HYPERGEOMETRIC SERIES II
(q–ANALOGUES)

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Contents

Foreword

This is the typewritten version of a handwritten manuscript which was completed by Ian G. Macdonald in 1987 or 1988. It is the sequel to the manuscript "Hypergeometric fucntions I". The two manuscripts are very informal working papers, never intended for formal publication. Nevertheless, copies of the manuscripts have circulated widely, giving rise to quite a few citations in the subsequent 25 years. Therefore it seems justified to make the manuscripts available for the whole mathematical community. The author kindly gave his permission that typewritten versions be posted on arXiv. These notes were typeset verbatim by Tierney Genoar and Plamen Koev, supported by the San Jose State University Planning Council and National Science Foundation Grant DMS-1016086.

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The classical definition (one variable) is

$$ {}_r\Phi_s(a_1,\ldots,a_r;b_1,\ldots,b_s;x;q)=\sum_{n\geq 0}\frac{(a_1;q)_n\ldots(a_r;q)_n}{(b_1;q)_n\ldots(b_s;q)_n}\cdot\frac{x^n}{(q;q)_n} $$

and we have in particular

$$ {}_0\Phi_0(x;q)=(x;q)_\infty^{-1}, \tag{1.1} $$

$$ {}_1\Phi_0(a;x;q)=(ax;q)_\infty/(x;q)_\infty. \tag{1.2} $$

In several variables the definitions should be such as to give

$$ {}_0\Phi_0(x_1,\ldots,x_n;q,t)=\prod_{i=1}^n(x_i;q)_\infty^{-1}, \tag{1.3} $$

$$ {}_1\Phi_0(a;x_1,\ldots,x_n;q,t)=\prod_{i=1}^n\frac{(ax_i;q)_\infty}{(x_i;q)_\infty}. \tag{1.4} $$

We have (any u)

$$ \varepsilon_{u,t}(J_\lambda(x;q,t))=\prod_{(i,j)\in\lambda}(t^{i-1}-q^{j-1}u)=t^{n(\lambda)}\prod_{i\geq 1}(t^{1-i}u;q)_{\lambda_i} $$

and we define

$$ (u;q,t)_\lambda=\prod_{i\geq 1}(ut^{1-i};q)_{\lambda_i} \tag{1.5} $$

so that we have

$$ \boxed{\varepsilon_{u,t}(J_\lambda)=t^{n(\lambda)}(u;q,t)_\lambda.} \tag{1.6} $$

In particular,

$$ \varepsilon_{0,t}(J_\lambda)=t^{n(\lambda)}. \tag{1.7} $$

More generally, if $\underline{a}=(a_1,\ldots,a_r)$, define

$$ (\underline{a};q,t)_\lambda=\prod_{i=1}^r(a_i;q,t)_\lambda. $$

Let

$$ J_\lambda^*(x;q,t)=J_\lambda(x;q,t)/\langle J_\lambda,J_\lambda\rangle_{q,t} $$

so that (J_λ^*) is the basis of Λ_F dual to the basis (J_λ).

(1.8) Definition. Let $\underline{a}=(a_1,\ldots,a_r)$, $\underline{b}=(b_1,\ldots,b_s)$. Then we define

$$ {}_r\Phi_s(\underline{a};\underline{b};x;q,t)=\sum_\lambda\frac{(\underline{a};q,t)_\lambda}{(\underline{b};q,t)_\lambda}t^{n(\lambda)}J_\lambda^*(x;q,t), $$

a formal power series with coefficients in $F(\underline{a},\underline{b})$, i.e., an element of $\hat{\Lambda}_{F(a,b)}$.

Here the number of variables x_i may be finite or infinite.

The next definition, however, seems to be relevant only when the number of variables is finite, say $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$.

