\geq k} \frac{(m-1)^{(\nu)}_{\kappa}}{(\alpha + m-1 + 1)_{\kappa}} \cdot \frac{1}{k!} \cdot C_{\kappa}^{(\nu)} I - \Sigma^{-1},
$$

where $z$ is arbitrary. Muirhead [23, p. 341] suggests the value $z = 2\sigma_1\sigma_m/(\sigma_1 + \sigma_m)$, where $\sigma_1 \geq \cdots \geq \sigma_m \geq 0$ are the eigenvalues of $\Sigma$.

The second sum in (4.5) is the marginal sum of $F_0^{(\alpha)}(\frac{1}{\nu}; I - \Sigma^{-1})$ and the truncation of (4.5) for $|\kappa| \leq M$ can be computed using $\text{mhg}$ as follows. If $s$ is the vector of eigenvalues of $\Sigma$, these marginal sums (for say $k = 0$ through $M$) are returned in the variable $c$ by the call:

$$
[f, c] = \text{mhg}(M, \alpha, n/\alpha, \{1-z/s\}),
$$

making the remaining computation trivial.

Empirically the trace can be generated as $\frac{2}{\nu} \sum_{i=1}^{m} t_i \sigma_i$, where $t_i \sim \chi^2_{2n/\alpha}$ are i.i.d.

The density of the trace of the Wishart matrix was obtained by Muirhead [23, p. 341] in the real, $\alpha = 2$, case. For other $\alpha > 0$ this is a new result, see Theorem 6.2.

[**Moments of det of Wishart**] [23, p. 115].

4.2. **$\beta$-Laguerre**. The distributions of the extreme eigenvalues of a $\beta$-Laguerre matrix $L \sim \mathcal{L}_m^{(\alpha)}$ are given by (4.1)–(4.4), where $n = \alpha \Sigma = I$ [8, Theorem 10.2.1, p. 146], [20].

For $t \equiv a - \frac{m-1}{\alpha} - 1 \in \mathbb{Z}_{\geq 0}$ the density of $\lambda_{\text{min}}(L)$ is known from [8, Theorem 10.1.1, p. 147] (note the $\frac{1}{\beta}$ factor in our definition of a $\beta$-Laguerre matrix), but the scaling factor is a new result (see Theorem 6.3):

$$
dens_{\lambda_{\text{min}}(L)}(x) = \frac{m}{\alpha} \cdot \frac{\Gamma_m \left( \frac{m-1}{\alpha} + 1 \right)}{\Gamma_m^{(\alpha)}(a)} \cdot \left( \frac{x}{\alpha} \right)^{tm} e^{-\frac{m-1}{\alpha}} \cdot 2F_0^{(\alpha)} \left( \frac{m}{\alpha} + 1, -t; -\frac{a}{\alpha} I_{m-1} \right).
$$

4.3. **Jacobi**. Let $C \sim \mathcal{J}_m^{(\alpha)}(a, b)$ be a $\beta$-Jacobi matrix.

Define $t \equiv b - \frac{m-1}{\alpha} - 1$ and

$$
D_{m,a,b}^{(\alpha)} = \frac{\Gamma_m^{(\alpha)}(a + b) \Gamma_m \left( \frac{m-1}{\alpha} + 1 \right)}{\Gamma_m^{(\alpha)}(a + \frac{m-1}{\alpha} + 1) \Gamma_m^{(\alpha)}(b)}.
$$

Then

$$
P(\lambda_{\text{max}}(C) < x) = D_{m,a,b}^{(\alpha)}, \quad x^{ma} \cdot 2F_1^{(\alpha)}(a, -t; a + \frac{m-1}{\alpha} + 1; xI);
$$

$$
dens_{\lambda_{\text{max}}(C)}(x) = D_{m,a,b}^{(\alpha)}, \quad ma(1-x)^t \cdot 2F_1^{(\alpha)}(a - \frac{1}{\alpha}, -t; a + \frac{m-1}{\alpha} + 1; xI_{m-1}).
$$
When \( t \in \mathbb{Z}_{\geq 0} \), (2.12) gives alternative (and seemingly numerically much more stable) expressions:

(4.9) \[ P(\lambda_{\text{max}}(C) < x) = x^{ma} \sum_{k=0}^{mt} \sum_{\kappa+\kappa_\kappa \leq t} \frac{(a)_{\kappa}^{(\alpha)}}{k!} (1 - x)^k; \]

(4.10) \[ \text{dens} \lambda_{\text{max}}(C)(x) = H \cdot (1 - x)^{x^{ma - 1}} 2F_1(a; -t; -t - \frac{2}{a}; x I_{m-1}), \]

where

\[ H = \Gamma(b - \frac{a-1}{a}) \Gamma(b + \frac{1}{a}) \Gamma(1) \]

\[ = \Gamma(b - \frac{a-2}{a}) \Gamma(t + 1) \Gamma(\frac{m}{a}) \Gamma(a) \]

The density and distribution of the smallest eigenvalue follow easily from the corresponding expressions for the largest ones. If \( C_{a,b} \sim J_m^{(\alpha)}(a,b) \), then \( I - C_{a,b} \sim J_m^{(\alpha)}(b,a) \). Therefore

(4.11) \[ P(\lambda_{\text{min}}(C^{a,b}) < x) = 1 - P(\lambda_{\text{max}}(C^{b,a}) < 1 - x) \]

(4.12) \[ \text{dens} \lambda_{\text{min}}(C^{a,b})(x) = \text{dens} \lambda_{\text{max}}(C^{b,a})(1 - x). \]

The results in this section are due to Constantine [7, (61)] ((4.7) for \( \alpha = 2 \)), Absil, Edelman, and Koev [1], ((4.8) for \( \alpha = 2 \)), and Dumitriu and Koev [9] for the generalization to any \( \alpha \). The \( \alpha = 2 \) version of (4.9) is also implied by [23, (37), p. 483].

[Density is unproven anywhere!]

4.3.1. Trace. [In progress]

The Laplace transform of the distribution of the trace of Jacobi is given in Muirhead [23, (29), p. 479]:

\[ G(t) = 1_{F_1}^{(2)} \left( \frac{a}{2}; \frac{m+\alpha}{2}; -tI \right) \]

\[ = \sum_{k=0}^{\infty} \frac{(-1)^k t^k}{k!} \sum_{\kappa+\kappa \leq t} \frac{(\kappa)_{2}^{(2)}}{2^k} C_{\kappa}^{(\alpha)}(I). \]

[Moments of the determinant of noncentral Jacobi] [23, p. 455].

4.4. Numerical experiments. Numerical experiments for the distributions in the previous sections are in Figure 1.

5. New results

We now prove (2.2), (2.7), (2.8), (2.16), (2.13), and (4.5). Identity (2.8) (identity (5.6) below) is due to Raymond Kan.

Theorem 5.1. With the notation from Section 2.1 the identity

(5.1) \[ _0F_0^{(\alpha)}(X,Y) = e^{tr X} _0F_0^{(\alpha)}(X,Y - I) = e^{y_1^{tr X + x_1 tr Y - m_1 Y_1} I_1 F_1^{(\alpha)} \left( \frac{m-1}{\alpha}; \frac{m}{\alpha}; \bar{X}, \bar{Y} \right)} \]

holds for any \( \alpha > 0 \).

Proof. For the first part, following Kaneko [17, Section 5.1], denote by \( g_{\mu,\lambda}^{(\alpha)} \) the (normalized) coefficient of \( J_{\mu}^{(\alpha)}(X) \) in the expansion

\[ J_{\mu}^{(\alpha)}(X) J_{\lambda}^{(\alpha)}(X) = \sum_{\kappa} j_{\lambda}^{-1} g_{\mu,\lambda}^{(\alpha)} J_{\kappa}(X), \]

where \( j_{\lambda} \) is defined in (1.1). Since \( g_{\mu,\lambda}^{(\alpha)} = 0 \) unless \( |\mu| + |\lambda| = |\kappa| \) [29, Corollary 6.4], the above along with (2.28) imply

(5.2) \[ C_{\mu}^{(\alpha)}(X) C_{\lambda}^{(\alpha)}(X) = \frac{\mu! |\lambda|!}{j_{\mu} j_{\lambda}} \sum_{\kappa} \frac{1}{|\kappa|!} g_{\mu,\lambda}^{(\alpha)} C_{\kappa}(X). \]
Also from Kaneko [17, Proposition 2],

\[
\binom{\kappa}{\mu} = \sum_{\lambda} \frac{1}{j_\lambda j_\mu} g_{\mu \lambda}^\kappa.
\]

Since \( \binom{\emptyset}{\mu} = 0 \) unless \( \mu \subseteq \kappa \), the summation in (2.30) can be thought of as being over all partitions \( \sigma \). We use this fact along with (5.3) to transform the left hand side of (5.1):

\[
\begin{align*}
\mathcal{O}F_0^{(\alpha)}(X, Y + I) & = \sum_{\kappa} \frac{1}{|\kappa|!} \frac{C_{\kappa}(X)C_{\kappa}(Y + I)}{C_{\kappa}(I)} \\
& = \sum_{\kappa} \frac{1}{|\kappa|!} C_{\kappa}(X) \sum_{\mu} \binom{\kappa}{\mu} \frac{C_{\mu}(Y)}{C_{\mu}(I)} \\
& = \sum_{\kappa} \frac{1}{|\kappa|!} C_{\kappa}(X) \sum_{\mu} \sum_{\lambda} \frac{g_{\mu \lambda}^\kappa}{j_\lambda j_\mu} \frac{C_{\mu}(Y)}{C_{\mu}(I)}.
\end{align*}
\]

