7 Homework Solutions

18.335 - Fall 2004

7.1 Compute the smallest eigenvalue of the $100 \times 100$ Hilbert matrix

$$H_{ij} = \frac{1}{(i + j - 1)}.$$ (Hint: The Hilbert matrix is also Cauchy. The

determinant of a Cauchy matrix $C(i, j) = \frac{1}{(x_i + y_j)}$ is

$$\det C = \prod_{i<j} (x_j - x_i)(y_j - y_i) / \prod_{i<j} (x_i + y_j).$$ Any submatrix of a Cauchy

determinant is also Cauchy. You can use Cramer’s rule in order to com-

pute accurate formulas for $H^1$ and then compute its largest eigen-

value.)

We use Cramer’s rule

$$H_{ij}^{-1} = \frac{(-1)^{i+j} \det(C_{ij})}{\det(H)}$$

together with the formula given for the determinant with $x_i = i$ and $y_j = j - 1$
to get $H_{ij}^{-1}$ :

$$H_{ij}^{-1} = \frac{\prod_{r<s} (x_s - x_r)(y_s - y_r)}{\prod_{r\neq i,s\neq j} (y_s + x_r)} \prod_{i<j} (x_i + y_j)$$

$$\prod_{i<j} (x_j - x_i)(y_j - y_i)$$

$$= \cdots$$

$$= (-1)^{i+j} (i + j - 1) \binom{n + i - 1}{n - j} \binom{n + j - 1}{n - i} \binom{i + j - 2}{i - 1}$$

Having computed the coefficients of $H^{-1}$ we may use any iterative scheme to

estimate the largest eigenvalue which can be inverted to obtain the smallest

eigenvale of $H$. Alternatively one could use a simple matlab command:

$$\lambda_{\text{min}}(H) = \frac{1}{\lambda_{\text{max}}(H^{-1})} = 1 / \max(eig(\text{invhilb}(100))) \approx 5.779700862834800e-151$$

7.2 Trefethen 30.2

- Jacobi algorithm

Calculation of $J : \mathcal{O}(1)$ flops

$J^TA$ alters 2 rows of $A$ only $\Rightarrow 3$ ops $\times 2m$ elements $\Rightarrow \mathcal{O}(6m)$ flops

$J^TJA$ alters 2 columns $\Rightarrow \mathcal{O}(6m)$ flops.

In total we need $\mathcal{O}(12m)$ flops for a single step of Jacobi algorithm (Half in case $A$ is symmetric)

In a single sweep we need $\sim m^2 \mathcal{O}(12m) / 2 = \mathcal{O}(6m^3)$ flops (not counting convergence iterations).

- QR

Requires $\mathcal{O}(4m^3/3)$, a much better algorithm!