Let

$$J_\lambda^*(x,y;q,t) = \frac{J_\lambda^*(x;q,t)J_\lambda^*(y;q,t)}{\varepsilon_{t^n,t}(J_\lambda^*)} \tag{1.9}$$

the denominator of which is $J_\lambda^*(1,t,\dots,t^{n-1},q,t)$.

$$= \frac{J_\lambda(x)J_\lambda^*(y)}{t^{n(\lambda)}(t^n)_\lambda}$$

(1.10) Definition. With $\underline{a},\underline{b}$ as in (1.8), we define

$${}_r\Phi_s(\underline{a};\underline{b};x,y;q,t) = \sum_\lambda \frac{(\underline{a};q,t)_\lambda}{(\underline{b};q,t)_\lambda} t^{n(\lambda)} J_\lambda^*(x,y;q,t).$$

Here the sum is over partitions of length $\le n$. (hypergeometric kernel)

The relationship between ${}_r\Phi_s(x,y)$ and ${}_r\Phi_s(x)$ is given by

$$\varepsilon_{t^n,t}^{(y)}\,{}_r\Phi_s(x,y) = {}_r\Phi_s(x) \tag{1.11}$$

Proof. This follows from the definitions, since

$$\varepsilon_{t^n,t}^{(y)} J_\lambda^*(x,y) = J_\lambda^*(x).$$

□

Each such ${}_r\Phi_s(x,y)$ determines a scalar product on $\Lambda_{F,n}$ for which the P_λ are pairwise orthogonal and

$$\langle P_\lambda, Q_\lambda\rangle = \frac{(\underline{a})_\lambda}{(\underline{b})_\lambda (t^n)_\lambda}$$

for

$${}_r\Phi_s = \sum_\lambda \frac{(\underline{a})_\lambda}{(\underline{b})_\lambda}\cdot\frac{P_\lambda(x)Q_\lambda(y)}{(t^n)_\lambda}. \qquad \text{(better definition)}$$

2. Particular cases

(2.1) $${}_0\Phi_0(x;q,t)=\prod_i (x_i;q)_\infty^{-1}.$$

Proof.

$$\begin{aligned}{}_0\Phi_0(x;q,t) &= \sum_\lambda t^{n(\lambda)} J^*_\lambda(x;q,t)\\ &= \sum_\lambda \varepsilon_{0,t}(J_\lambda)J^*_\lambda \qquad \text{by (1.7)}\\ &= \varepsilon^{(y)}_{0,t}\Pi(x,y;q,t).\end{aligned}$$

Since $\varepsilon_{0,t}(P_r)=(1-t^r)^{-1}$ $(r\geq 1)$, the effect of $\varepsilon_{0,t}$ on the y-variables is to specialize $y_i\mapsto t^{i-1}$ $(i\geq 1)$, + hence

$$\varepsilon^{(y)}_{0,t}\Pi(x,y;q,t)=\prod_{i,j}\frac{(x_it^j;q)_\infty}{(x_it^{j-1};q)_\infty}=\prod_i\frac{1}{(x_i,q)_\infty}.$$

□

(2.2) $${}_1\Phi_0(a;x;q,t)=\prod_i\frac{(ax_i;q)_\infty}{(x_i;q)_\infty}$$

Proof. By (1.6) and (1.8),

$${}_1\Phi_0(a;x;q,t)=\sum_\lambda \varepsilon_{a,t}(J_\lambda)J^*_\lambda=\varepsilon^{(y)}_{a,t}\Pi(x;y;q,t).$$

Now

$$\Pi(x,y;q,t)=\exp\sum_{r\geq 1}\frac{1}{r}\cdot\frac{1-t^r}{1-q^r}p_r(x)p_r(y)$$

and $\varepsilon_{a,t}p_r=\frac{1-a^r}{1-t^r}$, so that

$$\begin{aligned}\varepsilon^{(y)}_{a,t}\Pi(x,y;q,t) &= \exp\sum_{r\geq 1}\frac{1}{r}\cdot\frac{1-a^r}{1-q^r}p_r(x)\\ &= \Pi(x,1;q,a)\\ &= \prod_i\frac{(ax_i;q)_\infty}{(x_i;q)_\infty.}\end{aligned}$$

□

Notice that (2.1) is the case $a=0$ of (2.2), since $(0;q,t)_\lambda=1$ for all partitions λ. Thus

$${}_r\Phi_s(a_1,\dots,a_{r-1},\dots,0;b_1,\dots,b_s,x;q,t)={}_{r-1}\Phi_s(a_1,\dots,a_{r-1};b_1,\dots,b_s;x;q,t)$$

and likewise if one of the b_i is zero.