(5.4)
Next, we transform the right hand side of (5.1) using (2.1) and (5.2):

\[ 0 F^0_0(X, Y) = e^{\text{tr} X} \sum_{\mu} \frac{1}{|\mu|!} \frac{C^{(\alpha)}_\mu(Y) C^{(\alpha)}_\mu(X)}{C^{(\alpha)}_\mu(I)} \sum_{\lambda} \frac{1}{|\lambda|!} C^{(\alpha)}_\lambda(X) \]

\[ = \sum_{\mu} \sum_{\lambda} \frac{C^{(\alpha)}_\mu(Y)}{C^{(\alpha)}_\mu(I)} \frac{1}{|\mu|!|\lambda|!} C^{(\alpha)}_\mu(X) C^{(\alpha)}_\lambda(X) \]

\[ = \sum_{\mu} \sum_{\lambda} \frac{C^{(\alpha)}_\mu(Y)}{C^{(\alpha)}_\mu(I)} \frac{1}{|\mu|!|\lambda|!} \sum_{\kappa} g^{\kappa}_\mu C^{(\alpha)}_\kappa(X) \frac{1}{|\kappa|!}. \]

which is the same as (5.4), thus proving (5.1).

For the second part, let \( I_k \) be the identity matrix of size \( k \). Since \( C^{(\alpha)}_\kappa(X - x_1 I_m) = C^{(\alpha)}_\kappa(\bar{X}) \) and analogously for \( Y \), from (2.2)

\[ 0 F^0_0(X, Y) = e^{\text{tr} X} \cdot 0 F^0_0(X, Y - y_1 I_m) \]

\[ = e^{\text{tr} X} + \sum_{k=0}^{\infty} \frac{1}{k!} C^{(\alpha)}_\kappa(X - x_1 I_m) C^{(\alpha)}_\kappa(Y - y_1 I_m) \]

\[ = e^{\text{tr} X} Y^{-x_1 I_m} \cdot \sum_{k=0}^{\infty} \frac{1}{k!} C^{(\alpha)}_\kappa(X - x_1 I_m) C^{(\alpha)}_\kappa(Y - y_1 I_m) \]

\[ = e^{\text{tr} X} Y^{-x_1 I_m} \cdot \sum_{k=0}^{\infty} \frac{1}{k!} C^{(\alpha)}_\kappa(X - x_1 I_m) C^{(\alpha)}_\kappa(Y - y_1 I_m) \]

\[ = e^{\text{tr} X} Y^{-x_1 I_m} \cdot 1 F^1_1(m - 1) \cdot \frac{m - 1}{m - 1} \cdot \bar{X}, \bar{Y}. \]

The next two theorems use the fact that \( \left( \frac{1}{\alpha} \right)_\kappa \) of 0 for partitions \( \kappa \) in more than one part.

**Theorem 5.2.**

\[ p F^1_q(\alpha) \left( \frac{1}{\alpha}, a_{2:p}; b_{1:q}; X I \right) = p F^1_q \left( \frac{m}{\alpha}, a_{2:p}; b_{1:q}; x I \right). \]

**Proof.** Using (2.32),

\[ p F^1_q(\alpha) \left( \frac{1}{\alpha}, a_{2:p}; b_{1:q}; x I \right) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\left( \frac{1}{\alpha} \right)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} C_k^{(\alpha)}(x) \]

\[ = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\left( \frac{1}{\alpha} \right)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \cdot \frac{m}{\alpha} \]

\[ = p F^1_q \left( \frac{m}{\alpha}, a_{2:p}; b_{1:q}; x I \right). \]

**Theorem 5.3.** Let the matrices \( X \) and \( I \) be \( m \times m \). Let \( t \) and \( n \) be real numbers and let \( \alpha > 0 \). The following identities hold:

\[ 1 F^1_1 \left( \frac{1}{\alpha}, \frac{m}{\alpha}; X + t I \right) = 1 F^1_1 \left( \frac{1}{\alpha}, \frac{m}{\alpha}; X \right) \cdot e^t; \]

\[ 2 F^1_1 \left( \frac{1}{\alpha}, n; \frac{m}{\alpha}; X \right) = (1 - t)^n 2 F^1_1 \left( \frac{1}{\alpha}, n, \frac{m}{\alpha}, (1 - t)X + t I \right), \quad t \neq 1. \]
Proof. We transform the left hand side of (5.5) using (2.32) and (2.33):

\[ 1F_1^a\left(\frac{1}{\alpha}; \frac{m}{\alpha}; X + tI\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \left(\frac{1}{\alpha}\right)_k \cdot C_k^a(X + tI) \]

\[ = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot C_k^a(X + tI) \]

\[ = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{s=0}^{k} t^{k-s} \cdot \left(\frac{k}{s}\right) \cdot \frac{C_s^a(X)}{C_s^a(I)} \]

\[ = \sum_{i=0}^{\infty} \frac{C_i^a(X)}{C_i^a(I)} \sum_{j=i}^{\infty} \frac{t^j}{j!} \cdot \left(\frac{j}{i}\right) \]

\[ = \sum_{i=0}^{\infty} \frac{1}{i!} \cdot \left(\frac{1}{\alpha}\right)_k \cdot C_k^a(X) \cdot \sum_{j=i}^{\infty} \frac{t^j}{(j-i)!} \]

\[ = 1F_1^a\left(\frac{1}{\alpha}; \frac{m}{\alpha}; X\right) \cdot e^t. \]

We use (2.33) again to obtain (5.6):

\[ (1 - t)^n 2F_1^a\left(\frac{1}{\alpha}, n; \frac{m}{\alpha}; (1 - t)X + tI\right) = (1 - t)^n \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \left(\frac{1}{\alpha}\right)_k \cdot C_k^a((1 - t)X + tI) \]

\[ = (1 - t)^n \sum_{k=0}^{\infty} \frac{(n)_k}{k!} \sum_{s=0}^{k} t^{k-s} \cdot \left(\frac{k}{s}\right) \cdot \frac{C_s^a((1 - t)X)}{C_s^a(I)} \]

\[ = (1 - t)^n \sum_{i=0}^{\infty} \frac{C_i^a(X)}{C_i^a(I)} (1 - t)^i \sum_{j=i}^{\infty} \frac{t^j}{j!} \cdot \left(\frac{j}{i}\right) \cdot (n)_j \]

\[ = \sum_{i=0}^{\infty} \frac{1}{i!} \cdot \frac{C_i^a(X)}{C_i^a(I)} (n)_i \cdot (1 - t)^{n+i} \sum_{l=0}^{\infty} \frac{t^l}{l!} (n+i)_l \]

\[ = 2F_1^a\left(\frac{1}{\alpha}, n; \frac{m}{\alpha}; X\right). \]

\[ \square \]

Next, we prove that (2.16) is true for any \( X \).

**Theorem 5.4.** For \( r \in \mathbb{Z}_{\geq 0} \):

\[ 2F_0^a\left(\frac{m-1}{\alpha} + 1, -r; X\right) = \frac{\Gamma_m^a\left(\frac{m-1}{\alpha} + 1 + r\right)}{\Gamma_m^a\left(\frac{m-1}{\alpha} + 1\right)} \cdot (\det(-X))^r \cdot \sum_{k=0}^{mr} \sum_{k'k; k' \leq r} \frac{C_k^a(-X^{-1})}{k!}. \]

**Proof.** From (2.11),

\[ 2F_1^a(a, -r; c; I - X) = 2F_1^a(a - c, -r; c; I - X^{-1}) \cdot (\det X)^r. \]
For \( r \in \mathbb{Z}_{\geq 0} \) we have from (2.12):
\[
\frac{\Gamma_m^{(a)}(c-a+r)}{\Gamma_m^{(a)}(c-a)} \cdot {}_2F_1^{(a)}(a,-r;a-r+1+\frac{m-1}{\alpha} - c;X) = \frac{\Gamma_m^{(a)}(c-a+r)}{\Gamma_m^{(a)}(a)} \cdot {}_2F_1^{(a)}(c-a,-r;a-r+1+\frac{m-1}{\alpha};X^{-1}) \cdot (\det X)^r.
\]

Cancelling terms on both sides we get:
\[
\frac{\Gamma_m^{(a)}(c-a+r)}{\Gamma_m^{(a)}(c-a)} \cdot {}_2F_1^{(a)}(a,-r;a-r+1+\frac{m-1}{\alpha} - c;X) = \frac{\Gamma_m^{(a)}(a)}{\Gamma_m^{(a)}(r)} \cdot {}_2F_1^{(a)}(c-a,-r;a-r+1+\frac{m-1}{\alpha};X^{-1}) \cdot (\det X)^r.
\]

