5 Homework Solutions
18.335 - Fall 2004

5.1 Trefethen 20.1

⇒ If \( A \) has an LU factorization, then all diagonal elements of \( U \) are not zero. Since \( A = LU \) implies that \( A_{1:k,1:k} = L_{1:k,1:k}U_{1:k,1:k} \) we get that \( A_{1:k,1:k} \) is invertible.

⇐ We prove by induction that \( A_{1:k,1:k} = L_{1:k,1:k}U_{1:k,1:k} \) with

\[
L_{1:k+1,1:k+1} = \begin{pmatrix}
L_{1:k,1:k} & 0 \\
* & 1
\end{pmatrix} \quad \text{and} \quad U_{1:k+1,1:k+1} = \begin{pmatrix}
U_{1:k,1:k} & * \\
0 & u_{k+1}
\end{pmatrix}
\]

with all the elements on the diagonal of \( U_{1:k,1:k} \) are non-zero for any \( k \).

**Step 1** For \( k = 1 \) we have \( A_{1,1:1} = L_{1,1:1}U_{1,1:1} \) with \( L_{1,1:1} = 1, U_{1,1:1} = A_{1,1:1} \neq 0 \).

**Step 2** If that is true for \( k \leq m \) we prove it for \( m + 1 \). Simply choose:

\[
A_{1:m+1,1:m+1} = \begin{pmatrix}
L_{1:m,1:m} & 0 \\
X_m & 1
\end{pmatrix} \begin{pmatrix}
U_{1:m,1:m} & Y_m \\
0 & u_{m+1}
\end{pmatrix}
\]

with

\[
X_m = \begin{bmatrix} a_{m+1,1} & \cdots & a_{m+1,m} \end{bmatrix} U_{1:m,1:m}^{-1}
\]

\[
Y_m = L_{1:m,1:m}^{-1} \begin{bmatrix} a_{1,m+1} \\
\vdots \\
a_{m,m+1} \end{bmatrix}
\]

\[
u_{m+1} = -X_m Y_m
\]

Now we have \( u_{m+1} \neq 0 \) since \( \det(A_{1:m+1,1:m+1}) = \det(U_{1:m,1:m}) u_{m+1} \neq 0 \). Now since \( A = A_{1:n,1:n} = L_{1:n,1:n}U_{1:n,1:n} \) and \( L_{1:n,1:n} \) is unit lower diagonal, \( U_{1:n,1:n} \) is upper diagonal and we complete the proof.

5.2 Trefethen 21.6

Write

\[
A = \begin{pmatrix}
a_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\]

Proceed with the first step of Gaussian elimination:

\[
\begin{pmatrix}
a_{11} & A_{12} \\
0 & A_{22} - \frac{A_{21}}{a_{11}} A_{12}
\end{pmatrix}
\]
Now for $A_{22} - \frac{A_{21}}{a_{11}} A_{12}$ we show that it has the property of strictly diagonally dominant matrices.

$$\sum_{j\neq k} \left| \left(A_{22} - \frac{A_{21}}{a_{11}} A_{12} \right)_{jk} \right| \leq \sum_{j\neq k} \left| (A_{22})_{jk} \right| + \sum_{j\neq k} \left| \frac{1}{a_{11}} \left( (A_{21})_j (A_{12})_k \right) \right|$$

$A$ is strictly diagonally dominant, so we may write

$$\sum_{j\neq k} \left| (A_{22})_{jk} \right| < |(A_{22})_{kk}| - |(A_{12})_k| \text{ and } \sum_{j\neq k} \left| (A_{21})_j \right| < |a_{11}| - |(A_{21})_k|$$

so that in the end we get:

$$\sum_{j\neq k} \left| \left(A_{22} - \frac{A_{21}}{a_{11}} A_{12} \right)_{jk} \right| < |(A_{22})_{kk}| - |(A_{12})_k| + \frac{|(A_{12})_k|}{|a_{11}|} \left( |a_{11}| - |(A_{21})_k| \right)$$

$$< |(A_{22})_{kk}| - \frac{|(A_{12})_k| |(A_{21})_k|}{|a_{11}|} \leq |(A_{22})_{kk}| - \frac{\left( (A_{21})_k (A_{12})_k \right)}{a_{11}}$$

Hence by induction if the property is true for any matrix of dimension $\leq m - 1$ then it is true for any matrix $A$ of dim $A = n$. This means that the submatrices that are created by successive steps of Gaussian elimination are also strictly diagonally dominant and hence we have no need for row swappings.