(2.3) $${}_1\Phi_0(t^n;x,y;q,t)=\Pi(x,y;q,t)$$

Proof. We have

$$\begin{aligned} {}_1\Phi_0(t^n;x,y;q,t) &= \sum_\lambda t^{n(\lambda)}(t^n)_\lambda J^*_\lambda(x,y) \\ &= \sum_\lambda J_\lambda(x)J^*_\lambda(y) \end{aligned}$$

(since $t^{n(\lambda)}(t^n)_\lambda = \varepsilon_{t^n,t}(J_\lambda)$) whence the result. □

Next consider the scalar product of Ch.VI, §9:

$$\langle f,g\rangle'_{q,t} = \frac{1}{n!}[f\bar{g}\Delta]_1,$$

where

$$\Delta = \Delta(x;q,t) = \prod_{i\neq j}\frac{(x_ix_j^{-1};q)_\infty}{(tx_ix_j^{-1};q)_\infty}$$

and $[\ ]_1$ denotes the constant term.

We shall normalize this as follows

$$\langle f,g\rangle''_{q,t} = \langle f,g\rangle'_{q,t}/\langle 1,1\rangle'_{q,t}$$

[In fact (q–Dyson conjecture)

$$\langle 1,1\rangle'_{q,t} = \frac{(t;q)^n_\infty}{(t^n;q)_\infty(q;q)^{n-1}_\infty}\cdot\prod_{i=1}^{n-1}(1-t^i)^{-1}$$

–see e.g., Stembridge for a reasonably simple proof.]

Conjecture (C1). $\langle P_\lambda,P_\lambda\rangle''_{q,t} = \varepsilon_{t^n,t}(P_\lambda)/\varepsilon_{qt^{n-1},t}(Q_\lambda)$

We associate with this scalar product the power series

$$\Pi''(x,y;q,t) = \sum_\lambda u_\lambda(x)v_\lambda(y),$$

where $(u_\lambda),(v_\lambda)$ are dual bases of Λ_F for the scalar product. Taking $u_\lambda = P_\lambda$ we have

$$\begin{aligned} \Pi''(x,y;q,t) &= \sum_\lambda \frac{P_\lambda(x)P_\lambda(y)}{\langle P_\lambda,P_\lambda\rangle''} \\ &= \sum_\lambda \frac{P_\lambda(x)P_\lambda(y)\varepsilon_{qt^{n-1},t}(Q_\lambda)}{\varepsilon_{t^n,t}(P_\lambda)} \\ &= \sum_\lambda \frac{\varepsilon_{qt^{n-1},t}(J_\lambda)J^*_\lambda(x)J^*_\lambda(y)}{\varepsilon_{t^n,t}(J^*_\lambda)} \\ \text{(because } P_\lambda Q_\lambda = J_\lambda J^*_\lambda\text{)} \quad &= \sum_\lambda t^{n(\lambda)}(qt^{n-1})_\lambda J^*_\lambda(x,y) \\ &= {}_1\Phi_0(qt^{n-1};x,y;q,t). \end{aligned}$$

Then we have

$$\text{Conjecture (C1)} \iff \Pi''(x,y;q,t) = {}_1\Phi_0(qt^{n-1};x,y;q,t), \tag{2.4}$$