Plugging in \( a = \frac{m-1}{\alpha} + 1 \), replacing \( X \) by \(-cX\), and then moving a \( e^{nr} \) factor to the left we get:
\[
\frac{\Gamma_m^{(a)}(c-a+r)}{\Gamma_m^{(a)}(c-a)} \cdot {}_2F_1^{(a)}(m-1,1,-r;-r-cX) = \frac{\Gamma_m^{(a)}(m-1,1+1)}{\Gamma_m^{(a)}(m-1,1+1)} \cdot {}_2F_1^{(a)}(c-a,m-1,-r;-r;\frac{1}{2}X^{-1}) \cdot (\det(-X))^r.
\]

By letting \( c \to \infty \) and taking limits on both sides using (2.17) and \( \Gamma(z+1) = z\Gamma(z) \) to conclude that
\[
\lim_{c \to \infty} \frac{\Gamma_m^{(a)}(c-a+r)}{\Gamma_m^{(a)}(c-a)} \cdot e^{nr} = 1,
\]
we get (5.7).

\[
\square
\]

6. Eigenvalue distributions

**Theorem 6.1.** For the extreme eigenvalues of a Wishart matrix \( A \sim \mathcal{W}_m^{(a)}(n,\Sigma) \) we have:
\[
\begin{align*}
P(\lambda_{\text{max}}(A) < x) &= \frac{\Gamma_m^{(a)}(\frac{m-1}{\alpha} + 1)}{\Gamma_m^{(a)}(n+\frac{m-1}{\alpha} + 1)} \frac{1}{\det \left( \frac{2}{\alpha} \Sigma^{-1} \right)} \frac{1}{\Gamma(\frac{m}{\alpha})} \cdot {}_1F_1^{(a)} \left( \frac{n+\frac{m-1}{\alpha} + 1}{\alpha}; -\frac{2}{\alpha} \Sigma^{-1} \right) \\
\int_{[0,x]^n} \prod_{i=1}^m l_{\alpha_i} \prod_{j<k} |\lambda_k - \lambda_j|^{\frac{2}{\alpha}} \cdot {}_0F_0^{(a)} \left( -\frac{1}{\alpha} l, \Sigma^{-1} \right) dL \\
= & x^{\frac{mn}{\alpha}} \frac{(\det \Sigma)^{\frac{n}{\alpha}}}{\Gamma_m^{(a)}(\frac{n}{\alpha})} \cdot \frac{m!}{\alpha} \frac{\prod_{i=1}^m \Gamma \left( \frac{1}{\alpha} + i \right)}{\prod_{j<k} \Gamma \left( \frac{1}{\alpha} + j \right)} \cdot \frac{\prod_{i=1}^m \Gamma \left( \frac{m}{\alpha} - \frac{n+i-1}{\alpha} \right)}{\prod_{i=1}^m \Gamma \left( \frac{m}{\alpha} - \frac{n+i-1}{\alpha} \right)} \cdot {}_0F_0^{(a)} \left( Z, -\frac{2}{\alpha} \Sigma^{-1} \right) dZ
\end{align*}
\]

Simplifying the last expression using (1.3) and (1.9) we obtain (6.1).
For the second part, analogously, the change of variables is \( L = x(I + Z) \) with \( dL = x^m dZ \):

\[
P(\lambda_{\max} > x) = \frac{\det \Sigma}{\mathcal{K}_m^{(\alpha)}(\frac{\alpha}{2})} \int_{|x| = \infty}^m \prod_{i=1}^m \lambda_i \prod_{j < k} |\lambda_k - \lambda_j|^{\frac{\alpha}{2}} \cdot \alpha_0 F_0^{(\alpha)}(-\frac{1}{\alpha} L, \Sigma^{-1}) dL
\]

(6.3)

\[
x^{\frac{m}{\alpha}} \frac{\det \Sigma}{\mathcal{K}_m^{(\alpha)}(\frac{\alpha}{2})} \int_{\mathbb{R}^m_+} \det(I + Z)^t \prod_{j < k} |z_k - z_j|^{\frac{\alpha}{2}} \cdot \alpha_0 F_0^{(\alpha)}(I + Z, -\frac{\alpha}{\alpha} \Sigma^{-1}) dZ
\]

(6.4)

\[
e^{\text{tr}(-\frac{\alpha}{\alpha} \Sigma^{-1}) \frac{\det(x \Sigma^{-1})^{\frac{\alpha}{2}}}{\mathcal{K}_m^{(\alpha)}(\frac{\alpha}{2})} \int_{\mathbb{R}^m_+ |\kappa| \leq (m^t)} \frac{(-t)^{\kappa}(Z)}{|\kappa|!} \prod_{j < k} |z_k - z_j|^{\frac{\alpha}{2}} \cdot \alpha_0 F_0^{(\alpha)}(Z, -\frac{\alpha}{\alpha} \Sigma^{-1}) dZ
\]

(6.5)

\[
e^{\text{tr}(-\frac{\alpha}{\alpha} \Sigma^{-1}) \frac{\det(x \Sigma^{-1})^{\frac{\alpha}{2}}}{\mathcal{K}_m^{(\alpha)}(\frac{\alpha}{2})} \int_{\mathbb{R}^m_+ |\kappa| \leq (m^t)} \frac{(-1)^{|\kappa|}(-t)^{\kappa}}{|\kappa|!} \prod_{j < k} C_\kappa^{(\alpha)}(Z) \prod_{j < k} |z_k - z_j|^{\frac{\alpha}{2}} \cdot \alpha_0 F_0^{(\alpha)}(-Z, -\frac{\alpha}{\alpha} \Sigma^{-1}) dZ
\]

(6.6)

which along with (2.16) implies (6.2). We used (2.2) and (2.3) to go from (6.3) to (6.4) and (2.25) to go from (6.5) to (6.6).

The following theorem generalizes Theorem 8.3.4 in Muirhead [23, p. 339] to all \( \alpha > 0 \).

**Theorem 6.2.** If \( A \sim \mathcal{W}_m^{(\alpha)}(n, \Sigma) \), then

\[
dens_{\text{tr} A}(x) = \det(\frac{\alpha}{\alpha} \Sigma^{-1})^{\frac{\alpha}{2}} e^{-\frac{\alpha}{\alpha} \Sigma^{-1}} \sum_{k=0}^\infty \frac{1}{\Gamma(\frac{nm}{\alpha} + k)} \cdot \frac{1}{\alpha \lambda} \left(\frac{\alpha}{\alpha}\right)^{k-1} \sum_{k>\kappa} \left(\frac{\alpha}{\alpha}\right)_\kappa \cdot \frac{1}{k!} \cdot C_\kappa^{(\alpha)}(I - \lambda \Sigma^{-1}),
\]

where \( \lambda \) is arbitrary.

**Proof.** We follow the argument in Muirhead [23, Theorem 8.3.4, p. 339]. From (1.8), the moment generating function of \( \text{tr} A \) is:

\[
\phi(t) = E[e^{\text{tr}(tL)}] = \frac{\det \Sigma}{\mathcal{K}_m^{(\alpha)}(\frac{\alpha}{2})} \int_{\mathbb{R}^m_+} e^{-\text{tr}(-tL)} \prod_{i=1}^m \lambda_i^{n-m-1} \prod_{j < k} |\lambda_k - \lambda_j|^{\frac{\alpha}{2}} \cdot \alpha_0 F_0^{(\alpha)}(-tL, \frac{\alpha}{\alpha} \Sigma^{-1}) dL
\]

making a change of variables \( L \to -tL \) and using (2.23) we obtain

\[
= \frac{\det \Sigma}{\mathcal{K}_m^{(\alpha)}(\frac{\alpha}{2})} \cdot \frac{(-t)^{n-m}}{c_m^{(\alpha)}(\frac{\alpha}{2}) \mathcal{K}_m^{(\alpha)}(\frac{\alpha}{2})} \cdot \alpha_0 F_0^{(\alpha)}(\frac{\alpha}{\alpha} \Sigma^{-1})
\]

\[
= \det(-t \alpha \Sigma)^{-\frac{\alpha}{2}} \cdot \det(I - \frac{1}{\alpha} \Sigma^{-1})^{-\frac{\alpha}{2}}
\]

\[
= \det(I - \alpha \Sigma)^{-\frac{\alpha}{2}}.
\]
For $0 < \lambda < \infty$ write
\[
\phi(t) = \det(I - t\alpha \Sigma)^{-\frac{m}{\alpha}}
\]
\[
= (1 - t\alpha \lambda)^{-\frac{m}{\alpha}} \det(\lambda^{-1} \Sigma)^{-\frac{m}{\alpha}} \det \left[ I - \frac{1}{1 - t\alpha \lambda} (I - \lambda \Sigma^{-1}) \right]^{-\frac{m}{\alpha}}
\]
\[
= (1 - t\alpha \lambda)^{-\frac{m}{\alpha}} \det(\lambda^{-1} \Sigma)^{-\frac{m}{\alpha}} \cdot \frac{1}{1 - t\alpha \lambda} (I - \lambda \Sigma^{-1})
\]
(6.7)
\[
= (1 - t\alpha \lambda)^{-\frac{m}{\alpha}} \det(\lambda^{-1} \Sigma)^{-\frac{m}{\alpha}} \sum_{k=0}^{\infty} \frac{1}{(1 - t\alpha \lambda)^k} \sum_{n+k} \frac{1}{k!} (\frac{\lambda}{\alpha})^n C_r^{(\alpha)}(I - \lambda \Sigma^{-1}),
\]
where $t$ and $\lambda$ are such that $\|I - \lambda \Sigma^{-1}\| < |1 - t\alpha \lambda|$, so that the $1_F^{(\alpha)}$ function above converges.

