Smallest Eigenvalue of M-Matrices

- Def: M-Matrix

\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} ; \quad a_{ii} \geq 0 \]

\[ a_{ij} \leq 0, \ i \neq j \]

Row Sums \( s_i = \sum_{j=1}^{n} a_{ij} \geq 0 \)

- Given: Row sums \( s_i \) and off diagonals \( a_{ij}, i \neq j \).

- Diagonal elements computable accurately, sum of positives

\[ a_{ii} = s_i - \sum_{j \neq i} a_{ij} \]
GE on Weakly Diagonally Dominant M-Matrices

• Pivoting, if needed, is diagonal, preserves structure

• One step of GE:
  – Off diagonals: $a_{ij} = a_{ij} - \frac{a_{ik}a_{kj}}{a_{kk}}$
  – Row sums: $s_i = s_i - \frac{a_{ik}}{a_{kk}}s_k$

• Everything is preserved in Schur complementation
  – Weak diagonal dominance
  – M-matrix structure
  – High relative accuracy in $a_{ij}$ and $s_i$

• Yields Cholesky factors
Getting the inverse

- Again no subtractions in solving

\[
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & c_{14} \\
c_{22} & c_{23} & c_{24} & \\
c_{33} & c_{44} & \\
c_{44} & 
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 
\end{bmatrix} =
\begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 
\end{bmatrix}
\]

- Think of \( b \) as \( e_i \) or \( > 0 \) in general.

\[
x_4 = b_4/c_{44}
\]
\[
x_3 = (b_3 - c_{34}x_4)/c_{33}
\]
\[
x_2 = (b_2 - c_{24}x_4 - c_{23}x_3)/c_{22}
\]
\[
x_1 = (b_1 - c_{14}x_4 - c_{13}x_3c_{12}x_2)/c_{11}
\]

- Solving with \( C^T \) analogous \( \Rightarrow A^{-1} \) – positive.

- Accurate (Positive) Inverse = Accurate smallest eigenvalue (even in the nonsymmetric case)