From (2.4) and (1.11) we obtain (always assuming (C1)) that

$$\begin{aligned}\varepsilon^{(y)}_{t^n,t}\Pi''(x,y;q,t) &= {}_1\Phi_0(qt^{n-1};x;q,t)\\ &= \prod_i \frac{(qt^{n-1}x_i;q)_\infty}{(x_i,q)_\infty} \quad \text{by (2.2)}\\ &= \prod_i (x_i;q)^{-1}_{k(n-1)+1}\end{aligned}$$

if $t = q^k$:

(2.5) $$\text{Conjecture (C1)} \iff \varepsilon^{(y)}_{t^n,t}\,\Pi''(x,y;q,t) = \prod_{i=1}^{n} \frac{(qt^{n-1}x_i;q)_\infty}{(x_i;q)_\infty}$$

3. Selberg integrals

Set $t = q^k$ provisionally, where k is a positive integer. Define

$$W_{a,b}(x;q,t) = \prod_{i=1}^{n} x_i^{a-1}(qx_i;q)_{b-1} \cdot \prod_{1\le i<j\le n} \prod_{r=0}^{k-1} (x_i - q^r x_j)(x_i - q^{-r}x_j) \tag{3.1}$$

Let C_n denote the unit cube $[0,1]^n$ and define

$$I_{a,b}(f) = \int_{C_n} f(x) W_{a,b}(x) \mathrm{d}_q x \tag{3.2}$$

$$J_{a,b}(f) = I_{a,b}(f)/I_{a,b}(1). \tag{3.3}$$

[The multiple q-integral is defined as follows: if f is a function on C_n, then

$$\int_{C_n} f(x)\mathrm{d}_q(x) = (1-q)^n \sum_{\alpha\in\mathbb{N}^n} q^{|\alpha|} f(q^{\alpha_1},\dots,q^{\alpha_n}). \tag{3.4}$$

If f vanishes whenever some x_i is equal to 1, then

$$\int_{C_n} f(x)\mathrm{d}_q x = (1-q)^n \sum_{\alpha\in\mathbb{N}^n} q^{n+|\alpha|} f(q^{\alpha_1+1},\dots,q^{\alpha_n+1}),$$

i.e.,

$$\int_{C_n} f(x)\mathrm{d}_q x = q^n \int_{C_n} f(qx)\mathrm{d}_q x.] \tag{3.5}$$

Conjecture (C2). $J_{a,b}(P_\lambda) = \varepsilon_{u,t}(P_\lambda)\varepsilon_{t^n,t}(P_\lambda)/\varepsilon_{v,t}(P_\lambda)$

where $u = q^a t^{n-1}$, $v = q^{a+b}t^{2n-2}$.

$(t = q^k: \quad u = q^{a'}, v = q^{a'+b'}, a' = a + k(n-1), b' = b + k(n-1))$

Equivalently, by (1.6), (C2) $\Longleftrightarrow$

$$J_{a,b}(P_\lambda) = \frac{(q^a t^{n-1})_\lambda}{(q^{a+b}t^{2n-2})_\lambda} \varepsilon_{t^n,t}(P_\lambda). \tag{C2$'$}$$

Conjecture (C2) implies

$$I_{a,b}(1)^{-1} \int_{C_n} {}_r\Phi_s(\underline{a};\underline{b};x,y) W_{a.b}(x)\mathrm{d}_q x = {}_{r+1}\Phi_{s+1}(\underline{a},u;\underline{b},v;y), \tag{3.6}$$

where as above $u = q^a t^{n-1}$, $v = q^{a+b}t^{2n-2}$.

Proof. We have

$$\begin{aligned} I_{a,b}(1)^{-1} \int_{C_n} J^*_\lambda(x,y) W_{a,b}(x)\mathrm{d}_q x &= \frac{J_{a,b}(J^*_\lambda(x))J^*_\lambda(y)}{\varepsilon_{t^n,t}(J^*_\lambda)} \\ &= \frac{(u)_\lambda}{(v)_\lambda} J^*_\lambda(y) \qquad \text{by (C2$'$)}. \end{aligned}$$

∎

Thus integration against $W_{a,b}(x;q,t)$ raises both indices by 1.