Since $(1 - t\alpha \lambda)^{-\gamma}$ is the moment-generating function of the gamma distribution with parameters $r$ and $\alpha \lambda$ and density function
\[
gr_{r, \alpha \lambda}(u) = \frac{e^{-\frac{u}{\alpha \lambda} - 1}}{(\alpha \lambda)^r \Gamma(r)} (u > 0),
\]
the moment-generating function (6.7) can be inverted term by term to obtain the density of $\text{tr} A$:
\[
\text{dens}_{\text{tr} A}(x) = \det \left( \frac{\lambda}{\alpha} \Sigma^{-1} \right)^{-\frac{m}{\alpha}} e^{-\frac{x}{\lambda}} \sum_{k=0}^{\infty} \frac{1}{\Gamma(\frac{m}{\alpha} + k)} \cdot \frac{1}{\alpha \lambda} (\frac{\alpha}{\lambda})^k \sum_{n+k} \frac{1}{k!} (\frac{\lambda}{\alpha})^n C_r^{(\alpha)}(I - \lambda \Sigma^{-1}),
\]
where $\alpha$ is arbitrary and, as in Muirhead [23, p. 341], can be chosen as $\lambda = \frac{2X'X''}{X'}$, where $X'$ and $X''$ are the largest and the smallest eigenvalues of $\Sigma$, respectively. \(\square\)

We now derive the scaling constant in (4.6); the proportionality result is from [8, Theorem 10.1.1, p. 146].

**Theorem 6.3.** For the smallest eigenvalue of a $\beta$-Laguerre matrix $A \sim \mathcal{L}_m^{(\alpha)}(a)$, where $r \equiv a - \frac{m-1}{\alpha} - 1 \in \mathbb{Z}_{\geq 0}$, we have:

\[
P(\lambda_{\text{min}}(A) < y) = 1 - \frac{\Gamma_m^{(\alpha)}(\frac{m-1}{\alpha} + 1)}{\Gamma_m^{(\alpha)}(a)} \cdot e^{-\frac{m-1}{\alpha} \cdot \left( \frac{y}{\alpha} \right)^{mr} \cdot 2F_1^{(\alpha)}(\frac{m-1}{\alpha} + 1, -r; -\frac{a}{y})};
\]
\[
\text{dens}_{\lambda_{\text{min}}(A)}(y) = \frac{\Gamma_m^{(\alpha)}(\frac{m-1}{\alpha} + 1)}{\Gamma_m^{(\alpha)}(a)} \cdot e^{-\frac{m-1}{\alpha} \cdot \left( \frac{y}{\alpha} \right)^{mr} \cdot 2F_1^{(\alpha)}(\frac{m-1}{\alpha} + 1, -r; -\frac{a}{y}I_{m-1})}.
\]

**Proof.** Let $C \sim \mathcal{S}_m^{(\alpha)}(a, b)$.

From (4.7), (4.11), and the identity (2.11), we have:
\[
P(\lambda_{\text{min}}(C) < x) = 1 - D_m^{(\alpha)}(b - a + \frac{m-1}{\alpha} + 1; b + \frac{m-1}{\alpha} + 1; (1 - x)I)
\]
\[
= 1 - D_m^{(\alpha)}(b - a + \frac{m-1}{\alpha} + 1; b + \frac{m-1}{\alpha} + 1; (1 - x)I)
\]
Substituting $y = ab \frac{x^{\frac{1}{2}}}{1-x}$ we get
\[
P(\lambda_{\text{min}}(abc)(I - C)^{-1}) < y) = 1 - D_m^{(\alpha)}(\frac{m-1}{\alpha} + 1, -r; b + \frac{m-1}{\alpha} + 1; -\frac{by}{x}I).
\]

Using Proposition 1.11 and taking the limit as $b \to \infty$ we obtain the distribution of $\lambda_{\text{min}}$.

With the density we take the same approach. From (4.8), (4.12), and the identity (2.11) we have:
\[
\text{dens}_{\lambda_{\text{min}}(C)}(x) = mb \cdot D_m^{(\alpha)}(b - \frac{1}{\alpha} - r; b + \frac{m-1}{\alpha} + 1; (1 - x)I_{m-1})
\]
\[
= mb \cdot D_m^{(\alpha)}(b - \frac{1}{\alpha} - r; b + \frac{m-1}{\alpha} + 1; (1 - x)I_{m-1})
\]
Substituting $y = ab \frac{x}{1-x}$ we get:

$$
\text{dens}_{\lambda_{\text{min}}(\alpha b (I-C)^{-1})}(y) = \frac{m}{\alpha} \cdot \frac{\Gamma_m(a-b) \cdot b^{mr}}{\Gamma_m(a)} (1 + \frac{\alpha y}{b})^{-mr} \cdot \frac{\Gamma_m(b+\frac{m-1}{\alpha}+1)}{\Gamma_m(b)} (\frac{y}{a})^{mr} (1 + \frac{\alpha y}{b})^{-mr} \\
\times 2F_1(a+1, -r; b + \frac{m-1}{\alpha}; b - \frac{2}{y} I_{m-1}).
$$

Using Proposition 1.11 and taking the limit as $b \to \infty$, we obtain the density result. \qed

7. Largest Eigenvalue Asymptotics

In this section we present numerical evidence for the convergence of the largest eigenvalues of the Wishart and Jacobi ensembles to the Tracy–Widom limits.

7.1. The largest eigenvalue of the Wishart ensemble. Let $A \sim \mathcal{W}_m(n,I)$. If $(m,n) \to \infty$ in such a way that $n/m \to \gamma \geq 1$, then [15, Theorem 1.1]

$$
\frac{\lambda_{\text{max}}(A) - \mu}{\sigma} \xrightarrow{d} W_1,
$$

where

$$
\mu = (\sqrt{n-1} + \sqrt{m})^2, \quad \sigma = (\sqrt{n-1} + \sqrt{m}) \left( \frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{m}} \right)^{1/3}.
$$

Johnstone suggests a slight modification of the rescaling and recentering constants, which yields faster convergence (see El Karoui [18]):

$$
\tilde{\mu} = (\sqrt{n-\frac{1}{2}} + \sqrt{m-\frac{1}{2}})^2, \quad \tilde{\sigma} = (\sqrt{n-\frac{1}{2}} + \sqrt{m-\frac{1}{2}}) \left( \frac{1}{\sqrt{n-\frac{1}{2}}} + \frac{1}{\sqrt{m-\frac{1}{2}}} \right)^{1/3}.
$$

In Figure 2 we plot (7.1) for $m = 2, 3, \ldots, 6$, and $n = 4m$ for the constants (7.2) and (7.3).

7.2. The smallest eigenvalue of the complex Wishart ensemble. Let $A \sim \mathcal{W}_{n}(1,n\Sigma)$ and let $\lambda_m$ be the smallest eigenvalue of $A$. Define the constants $\tilde{\mu}$ and $\tilde{\sigma}$ as in [15, (5.2), p. 316]:

$$
\tilde{\mu} = (\sqrt{n} - \sqrt{m})^2, \quad \tilde{\sigma} = (\sqrt{n} - \sqrt{m}) \left( \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{m}} \right)^{1/3}.
$$

Then, as $n, m \to \infty$ in such a way that $n/m \to \gamma > 1$, we have

$$
\frac{\lambda_m - \tilde{\mu}}{\tilde{\sigma}} \xrightarrow{d} W_2.
$$

In Figure 3 we plot the limiting distribution (Painlevé, for $\beta = 2$), the formula (4.3), and a Monte-Carlo experiment for $m = 400$.

---

2I thank Brian Sutton for pointing out that $\tilde{\sigma}_N$ in [15] has an incorrect sign.
7.3. The largest eigenvalue of the Jacobi ensemble. Let $C \sim J_p^{(2)}(\frac{q}{2}, \frac{n-q}{2})$, where $p \leq \min(q, n-q)$. Following [16], let the angles $\gamma_p \leq \phi_p$ in $(0, \pi)$ be defined via the equations

$$\sin^2\left(\frac{\gamma_p}{2}\right) = \frac{p - \frac{1}{2}}{n - 1}, \quad \sin^2\left(\frac{\phi_p}{2}\right) = \frac{q - \frac{1}{2}}{n - 1}.$$ 

Let

$$\mu_{p+} = \sin^2\left(\frac{\phi_p + \gamma_p}{2}\right), \quad \sigma_{p+}^3 = \frac{1}{(2n-2)^2} \cdot \frac{\sin^4(\phi_p + \gamma_p)}{\sin \phi_p \sin \gamma_p}.$$ 

Now if $(p, q, n) \to \infty$ in such a way that $(p/n, q/n)$ converge to positive, finite constants, then

$$\frac{\lambda_{\text{max}}(C) - \mu_{p+}}{\sigma_{p+}} \xrightarrow{d} W_1.$$ 

For our experiment we fix $p = 4t + 2$, $q = 6t + 3$, $n = 12t + 6$, and let $t = 1, 2, \ldots$ (Thus $(p/n, q/n) = (1/3, 1/2)$ and $t = (n - p - q - 1)/2$ is an integer).