**5.3 Trefethen 22.1**

Apply 1 step of Gaussian elimination to $A$:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \xrightarrow{1 \text{ Step of GE}} \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ 0 & a_{22}^{(1)} & \cdots & a_{2m}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{m2}^{(1)} & \cdots & a_{mm}^{(1)} \end{pmatrix}$$

where the entries $a_{ij}^{(1)} = a_{ij} - l_{ik} a_{kj}$. Since we used partial pivoting in our calculation, we must have $|l_{ik}| \leq 1$,

$$|\tilde{a}_{ij}| = |a_{ij} - l_{ik} a_{kj}| \leq |a_{ij}| + |l_{ik}| |a_{kj}| \leq |a_{ij}| + |a_{kj}| \leq 2 \max_{i,j} |a_{i,j}|$$

In order to form $A$ we need $m - 1$ such steps, so in the end we have:

$$|u_{ij}| \leq 2 \max_{i,j} \left| a_{i,j}^{(m-2)} \right| \leq 2 \max_{i,j} \left| a_{i,j}^{(m-3)} \right| \leq \cdots \leq 2 \max_{i,j} |a_{i,j}|$$

so that we obtain $|u_{ij}| \leq 2^{m-1} \max_{i,j} |a_{i,j}|$. Therefore

$$\rho = \frac{\max_{i,j} |u_{i,j}|}{\max_{i,j} |a_{i,j}|} \leq 2^{m-1}$$
5.4 Let $A$ be symmetric and positive definite. Show that $|a_{ij}|^2 < a_{ii} a_{jj}$.

Since $A$ is symmetric and positive definite, it has all $a_{ii}$ positive and for any vector $x$ we have $x^T Ax > 0$. Choose $x$ such that $x_k = \delta_{ik} a_{jj} - \delta_{kj} a_{ii}$, where $\delta_{lm}$ is the Kronecker delta, meaning that all the entries of $x$ are zero except the $i$-th and the $j$-th entries which equal to $-a_{ij}$ and $a_{jj}$ respectively. Carrying out the calculation gives $x^T Ax = a_{ii} (a_{ii} a_{jj} - a_{ij}^2) > 0$ thus completing the proof.

5.5 Let $A$ and $A^{-1}$ be given real $n$-by-$n$ matrices. Let $B = A + xy^T$ be a rank-one perturbation of $A$. Find an $O(n^2)$ algorithm for computing $B^{-1}$. Hint: $B^{-1}$ is a rank-one perturbation of $A^{-1}$.

Since $B^{-1}$ is a rank-one perturbation of $A^{-1}$ we may write $B^{-1} = A^{-1} + uv^T$. Then

$$BB^{-1} = (A + xy^T) (A^{-1} + uv^T)$$

$$I = I + Auv^T + xy^T A^{-1} + xy^T uv^T$$

$$0 = Auv^T + xy^T A^{-1} + xy^T uv^T$$

Choosing $u = A^{-1} x$, allows us to write:

$$0 = xv^T + xy^T A^{-1} + xy^T uv^T$$

$$0 = v^T + y^T A^{-1} + y^T uv^T$$

$$0 = v^T (1 + y^T u) + y^T A^{-1}$$

$$v^T = \frac{-y^T A^{-1}}{1 + y^T A^{-1} x}$$

Hence $B^{-1}$ is given by:

$$B^{-1} = A^{-1} - \frac{A^{-1} xy^T A^{-1}}{1 + y^T A^{-1} x}$$

It is easy to see that the inverse can be computed in $O(n^2)$ operations.