We can rewrite the Selberg kernel $W_{a,b}$ in terms of $\Delta(x;q,t)$:–

We have

$$\prod_{r=0}^{k-1}(x_i-q^r x_j)(x_i-q^{-r}x_j)=(-1)^k q^{-k(k-1)/2}(x_i x_j)^k (x_i x_j^{-1};q)_k (x_i^{-1}x_j;q)_k$$

and therefore

$$W_{a,b}(x;q,t)=(-1)^{\alpha}q^{-\beta}\prod_{i=1}^{n} x_j^{a+k(n-1)-1}(qx_i;q)_{b-1}\Delta(x;q,t), \tag{3.7}$$

where $\alpha=k\binom{n}{2}$, $\beta=\binom{k}{2}\binom{n}{2}$.

So if we define

$$\widetilde{W}_{a,b}(x;q,t)=\prod_{i=1}^{n} x_i^{a-1}(qx_i;q)_{b-1}\Delta(x;q,t) \tag{3.8}$$

and

$$\widetilde{I}_{a,b}(f)=\int_{C_n} f(x)\widetilde{W}_{a,b}(x)\mathrm{d}_q x, \tag{3.9}$$

$$\widetilde{J}_{a,b}(f)=\widetilde{I}_{a,b}(f)/\widetilde{I}_{a,b}(1), \tag{3.10}$$

we have

$$\widetilde{I}_{a,b}(f)=(-1)^{\alpha}q^{-\beta}I_{a-k(n-1),b}(f) \tag{3.11}$$

$$\widetilde{J}_{a,b}(f)=J_{a-k(n-1),b}(f) \tag{3.12}$$

so that (C2) now takes the form

$$\widetilde{J}_{a,b}(P_\lambda)=\frac{(q^a)_\lambda}{(q^{a+b}\,t^{n-1})_\lambda}\varepsilon_{t^n,t}(P_\lambda). \tag{C2$''$}$$

The value of $I_{a,b}(1)$ is in fact

$$I_{a,b}(1)=n!q^{\gamma}\prod_{i=1}^{n}\frac{\Gamma_q(ik)\Gamma_q(a+(r-i)k)\Gamma_q(b+(r-i)k)}{\Gamma_q(k)\Gamma_q(a+b+(2r-i-1)k)}, \tag{3.13}$$

where $\gamma=ka\binom{n}{2}+2k^2\binom{n}{3}$.

If we define

$$\Gamma_{q,n}(a')=\prod_{i=1}^{n}\Gamma_q(a'-k(i-1)) \tag{3.14}$$

then (3.13) takes the form

$$I_{a,b}(1)=n!q^{\gamma}\frac{\Gamma_{q,n}(nk)}{\Gamma_q(k)^n}\cdot\frac{\Gamma_{q,n}(a')\Gamma_{q,n}(b')}{\Gamma_{q,n}(a'+b')}, \tag{3.13$'$}$$

where $a'=a+k(n-1)$, $b'=b+k(n-1)$.

4. Gauss summation for ${}_2\Phi_1$

(4.1)
$$
\begin{aligned}
{}_2\Phi_1(a_1,a_2;b;c,ct^{-1},\dots,ct^{1-n};q,t) &= \prod_{i=1}^{n}\frac{(a_1^{-1}bt^{1-i};q)_\infty(a_2^{-1}bt^{1-i};q)_\infty}{(bt^{1-i};q)_\infty(a_1^{-1}a_2^{-1}bt^{1-i};q)_\infty}\\
&\left(=\frac{\Gamma_{n,q}(\beta-\alpha_1)\Gamma_{n,q}(\beta-\alpha_2)}{\Gamma_{n,q}(\beta)\Gamma_{n,q}(\beta-\alpha_1-\alpha_2)}\right),
\end{aligned}
$$

where $c=b/(a_1a_2)$.