On Figure 4 we plot the density of

$$\frac{\lambda_{\text{max}}(C) - \mu_{p+}}{\sigma_{p+}}$$

vs the Tracy–Widom limit for $t = 1, 2, \ldots, 8.$

\footnote{In other words, if $A \sim W_p^{(2)}(q, I)$ and $B \sim W_p^{(2)}(n-q, I)$ have independent real Wishart distributions, then the eigenvalues of $C$ are distributed as the roots $u_i$ of the determinantal equation $|A - u_i(A + B)| = 0.$}
7.3.1. Incomplete expressions. It appears that an incomplete expression for the distribution of $\lambda_{\text{max}}(C)$ yields results that appear to converge faster (see Figure 5).

It appears that the largest eigenvalue in the finite case have a Tracy–Widom component (smaller partitions) and a correction.

It would be interesting to understand why such a truncation, essentially, is closer to the limit.
One needs to sum up to $pt = (4t + 2)t$ in (7.5). Instead we sum up to $M$ as indicated. It is interesting that we need to increase $M$ by about $t$ as we move from $t$ to $t + 1$; thus $M = O(t^2/2)$.

If we look at the expression (7.5) and simplify it, it does not seem to shed any light on why this phenomenon occurs.

\begin{align*}
P(\lambda_{\max}(C) < x) &= x^2 \sum_{k=0}^{pt} \sum_{\kappa \vdash k, \kappa_1 \leq t} \frac{(q^{(2)}_\kappa) C^{(2)}_\kappa ((1 - x)I)}{k!} \\
(7.6) &= x^{3(2t+1)^2} \sum_{k=0}^{pt} \sum_{\kappa \vdash k, \kappa_1 \leq t} \frac{(6t+3)^{(2)}_\kappa C^{(2)}_\kappa ((1 - x)I)}{k!} \\
(7.7) &= x^{3(2t+1)^2} \sum_{k=0}^{(4t+2)t} (1 - x)^k \sum_{\kappa \vdash k, \kappa_1 \leq t} \frac{(6t+3)^{(2)}_\kappa k! 2^k j^{(2)}_\kappa (I)}{j_\kappa k!} \\
(7.8) &= x^{3(2t+1)^2} \sum_{k=0}^{(4t+2)t} (1 - x)^k \sum_{\kappa \vdash k, \kappa_1 \leq t} \frac{(6t+3)^{(2)}_\kappa k! 2^k j^{(2)}_\kappa (I)}{j_\kappa},
\end{align*}

where $j_\kappa$ is defined in (1.1).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{truncated_polynomial_expressions.png}
\caption{Truncated polynomial expressions. Seems like summing up to $O(t^2/2)$ converges to the limit faster.}
\end{figure}
8. The William Chen tables

The MATLAB routine qmaxeigjacobi produces the entries of the William Chen tables [6, 5]. For example for $m = 7$, $n = 4$, $s = 6$, and $\alpha = 2$ ($\alpha = 2$ means we are working with real Jacobi matrices)

$$qmaxeigjacobi(\alpha, 2m+s+1, 2n+s+1, s, 0.9, 6)$$

returns 0.9276 for the 0.900 percentage point to 6 digits, as expected.

The matrices $A$ and $B$ in William Chen’s papers are distributed as $W:\beta_i(2m+s+1, I)$, and $W:\beta_s(2n+s+1, I)$. Then the percentage points are computed for the largest eigenvalue of $A(A+B)^{-1}$, which is distributed as $J:\beta_s(2m+s+1, 2n+s+1, I)$.

9. What Jacobi implies about Laguerre when $b \to \infty$

This approach lead to discovering new results, e.g., (2.16) and Theorem 5.4. Let $C \sim J:\beta_s(a, b)$, $C = A(A+B)^{-1}$, where $A \sim L_m(a, I)$ and $B \sim L_m(b, I)$ are independently distributed Laguerre matrices. If $\lambda$ is an eigenvalue of $C$ then $\alpha b^{\frac{a}{\alpha}}$ is an eigenvalue of $abAB^{-1}$.

Also, if $b \to \infty$, the eigenvalues of $abAB^{-1}$ approach those of $A$, a Laguerre matrix, see Proposition 1.11. We transform (4.7) using (2.11)

$$P(\max(C) < x) = D_{m, a, b}^{(a)} \cdot x^{ma} \cdot 2 \cdot F_1^{(a)}(a, -b + \frac{m-1}{\alpha} + 1; a + \frac{m-1}{\alpha} + 1; xI)$$

$$= D_{m, a, b}^{(a)} \cdot x^{ma} \cdot (1 - x)^{-ma} \cdot 2 \cdot F_1^{(a)}(a, a + b; a + \frac{m-1}{\alpha} + 1; -\frac{x}{1-x}I).$$

We make a change of variables $y = \alpha b^{\frac{a}{\alpha}} \cdot x$ to obtain the distribution of the largest eigenvalue of $abAB^{-1}$

$$P(\max(abAB^{-1}) < y) = D_{m, a, b}^{(a)} \cdot \left[ \frac{\alpha}{b} \right] \cdot 2 \cdot F_1^{(a)}(a, a + b; a + \frac{m-1}{\alpha} + 1; -\frac{\alpha}{b} y I)$$

(9.1)

$$= D_{m, a, b}^{(a)} \cdot \left[ \frac{\alpha}{b} \right] \cdot 2 \cdot F_1^{(a)}(a, a + b; a + \frac{m-1}{\alpha} + 1; \frac{1}{b} \cdot (-\frac{\alpha}{b} I)).$$

Since $\lim_{b \to \infty} (a + b) \cdot b^{-|n|} = 1$, we have

$$\lim_{b \to \infty} 2 \cdot F_1^{(a)}(a, a + b; a + \frac{m-1}{\alpha} + 1; \frac{1}{b} \cdot (-\frac{\alpha}{b} I)) = 1 \cdot F_1^{(a)}(a; a + \frac{m-1}{\alpha} + 1; \frac{\sigma}{\alpha} I).$$

By taking the limit in (9.1) as $b \to \infty$, we obtain (4.1) for $\Sigma = I$ and $n = a \alpha$.

10. Any beta research

We consider the joint eigenvalue density of the $2 \times 2$ matrix $A = X^T X$ where

$$X \sim \begin{bmatrix} \chi_{n, \beta} & G_{\beta} \\ 0 & \chi_{(n-1), \beta} \end{bmatrix} \times \begin{bmatrix} 1 \\ \sigma \end{bmatrix}.$$ 

The matrix $X$ has the same distribution as an $n \times 2$ matrix of $G_{\beta}$’s, i.e., $A$ has the same distribution as a $2 \times 2$ $\beta$-Wishart matrix with $n$ DOF and covariance matrix $\Sigma = \text{diag}(1, \sigma^2)$. For complex-values $G_{\beta}$, we have

$$X \sim \begin{bmatrix} 1 \\ \text{sign}(G_{\beta}) \end{bmatrix} \begin{bmatrix} \chi_{n, \beta} & \chi_{\beta} \\ 0 & \chi_{(n-1), \beta} \end{bmatrix} \begin{bmatrix} 1 \\ \sigma \end{bmatrix} \begin{bmatrix} 1 \\ \text{sign}(G_{\beta}) \end{bmatrix},$$

i.e., without loss of generality we may assume that

$$X \sim \begin{bmatrix} \chi_{n, \beta} & \chi_{\beta} \\ 0 & \chi_{(n-1), \beta} \end{bmatrix} \begin{bmatrix} 1 \\ \sigma \end{bmatrix} = \begin{bmatrix} \chi_{n, \beta} & \sigma \chi_{\beta} \\ 0 & \sigma \chi_{(n-1), \beta} \end{bmatrix} = \begin{bmatrix} x_1 & y \\ 0 & x_2 \end{bmatrix},$$

where $x_1 \sim \chi_{n, \beta}, x_2 \sim \chi_{(n-1), \beta}$, and $y \sim \chi_{\beta}$ are i.i.d.

The joint eigenvalue density of $A$ is given by (1.8) with $m = 2$ and we will obtain it from direct arguments following Dumitriu’s thesis [8].
If $a$ is a random variable with density $f(x)$, then $ca$ has density $cf(x)$. The $\chi_k$ distribution has density 
\[ x^{k-1}e^{-x^2/2} \frac{1}{2^{k/2-1}\Gamma(k/2)}. \]

Thus the joint density of $X$ is 
\[ dX = \frac{1}{2n^{\beta-3}\Gamma((n-1)/2)\Gamma(n/2)\Gamma(\beta/2)}x_1^{n\beta-1}\left(\frac{x_2}{\sigma}\right)^{(n-1)\beta-1}\left(\frac{y}{\sigma}\right)^{\beta-1}e^{-\frac{x_2^2}{2\sigma^2}-\frac{y^2}{2\sigma^2}}dx_1dx_2dy. \]

Going from here via the Jacobian $J_{X\to A} = 2^4x_1^2x_2^2/\sigma$ (see \[8, (4.2), p. 43\]), we obtain for the density of $A$
\[ dA = J_{X\to A}^{-1}dX = \frac{1}{2n^{\beta-3}\Gamma((n-1)/2)\Gamma(n/2)\Gamma(\beta/2)}x_1^{n\beta-3}\left(\frac{x_2}{\sigma}\right)^{(n-1)\beta-2}\left(\frac{y}{\sigma}\right)^{\beta-1}e^{-\frac{x_2^2}{2\sigma^2}-\frac{y^2}{2\sigma^2}}. \]

From here we do eigendecomposition and get the joint density of the eigenvalues and eigenvectors. Integrating the eigenvectors out, this should jibe with the joint eigenvalue density.

11. Other

See James [14, p. 496] for additional distributions.

Khatri for moments of trace or determinant?

“Connection formula” from Asymptotics of Special Functions?

“Reflection formula” for $\Gamma$ function?

Change the values of $t$ and $r$ (section 4.3) in software as well.

Hypergeometric Functions of $2 \times 2$ matrix argument are expressible in terms of Appell’s functions $F_4$.


Approximating the matrix Fisher and Bingham distributions: applications to spherical regression and Procrustes analysis

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\[ U(a,b,z) = \frac{\pi}{\sin \pi b} \left[ \frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b}\frac{M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)} \right], \]

where $M(a,b,z) = 1_{F_1}(a,b,z)$.

This section outlines a faster way to compute the updates in the computation of the hypergeometric function of a matrix argument in [19]. The results in this section are due to Raymond Kan and Vesselin Drensky.

In what follows, it will be convenient to use certain additional definitions. We consider the partition
\[ \kappa = (\kappa_1, \kappa_2, \ldots, \kappa_h), \]
where \( h = \kappa'_1 \) is the number of nonzero parts of \( \kappa \). When \( \kappa_i > \kappa_{i+1} \) we define the partition
\[ \kappa(i) \equiv (\kappa_1, \kappa_2, \ldots, \kappa_{i-1}, \kappa_i - 1, \kappa_{i+1}, \ldots, \kappa_h). \]
Also, let \( \tilde{\kappa}_i \equiv \alpha \kappa_i - i \) and \( \bar{\kappa}_i \equiv \kappa_i - 1 - \frac{i-1}{\alpha} \).

Next we express \( H^*_\kappa \) in a way that does not involve the conjugate partition:
\[
H^*_\kappa = \prod_{r=1}^h \prod_{\kappa_r} (\kappa'_r - r + \alpha(\kappa_r - c + 1)) = \prod_{r=1}^h \prod_{j=1}^r \prod_{c=\kappa_{j+1}} \prod_{j=r+1} (j - r + \alpha(\kappa_r - c + 1)) \]
\[
= \alpha^{|\kappa|} \prod_{j=1}^h \prod_{r=1}^j \left( \frac{j - r + 1}{\alpha} + \kappa_r - \kappa_j \right)_{\kappa_j < \kappa_{j+1}}.
\]

Analogously,
\[
H^{\tilde{\kappa}}_\kappa = \alpha^{|\tilde{\kappa}|} \prod_{j=1}^h \prod_{r=1}^j \left( \frac{j - r + 1}{\tilde{\kappa}_j + \alpha - \kappa_j} \right)_{\kappa_j < \kappa_{j+1}}.
\]

From these expressions, we obtain, for \( h = \kappa'_1 \), the number of parts of \( \kappa \),
\[
\frac{H^*_\kappa}{H^*_\kappa} = \frac{1}{\alpha \kappa_h} \prod_{j=1}^{h-1} \frac{\tilde{\kappa}_j - \tilde{\kappa}_h + \alpha - 1}{\tilde{\kappa}_j + \alpha}, \tag{12.1}
\]
\[
\frac{H^{\tilde{\kappa}}_\kappa}{H^*_\kappa} = \frac{1}{\alpha \kappa_h - \alpha + 1} \prod_{j=1}^{h-1} \frac{\tilde{\kappa}_j - \tilde{\kappa}_h}{\tilde{\kappa}_j + \tilde{\kappa}_h + 1}. \tag{12.2}
\]

12.1. The case \( X = xI \). Using (2.27) and (2.28) we have:
\[
p E_q^{(\alpha)}(a_1:p, b_1:q; X) = \sum_{k=0}^{\infty} \sum_{\kappa} x^k Q_\kappa,
\]
where
\[
Q_\kappa = \frac{(a_1)_\kappa^{(\alpha)} \cdots (a_p)_\kappa^{(\alpha)} \cdot \frac{\alpha^2}{\kappa} \cdot \frac{m}{\alpha}^{(\alpha)}}{(b_1)_\kappa^{(\alpha)} \cdots (b_q)_\kappa^{(\alpha)}}.
\]

Denoting \( \delta_{ij} \equiv \tilde{\kappa}_i - \tilde{\kappa}_j \) and using (12.1) and (12.2) we get the following update for \( Q_\kappa \):
\[
\frac{Q_\kappa}{Q^{\tilde{\kappa}}_\kappa} = \prod_{j=1}^p (a_j + \tilde{\kappa}_h) \cdot \frac{m + \alpha \tilde{\kappa}_h}{\kappa_h (\alpha \kappa_h - \alpha + 1)} \cdot \prod_{j=1}^{h-1} \frac{\delta_{ij} (\delta_{ij} + \alpha - 1)}{(\delta_{ij} + 1)(\delta_{ij} + \alpha)}.\]
12.2. Single matrix argument. In this case we use the “Schur” normalization of the Jack function $S_{\kappa}^{(\alpha)}(X) = Q_{\kappa}^{(\alpha)}(X)$, defined in (2.28). The choice to use $Q_{\kappa}^{(\alpha)}$ (as opposed to $F_{\kappa}^{(\alpha)}$, $J_{\kappa}^{(\alpha)}$, or $C_{\kappa}^{(\alpha)}$) yields the simplest update for the coefficients $\sigma_{\kappa \mu}$ below. Then

$$p_F^{(\alpha)}(a_1:p; b_1:q; X) = \sum_{k=0}^{\infty} \sum_{\kappa \times k} Q_{\kappa} S_{\kappa}^{(\alpha)}(X),$$

where

$$Q_{\kappa} = \frac{(a_1)_{(\alpha)} \cdots (a_p)_{(\alpha)}}{(b_1)_{(\alpha)} \cdots (b_q)_{(\alpha)}} \frac{\alpha^{[\kappa]}}{H_{\kappa}^{*}}.$$ 

From (12.2), the update for $Q_{\kappa}$ is

$$\frac{Q_{\kappa}}{Q_{\kappa(\kappa)}} = \frac{\prod_{j=1}^{p}(a_j + \tilde{\kappa}_i)}{\prod_{j=1}^{q}(b_j + \tilde{\kappa}_i)} \cdot \frac{\alpha}{\alpha \kappa_h - \alpha + 1} \cdot \frac{\prod_{j=1}^{h-1} \tilde{\kappa}_j - \tilde{\kappa}_h + 1}{\prod_{j=1}^{h-1} \tilde{\kappa}_j - \kappa_h + 1}.$$ 

This update costs only $2(p+q) + 4h + 1$, representing savings of $10\kappa_h + 5h - 12$ arithmetic operations over the one in [19, Lemma 3.1].

The coefficient update in the computation of the Jack function can also be done more efficiently than in [19]. From (1.4) we have:

(12.3) $$S_{\kappa}^{(\alpha)}(x_1, \ldots, x_n) = \sum_{\mu} S_{\mu}^{(\alpha)}(x_1, \ldots, x_{n-1}) x_n^{[\kappa/\mu]} \sigma_{\kappa \mu},$$

where the summation is over all partitions $\mu \leq \kappa$ such that $\kappa/\mu$ is a horizontal strip and (see (1.5))

(12.4) $$\sigma_{\kappa \mu} = \frac{\beta_{\kappa \mu} H_{\mu}^{*}}{H_{\kappa}^{*}} = \prod_{(i,j) \in \kappa} h_{\kappa}^{\kappa}(i,j) \prod_{(i,j) \in \mu} h_{\mu}^{\mu}(i,j),$$

where both products are over all $(i, j) \in \kappa$ such that $\kappa_j = \mu_j + 1$.

Following the assumptions of Lemma 3.2 in Koev and Edelman [19], let $\nu = \mu(\kappa)$, where $\kappa_j = \mu'_j$ for $j = 1, 2, \ldots, \mu_k - 1$. Using (12.4) we update a more efficient way to obtain $\sigma_{\kappa \mu}$:

$$\frac{\sigma_{\kappa \nu}}{\sigma_{\kappa \mu}} = \frac{\prod_{r=1}^{k} h_{\nu}^{\nu}(r, \mu_k) \prod_{r=1}^{k-1} h_{\nu}^{\nu}(r, \mu_k) \prod_{r=1}^{k} h_{\mu}^{\mu}(r, \mu_k) \prod_{r=1}^{k-1} h_{\mu}^{\mu}(r, \mu_k)}{\prod_{r=1}^{k} 1 + \tilde{\kappa}_r - \tilde{\mu}_k \prod_{r=1}^{k-1} \tilde{\kappa}_r - \tilde{\mu}_k + \alpha - 1 \prod_{r=1}^{k} \tilde{\mu}_r - \tilde{\mu}_k}.$$ 

This update costs $10k$ arithmetic operations (assuming the $\tilde{\mu}_i$ and $\tilde{\kappa}_i$ are stored and only updated, and the quantities $\tilde{\kappa}_r - \tilde{\mu}_k$ and $\tilde{\mu}_r - \tilde{\mu}_k$ are computed only once and reused), representing savings of $2k + 6\mu_k - 7$ arithmetic operations over the update in [19, Lemma 3.2].