Proof. From (3.6) we have

$$
{}_2\Phi_1(a_1,a_2;b;y;q,t)=I_{\alpha,\beta}(1)^{-1}\int_{C_n}{}_1\Phi_0(a_2;x,y)W_{\alpha,\beta}(x)\mathrm{d}_qx
$$

where $a_1=q^\alpha t^{n-1}$, $b=q^{\alpha+\beta}t^{2n-2}$ and the effect of replacing y_i by ct^{1-i} replaces ${}_1\Phi_0(a_2;x,y)$ by

$$
{}_1\Phi_0(a_2;ct^{1-n}x)=\prod_i(a_2ct^{1-n}x_i;q)_\infty/(ct^{1-n}x_i;q)_\infty.
$$

Hence the integral becomes

$$
\int_{C_n}\prod_{i=1}^{n}x_i^{\alpha-1}\cdot\prod_{i=1}^{n}\frac{(qx_i,q)_\infty}{(q^\beta x_i,q)_\infty}\cdot\frac{(a_2ct^{1-n}x_i;q)_\infty}{(ct^{1-n}x_i;q)_\infty}\prod_{i<j}\prod_{r=0}^{k-1}(\ )(\ )\mathrm{d}_qx
$$

Now $a_2ct^{1-n}=ba_1^{-1}t^{1-n}=q^\beta$, and $ct^{1-n}=a_2^{-1}q^\beta=q^\gamma$ say.

So finally we have

$$
\begin{aligned}
{}_2\Phi_1(a_1,a_2;b;(ct^{1-i})_{1\le i\le n};q,t) &= \frac{I_{\alpha,\gamma}(1)}{I_{\alpha,\beta}(1)}\\
&= \frac{\Gamma_{q,n}(\alpha')\Gamma_{q,n}(\gamma')}{\Gamma_{q,n}(\alpha'+\gamma')}\cdot\frac{\Gamma_{q,n}(\alpha'+\beta')}{\Gamma_{q,n}(\alpha')\Gamma_{q,n}(\beta')}
\end{aligned}
$$

by (3.13′), where $\alpha'=\alpha+k(n-1)$, $\beta'=\beta+k(n-1)$, $\gamma'=\gamma+k(n-1)$ so that

$$
q^{\alpha'}=a_1,\ q^{\beta'}=b/a_1,\ q^{\gamma'}=a_2^{-1}q^{\beta'}=c=b/a_1a_2,\ q^{\alpha'+\gamma'}=b/a_2.
$$

Hence we obtain the formula (4.1). □

Additional observation.

$$
\begin{aligned}
\Gamma_{n,q}(a) &= \prod_{i=1}^{n}\Gamma_q(a-k(i-1))\\
&= \prod_{i=1}^{n}\frac{(q;q)_\infty}{(q^{a-k(i-1)};q)_\infty}\cdot(1-q)^{-a+k(i-1)}
\end{aligned}
$$

q–Saalschutz should be

$$
{}_3\Phi_2(\underline{a};\underline{b};(qt^{n-i})_{1\le i\le n};q,t)=\dots,
$$

where some $a_i=q^{-N}$ and $a_1a_2a_3t^{n-1}q=b_1b_2$.
This will $\Rightarrow$ Gauss as $N\to\infty$.

5. Laplace transform

The q–analogue of e^x is

$$\sum \frac{x^n}{(q;q)_n} = (x;q)_\infty^{-1}$$

and so the analogue of e^{-x} is $(x;q)_\infty$. Let $b \to \infty$ in Conjecture (C2), then $v = q^{a+b}t^{2n-2} \to 0$ and so we have

$$\int_{C_n} P_\lambda(x;q,t) \prod_{i=1}^{n} x_i^{a-1}(qx_i;q)_\infty \prod_{i<j}\prod_{r=0}^{k-1}(x_i - q^r x_j)(x_i - q^{-r}x_j)\mathrm{d}_q x = I_{a,\infty}(1)^{(q^a t^{n-1})_\lambda} \cdot \varepsilon_{t^n,t}(P_\lambda). \tag{5.1}$$