We use (12.3) to compute $S_{\kappa}^{(\alpha)}(x_1, \ldots, x_i)$ for $i = h + 1, \ldots, n$. For $i = h$, we use a result of Stanley [29, Propositions 5.1 and 5.5] to obtain:

$$S_{\kappa}^{(\alpha)}(x_1, \ldots, x_h) = (x_1 \cdots x_h)^{\kappa_h} \cdot S_{\kappa-k_h I}^{(\alpha)}(x_1, \ldots, x_h) \prod_{j=1}^{\kappa_h} h_i - i + 1 + \alpha(\kappa_i - j),$$

where $\kappa - \kappa_h I \equiv (\kappa_1 - \kappa_h, \kappa_2 - \kappa_h, \ldots, \kappa_{h-1} - \kappa_h)$.

12.3. Two matrix arguments. We write the hypergeometric function of two matrix arguments (1.7) as:

(12.5) $$p_F^{(\alpha)}(a_1:p; b_1:q; X, Y) = \sum_{k=0}^{\infty} \sum_{\kappa \times k} Q_{\kappa} S_{\kappa}^{(\alpha)}(X) S_{\kappa}^{(\alpha)}(Y),$$
where we define $Q_\kappa$ slightly differently from before:

$$Q_\kappa = \frac{(a_1 \kappa \cdots (a_p \kappa)}{(b_1 \kappa \cdots (b_q \kappa)} \cdot \frac{H_\kappa}{\alpha^{|\kappa|} (\frac{m}{\alpha}) \kappa \lambda},$$

where we have used

$$C^\kappa(I) = \frac{H_\kappa}{\alpha^{|\kappa|} (\frac{m}{\alpha}) \kappa \lambda} \cdot S^\kappa(I).$$

From (12.1) and (12.2), the updating scheme for $Q_\kappa$ is:

$$\frac{Q_\kappa}{Q_{\kappa(h)}} = \frac{\prod_{j=1}^a (a_j + \tilde{k}_h)}{\prod_{j=1}^a (b_j + \tilde{k}_h)} \cdot \frac{\alpha^2 \kappa h}{(m + \alpha \tilde{k}_h)(\tilde{k}_h + h - \alpha + 1)} \prod_{j=1}^{h-1} \frac{(\tilde{k}_j - \tilde{k}_h)(\tilde{k}_j - \tilde{k}_h + \alpha)}{(\tilde{k}_j - \tilde{k}_h + 1)(\tilde{k}_j - \tilde{k}_h + \alpha - 1).$$

12.4. Complexity analysis. In this section we attempt to give some useful estimates of the cost of computing the hypergeometric function of a matrix argument.

In practice, we truncate the hypergeometric series (1.6) by summing over partitions that do not exceed a certain size, $|\kappa| \leq M$, for some $M$. Our experience shows that in order to reach convergence, one needs a rather large $M$, often several times the size of the matrix argument $m$. For such $m \ll M$, we count the number of partitions $|\kappa| \leq M$. If $|\kappa| \leq M$, then $\kappa_i \leq \frac{M}{\tau} + 1$, meaning

$$|\lambda_{M,m}| \leq \left( \frac{M}{m} \right),$$

where

$$\lambda_{M,m} = \{ \kappa \mid |\kappa| \leq M, \kappa_{m+1} = 0 \}.$$

On the other side, if we choose an $m$-tuple $(\kappa_1, \kappa_2, \ldots, \kappa_m)$ such that $\kappa_i \leq \frac{M}{\tau} + 1, i = 1, 2, \ldots, m$, then the number of all such $m$-tuples such that

[To be completed]

13. Cooley–Tukey Ideas for Jack Functions

The results of this section belong to Vesselin Drensky.

13.1. Normalization of Jack Polynomials. The Jack polynomials $J_{\kappa}(x_1, \ldots, x_n)$ satisfy the relation

$$J_{\kappa}(x_1, \ldots, x_n) = \sum_{\mu \in \Lambda_{M,m}} J_{\mu}(x_1, \ldots, x_{n-1}) x_{n}^{\kappa/\mu} \beta_{\kappa \mu},$$

where the sum runs on all partitions $\mu$ such that $\kappa/\mu$ is a horizontal strip,

$$\beta_{\kappa \mu} = \prod_{(i,j) \in \kappa} B_{\kappa \mu}(i,j) \prod_{(i,j) \in \mu} B_{\kappa \mu}(i,j),$$

$B_{\kappa \mu}(i,j) = h_{\kappa}(i,j) = \nu_i' - i + \alpha(\nu_i - j + 1)$, the upper hook length, if $\kappa_j' = \mu_j'$ and $B_{\kappa \mu}(i,j) = h_{\kappa}(i,j) = \nu_i' - i + 1 + \alpha(\nu_i - j)$, the lower hook length, if $\kappa_j' = \mu_j' + 1$.

We normalize $J_{\kappa}(\alpha)$ in the following way:

$$S_{\kappa}(x_1, \ldots, x_n) = \frac{J_{\kappa}(x_1, \ldots, x_n)}{\prod_{(i,j) \in \kappa} h_{\kappa}(i,j)}.$$

In this way we obtain for $\alpha = 1$ that

$$S_{\kappa}(x_1, \ldots, x_n) = S_{\kappa}(x_1, \ldots, x_n),$$
the usual Schur function. The equation (13.1) goes to
\begin{equation}
S_\kappa^{(\alpha)}(x_1, \ldots, x_n) = \sum S_\mu^{(\alpha)}(x_1, \ldots, x_{n-1}) a_{n|\kappa/\mu|} \sigma_{\kappa\mu},
\end{equation}
under the same restrictions on the summation and where
\[ \sigma_{\kappa\mu} = \beta_{\kappa\mu} \prod_{(i,j)\in\kappa} h_\kappa^{*}(i,j) \prod_{(i,j)\in\mu} h_\mu^{*}(i,j). \]

Since the nominator and the denominator of $\beta_{\kappa\mu}$ contain, respectively, as factors $h_\kappa^{*}$ and $h_\mu^{*}$ for all $\kappa'_j = \mu'_j$, we obtain that
\begin{equation}
\sigma_{\kappa\mu} = \prod_{(i,j)\in\kappa} h_\kappa^{*}(i,j) \prod_{(i,j)\in\mu} h_\mu^{*}(i,j)
\end{equation}
where the products are on all $(i,j)$ such that $\kappa'_j = \mu'_j + 1$. This seems to be a simplification from computational point of view.

We can rewrite (13.4) in the form
\begin{equation}
\sigma_{\kappa\mu} = \prod_{(i,j)\in\kappa/\mu} h_\kappa^{*}(i,j) \prod_{(i,j)\in\kappa/\mu} h_\kappa^{*}(i,j)
\end{equation}
where the first product is again on all $(i,j)$ such that $\kappa'_j = \mu'_j + 1$. Using the formula
\[ \left(\frac{-\delta}{k} - b\right) = \frac{1}{k!} (-\frac{a}{\alpha} - b) \left(\frac{a}{\alpha} - (b-1)\right) \cdots \left(\frac{a}{\alpha} - (b-k+1)\right) \]
\[ \quad = (-1)^k \frac{1}{k!\alpha^k} (a + ab)(a + \alpha(b+1)) \cdots (a + \alpha(b-k+1)) \]
we obtain
\[ \prod_{(i,j)\in\kappa/\mu} h_\kappa^{*}(i,j) = \left(-\frac{1}{\alpha}\right)_{\kappa_1 - \mu_1} \cdots \left(-\frac{1}{\alpha}\right)_{\kappa_{n-1} - \mu_{n-1}} \left(-\frac{1}{\alpha}\right)_{\kappa_n - \mu_n} (-1)^{|\kappa/\mu|}, \]
\[ \prod_{(i,j)\in\mu} h_\mu^{*}(i,j) \]
13.2. Counting the Partitions. We order the partitions $\kappa = (\kappa_1, \ldots, \kappa_n)$ in the following way: $\mu \leq \kappa$ if

$$\mu_{i+1} = \kappa_{i+1}, \ldots, \mu_n = \kappa_n, \quad \mu_i < \kappa_i.$$  

We want to find the number $r(\kappa)$ of the place of $\kappa$ in the sequence of all partitions $\lambda = (\lambda_1, \ldots, \lambda_n), |\lambda| \leq N$. If $p_n(N)$ is the number of all partitions of $N$ in $\leq n$ parts, then the generating function satisfies

$$p_n(t) = \sum_{N \geq 0} p_n(N)t^N = \prod_{i=1}^n \frac{1}{1 - t^i}. $$

It is a standard trick to show for the generating function

$$q_n(t) = \sum_{N \geq 0} \left( \sum_{M=0}^N p_n(M) \right) t^N = \frac{p_n(t)}{1 - t},$$

which gives the expression $q_n(N)$ for the number of all partitions $\kappa$ in $\leq n$ parts with $|\kappa| \leq N$. In order to find the exact number $r(\kappa)$ of $\kappa$, let us assume that $\kappa_m \neq 0, \kappa_{m+1} = 0$. Then

$$r(\kappa_1, \ldots, \kappa_m) = q_{m-1}(N) + q_{m-1}(N-m) + \cdots + q_{m-1}(N-m(k_m-1)) + r(\kappa_1 - \kappa_m, \ldots, \kappa_{m-1} - \kappa_m)$$

because $q_{m-1}(N - km)$ is the number of partitions $(\mu_1, \ldots, \mu_m)$ with $\mu_m = k$.

13.3. Relations between the Coefficients. We shall write

$$\sigma_{\kappa\mu} = \sigma(\kappa, \mu) = \sigma((\kappa_1, \ldots, \kappa_n), (\mu_1, \ldots, \mu_n)).$$

The following relation is obvious (follows from (13.4)) because $(\kappa + 1)'_1 = (\mu + 1)'_1$:

$$\sigma_{\kappa_1+1, \mu+1} = \sigma((\kappa_1 + 1, \ldots, \kappa_n + 1), (\mu_1 + 1, \ldots, \mu_n + 1)) = \sigma_{\kappa\mu}.$$  

The following also follows from (13.4):

$$\sigma((\kappa_1 + 1, \ldots, \kappa_n + 1), (\mu_1 + 1, \ldots, \mu_n + 1, 0)) = \prod_{i=1}^n \frac{n-i+1+\alpha}{n-i+\alpha_1} \prod_{i=1}^{n-1} \frac{n-i-\alpha_{\mu+1}}{n-i+\alpha_{\mu_i}} \sigma((\kappa_1, \ldots, \kappa_n), (\mu_1, \ldots, \mu_n-1, 0)).$$  

14. Raymond Kan’s Note 21

The results in this section are due to Raymond Kan.

Let $\alpha > 0$ and $A$ be a real symmetric $n \times n$ matrix. We define $d_k^{(\alpha)}(A)$ as the power series expansion of

$$|I_n - tA|^{-\frac{1}{\alpha}} = \sum_{k=0}^{\infty} d_k^{(\alpha)}(A)t^r.$$  

Note that $d_k^{(\alpha)}(A)$ is a normalized version of the top-order Jack polynomial, it is related to $C_k^{(\alpha)}(A)$ and $J_k^{(\alpha)}(A)$ by the following relation

$$d_k^{(\alpha)}(A) = \frac{J_k^{(\alpha)}}{\alpha^k k!} = \frac{1}{k!} \left( \frac{1}{\alpha} \right)_k C_k^{(\alpha)}(A).$$
**Lemma:** Let $x$ be a real scalar. We have

\begin{equation}
\tag{14.3}
d_k^{(\alpha)}(A + xI_n) = \sum_{j=0}^{k} \binom{\frac{n}{\alpha} + k - 1}{j} d_{k-j}^{(\alpha)}(A)x^j,
\end{equation}

\begin{equation}
\tag{14.4}
_{1}F_{1}^{(\alpha)}\left(1; \frac{n}{\alpha}; A + xI_n\right) = e^{x} _{1}F_{1}^{(\alpha)}\left(\frac{1}{\alpha}; \frac{n}{\alpha}; A\right),
\end{equation}

\begin{equation}
\tag{14.5}
_{2}F_{1}^{(\alpha)}\left(1, m; \frac{n}{\alpha}; A + xI_n\right) = (1 - x)^{-m} _{2}F_{1}^{(\alpha)}\left(1, m; \frac{n}{\alpha}; (1 - x)^{-1}A\right).
\end{equation}

**Proof:** The first identity can be proven as follows:\(^4\)

\[
|I_n - t(A + xI_n)|^{-\frac{s}{\alpha}} = (1 - tx)^{-\frac{n}{\alpha}} \left|I_n - \frac{t}{1 - tx}A\right|^{-\frac{s}{\alpha}}
\]

\[
\Rightarrow \sum_{k=0}^{\infty} d_k^{(\alpha)}(A + xI_n)t^k = (1 - tx)^{-\frac{n}{\alpha}} \sum_{i=0}^{\infty} d_i^{(\alpha)}(A) \left(\frac{t}{1 - tx}\right)^i
\]

\[
\Rightarrow \sum_{k=0}^{\infty} d_k^{(\alpha)}(A + xI_n)t^k = \sum_{i=0}^{\infty} d_i^{(\alpha)}(A)t^i(1 - tx)^{-\frac{n}{\alpha}+i}
\]

\[
\Rightarrow \sum_{k=0}^{\infty} d_k^{(\alpha)}(A + xI_n)t^k = \sum_{i=0}^{\infty} d_i^{(\alpha)}(A)t^i \sum_{j=0}^{\infty} \binom{\frac{n}{\alpha} + i + j - 1}{j} t^j x^j
\]

\[
\Rightarrow \sum_{k=0}^{\infty} d_k^{(\alpha)}(A + xI_n)t^k = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{\frac{n}{\alpha} + k - 1}{j} d_{k-j}^{(\alpha)}(A)x^j t^k, \quad (k = i + j).
\]

(14.6)

Comparing the coefficient of $t^k$ on both sides, we prove the first identity.

When one of the numerator coefficients in $pF_q^{(\alpha)}$ is $1/\alpha$, only the top-order jack polynomials are involved. Using the fact that (1) $C_\kappa^{(\alpha)}(A) = 0$ when $\kappa$ has more than one part, and (2) $(c)_k^{(\alpha)} = (c)_k$ for partition that has only part, we have

\begin{equation}
\tag{14.7}
pF_q^{(\alpha)}\left(\frac{1}{\alpha}, \alpha_2, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; A\right) = \sum_{k=0}^{\infty} \frac{(\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdots (\beta_q)_k} d_k^{(\alpha)}(A).
\end{equation}

\(^4\)For the case of $\alpha = 2$, the first identity is given in Robbins (1948, Eq.28).
Using this result and the first identity, we can prove the second identity as follows:

\[
{}_1F_1^{(\alpha)}\left(\frac{1}{\alpha}; \frac{n}{\alpha}; A + xI_n\right) = \sum_{k=0}^{\infty} d_k^{(\alpha)}(A + xI_n) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{\Gamma\left(\frac{n}{\alpha}\right)}{j!} d_{k-j}^{(\alpha)}(A)x^j
\]

This completes the proof.

There are some potential applications of this identity. For example, suppose \( A \) has two distinct eigenvalues \( \lambda_1 \) and \( \lambda_2 \), with multiplicities \( n_1 \) and \( n_2 \), respectively. Then putting \( x = -\lambda_2 \), we have

\[
{}_1F_1^{(\alpha)}\left(\frac{1}{\alpha}; \frac{n}{\alpha}; A\right) = e^{\lambda_2}{}_1F_1^{(\alpha)}\left(\frac{1}{\alpha}; \frac{n}{\alpha}; (\lambda_1 - \lambda_2)I_n\right) = e^{\lambda_2}{}_1F_1^{(\alpha)}\left(\frac{n_1}{\alpha}; \frac{n}{\alpha}; \lambda_1 - \lambda_2\right)
\]

by using the fact that \( d_k^{(\alpha)}(xI_{n_1}) = \left(\frac{n_1}{\alpha}\right)_k x^k/k! \). This identity allows us to write the result as a hypergeometric function with scalar argument.

The third identity can be similarly obtained.

\[
{}_2F_1^{(\alpha)}\left(\frac{1}{\alpha}, m; \frac{n}{\alpha}; A + xI_n\right) = \sum_{k=0}^{\infty} \frac{(m)_k d_k^{(\alpha)}(A + xI_n)}{(n\alpha)_k}
\]

\[
= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(n\alpha)}{j!} d_{k-j}^{(\alpha)}(A)x^j
\]

\[
= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(n\alpha)}{j!} d_{k-j}^{(\alpha)}(A)x^j
\]

\[
= \sum_{p=0}^{\infty} \frac{(m)_p d_p^{(\alpha)}(A)}{(n\alpha)_p} (1 - x)^{-m-p}
\]

This completes the proof.
Let $A = (1 - x)B$ for $x \neq 1$, we can write the last identity as

$$2F_1^{(a)} \left( \frac{1}{\alpha}, m; \frac{n}{\alpha}; x \right) = (1 - x)^m 2F_1^{(a)} \left( \frac{1}{\alpha}, m; \frac{n}{\alpha}; (1 - x)B + xI_n \right).$$

This identity can be used for the purpose of dimension reduction. Suppose $B$ has two distinct eigenvalues $\lambda_1$ and $\lambda_2$, with multiplicities $n_1$ and $n_2$, respectively. Without loss of generality, we assume $\lambda_2 \neq 1$. Letting $x = -\lambda_2/(1 - \lambda_2)$, we obtain

$$2F_1^{(a)} \left( \frac{1}{\alpha}, m; \frac{n}{\alpha}; B \right) = \left(1 + \frac{\lambda_2}{1 - \lambda_2}\right)^m 2F_1^{(a)} \left( \frac{1}{\alpha}, m; \frac{n}{\alpha}; \lambda_1 - \lambda_2 n \right),$$

$$= \frac{1}{(1 - \lambda_2)^m} 2F_1^{(a)} \left( \frac{n_1}{\alpha}, m; \frac{n}{\alpha}; \frac{\lambda_1 - \lambda_2}{1 - \lambda_2} \right).$$

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References

5. William R. Chen, *Table for upper percentage points of the largest root of a determinantal equation with five roots*.