Since $\Gamma_q(b) = (q;q)_b/(1-q)^{b-1}$, it follows that

$$\lim_{b\to\infty} \frac{\Gamma_q(b+(n-1)k)}{\Gamma_q(a+b+(2n-i-1)k)} = (1-q)^{a+(n-1)k}$$

and hence from (3.13) that

$$I_{a,\infty}(1) = n!q^\gamma \prod_{i=1}^{n} \frac{\Gamma_q(ik)\Gamma_q(a+(n-i)k)}{\Gamma_q(k)} \cdot (1-q)^{na+n(n-1)k}.$$

In the product of gammas the exponent of $(1-q)$is

$$\begin{aligned} n(k-1) - \sum_{i=1}^{n} \big((ik-1) + a + (n-i)k - 1\big) &= nk - n - n(nk+a-2) \\ &= -(na + n(n-1)k) + n \end{aligned}$$

so the total exponent is n. So we obtain

$$I_{a,\infty}(1) = n!q^\gamma (1-q)^n \prod_{i=1}^{n} \frac{(q,q)_\infty (q^k;q)_\infty}{(q^{ik};q)_\infty (q^{a+(n-i)k};q)_\infty}. \tag{5.2}$$

From (5.1) it follows that

$$I_{\alpha,\infty}(1)^{-1} \int_{C_n} {}_r\Phi_s(\underline{a},\underline{b};x,y) W_{a,\infty}(x)\mathrm{d}_q x = {}_{r+1}\Phi_s(\underline{a}, q^\alpha t^{n-1}; \underline{b}; y), \tag{5.3}$$

raising the r–index by 1.

In particular, assuming (C1) we have by (2.3) and (2.4)

$$\begin{aligned} I_{1,\infty}(1)^{-1} \int_{C_n} \Pi(x,y;q,t) W_{1,\infty}(x)\mathrm{d}_q x &= {}_2\Phi_0(t^n, qt^{n-1}; y) \\ &= I_{k,\infty}(1)^{-1} \int_{C_n} \Pi''(x,y;q,t) W_{k,\infty}(x)\mathrm{d}_q x \end{aligned}$$

and

$$\int_{C_n} \Pi(x,y;q,t) W_{1,\infty}(x)\mathrm{d}_q x = \int_{C_n} \prod_{i,j} \frac{(x_i y_j;q)_\infty}{(tx_i y_j;q)_\infty} \prod_i (qx_i;q)_\infty \prod_{i<j}\prod_{r=0}^{k-1} (x_i - q^{\pm r} x_j)\mathrm{d}_q x.$$

Here

$$\begin{aligned}
\frac{I_{k,\infty}(1)}{I_{1,\infty}(1)} &= q^{k(k-1)\binom{n}{2}} \prod_{i=1}^{n} \frac{(q^{1+(n-i)k};q)_\infty}{(q^{(n-i+1)k};q)_\infty} \\
&= q^{k(k-1)\binom{n}{2}} \frac{(q;q)_\infty}{(t^n;q)_\infty} \prod_{i=1}^{n} (1-t^i)^{-1} \\
&= q^{k(k-1)\binom{n}{2}} \frac{(q;q)_\infty^n}{(t;q)_\infty^n} \langle 1,1\rangle'_{q,t}.
\end{aligned}$$

Hahn polynomials in one variable

We shall use the notation

$$(x+1)_a = \frac{(x+a)!}{x!}$$

even if a, x are not integers.

Consider the function

(H1) $$F_n^{(a,b)}(x;N) = F_n(x) = (x-n+1)_{a+n}\,(N+1-x)_{b+n},$$

where n, N are integers such that $0 \le n \le N$. Then

(H2) $$\Delta^n F_n(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} (x-r+1)_{a+n}\,(N+1-x-n+r)_{b+n},$$

in which each term is of the form $(x+1)_a(N+1-x)_b$ multiplied by a polynomial in x (and a).

$$\begin{aligned} \frac{(x-r+1)_{a+n}}{(x+1)_a} &= \frac{(x-r+a+n)!}{(x+a)!} \cdot \frac{x!}{(x-r)!} \\ &= (x+a+1)_{n-r}(x-r+1)_{n-r} \end{aligned}$$

and likewise

$$\begin{aligned} \frac{(N-x+1-n+r)_{b+n}}{(N-x+1)_a} &= \frac{(N-x+r+b)!}{(N-x+b)!} \cdot \frac{(N-x)!}{(N-x-n+r)!} \\ &= (N-x+b+1)_r(N-x-n+r+1)_r. \end{aligned}$$

Putting $x = 0$ in (H2), we obtain

(H3) $$\begin{aligned} \Delta^n F_n(0) &= (1)_{a+n}(N+1-n)_{b+n} \\ &= (a+n)!(N+b)!/(N-n)!. \end{aligned}$$

We define the <u>nth Hahn polynomial</u> with parameters a, b to be

(H4) $$G_n^{(a,b)}(x,N) = \frac{(N-n)!}{N!} \cdot \frac{\Delta_x^n\big((x-n+1)_{a+n}(N+1-x)_{b+n}\big)}{(x+1)_a(N+1-x)_b}.$$

It has the following properties:–

(1) <u>Symmetry.</u>

If $g(x) = f(N-x)$ then

$$\begin{aligned} \Delta^n g(x) &= \sum_{r=0}^{n} (-1)^r \binom{n}{r} g(x+n-r) \\ &= \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(N-x-n+r) \\ &= (-1)^n (\Delta^n f)(N-x-n) \end{aligned}$$

Hence

$$G_n^{(a,b)}(N-x;N) = \frac{(N-n)!}{N!}(-1)^n \frac{\Delta^n((N+1-x)_{a+n}(x-n+1)_{b+n})}{(N+1-x)_a(x+1)_b}$$

i.e.,

(H5) $$G_n^{(a,b)}(N-x;N) = (-1)^n G_n^{(b,a)}(x;N).$$

Explicitly, we have

(H6) $$G_n^{(a,b)}(x;N) = \binom{N}{n}^{-1} \sum_{q+r=n} \frac{(-1)^r}{q!r!}(x+a+1)_q(y+b+1)_r(x-r+1)_q(y-q+1)_r,$$

where $y = N - x$.

(2) Leading term.

(H7) $G_n^{(a,b)}(x;N)$ is a polynomial in x of degree n, with leading coefficient $(-1)^n \binom{N}{n}^{-1}\binom{a+b+2n}{n}$.

Proof. . When a, b are positive integers, clearly $(x-n+1)_{a+n}(N+1-x)_{b+n}$ is a polynomial in x of degree $a+b+2n$, with leading coefficient $(-1)^{b+n}$. Hence $G_n^{(a,b)}(x;N)$ is a polynomial of degree

$$(a+b+2n) - n - (a+b) = n$$

with leading coefficient

$$(-1)^n \frac{(N-n)!}{N!}(a+b+n+1)_n = (-1)^n \binom{N}{n}^{-1}\binom{a+b+2n}{n}.$$

Since this is true whenever $a, b \in \mathbb{N}$, it holds generally. □

(3) Values of $G_n^{(a,b)}(x;N)$ at $x = 0$ and $x = N$.

From (H3) we have

$$G_n^{(a,b)}(0;N) = \frac{(N-n)!}{N!} \cdot \frac{(a+n)!}{(N-n)!} \cdot \frac{N!}{a!} = \frac{(a+n)!}{a!},$$

i.e.,

(H8) $$G_n^{(a,b)}(0;N) = (a+1)_n$$

and hence by symmetry

(H9) $$G_n^{(a,b)}(N;N) = (-1)^n(b+1)_n.$$

(4) Orthogonality

Lemma. *("integration by parts") For any two functions f, g, we have*

$$\sum_{x=0}^{N}(\Delta f)(x)g(x) + \sum_{x=0}^{N} f(x+1)(\Delta g)(x) = [fg]_0^{N+1}.$$

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