(2) Möbius strip.

There exists no differentiable field of unit vectors.

III. The Gauss map

Measure how a regular surface $S$ "pulls away" from $T_p S$ near a point $p \in S$. Idea: Look at how a normal vector to $S$ changes near $p \in S$.

Def. Let $S$ be a surface with an orientation $N$ (= choice of smooth field of unit normal vectors). Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Then the map $N : S \to S^2$ is the Gauss map of $S$.

The map $N$ is smooth, thus we can consider its differential $dN_p : T_p S \to T_{N(p)} S^2$.

Lemma $T_{N(p)} S^2 = T_p S \subseteq \mathbb{R}^3$.

Proof. $T_p S = \{ \mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{v}, N(p) \rangle_{\mathbb{R}^3} = 0 \}$.

On the other hand, if $\mathbf{x} : (-1, 0) \to S^2$, $\mathbf{x}(0) = N(p)$, then

1. $d\mathbf{x}(0) \mathbf{x}'(0) = 2 \mathbf{x}(0) \cdot \mathbf{x}'(0) = 2 N(p) \cdot \mathbf{x}'(0)$,

2. $T_{N(p)} S^2 = \{ \mathbf{v} \in \mathbb{R}^3 : \langle \mathbf{v}, N(p) \rangle_{\mathbb{R}^3} = 0 \}$.

Therefore, $dN_p : T_p S \to T_p S$.  

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Even if $S$ is non-orientable, every $p \in S$ has an open neighborhood $V \subset S$ which is orientable, thus $\exists$ unit normal field $N : V \to S^2$ on $V$, and $dN_p : T_p S \to T_p S$, $p \in V$.

**Examples**

1. $P = \left\{ \frac{r}{r + u} \mathbf{w}_1 + v \mathbf{w}_2, \right\}$, $\mathbf{w}_1, \mathbf{w}_2$ orthonormal
   \[ N(p) = \mathbf{w}_1 \mathbf{x} \mathbf{w}_2, \text{ thus } dN_p = 0. \text{ (That is, } dN_p(\mathbf{w}) = 0 \text{ for all } \mathbf{w} \in T_p S).\]

2. $S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$ has unit normal vector fields
   \[ N(x,y,z) = (x,y,z), \quad \tilde{N}(x,y,z) = (-x,-y,-z). \]

Consider a curve $\alpha : (-1,1) \to S^2$, $\alpha(t) = (x(t),y(t),z(t))$.

Then $dN_{\alpha(t)}(\alpha'(t)) = \frac{d}{dt} N(\alpha(t))|_{t=0} = \frac{d}{dt} \alpha(t)|_{t=0} = \alpha'(0)$.

$\Rightarrow dN_{\tilde{N}}(\mathbf{v}) = -\mathbf{v} \quad \forall \mathbf{v} \in T_p S^2$ : differs from $dN_p$ by an overall sign.

3. Let $S = \{(x,y,z) : x^2+y^2 = z^2, z > 0\}$. Then $N = (x,y,0)/\varepsilon$ and $\tilde{N} = -N$ are two fields of unit normal vectors.

Let $\alpha(t) = (x(t),y(t),z(t))$ be a smooth curve, $p = \alpha(0)$. Then
\[ dN_p(\alpha'(t)) = \frac{dx}{dt} N(\alpha(t))|_{t=0} = \frac{dx}{dt} (x(0),y(0),0)|_{t=0} = \varepsilon (x'(0),y'(0),0), \]

Thus, if $\mathbf{v} = \mathbf{z}'(0) \in T_p S$ is parallel to the $z$-axis (i.e., $x'(0) = y'(0) = 0$),
then $dN_p(\mathbf{v}) = 0$,
if $\mathbf{v} = \mathbf{z}'(0) \in T_p S$ is parallel to the $xy$-plane (i.e., $z'(0) = 0$),
then $dN_p(\mathbf{v}) = \varepsilon \mathbf{v}$.

$\Rightarrow \mathbf{v}, \mathbf{w}$ are eigenvectors of $dN_p$ with eigenvalues $\varepsilon, \varepsilon$, respectively.

Moreover, $d\tilde{N}_p = -dN_p$, so $\mathbf{v}, \mathbf{w}$ are eigenvectors of $d\tilde{N}_p$. 

\( dN_p \) with eigenvalues \( 0, \pm \frac{1}{2} \).

\[ S = \{ z = y^2 - x^2 \} = \mathbb{R}^3, \quad \mathbf{x}(u,v) = (u,v,v^2 - u^2) \] - hyperbolic paraboloid,

\( p = (0,0,0) \in S \).

Normalized normal \( \mathbf{N}(u,v) = \frac{\mathbf{z}_u \times \mathbf{z}_v}{\lVert \mathbf{z}_u \times \mathbf{z}_v \rVert} = \left( \frac{2u}{1+4u^2}, -\frac{2v}{1+4u^2}, \frac{1}{1+4u^2} \right) \).

Consider the curve \( \mathbf{z}(t) = \mathbf{x}(t; 0) \), then \( \mathbf{z}'(0) = (1,0,0) \),

\[ dN_p(\mathbf{z}(0)) = N_{\mathbf{x}}(0,0) = \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) = \mathbf{z}'(0). \]

For \( \beta(t) = \mathbf{z}(0,t) \), \( \beta'(0) = (0,1,0) \),

\[ dN_p(\beta'(0)) = N_{\mathbf{z}}(0,0) = \left( \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right) = \beta'(0). \]

Therefore, \( dN_p \) has eigenvalues \( \pm \frac{1}{2} \) with eigenvectors \( \{ (1,0,0) \} \) and \( \{ (0,1,0) \} \).

(5) Relationship with curvature of curves:

Consider \( p \in S \), \( S = \text{regular surface} \). Let \( \mathbf{v} \in T_p S \), \( k \equiv 1 \),

and let \( P \) = plane through \( p \) spanned by \( \mathbf{v}, \mathbf{N}(p) \).

Let \( \mathbf{z} : I \to \partial S \cap P \) be an arc length parameterization near \( p \), with \( \mathbf{z}(0) = p, \mathbf{z}'(0) = \mathbf{v} \).

Since \( \mathbf{z}(s) \in \partial S \cap P \), the normal vector of \( \mathbf{z} \) satisfies \( \mathbf{n}(0) = \pm \mathbf{N}(p) \).

\[ \Rightarrow \pm k(0) = \langle k(0) \mathbf{n}(0), \mathbf{N}(p) \rangle = \langle \mathbf{z}'(0), \mathbf{N}(\mathbf{z}(0)) \rangle_{\mathbb{R}^3} \]

\[ = \frac{ds}{ds} \left. \langle \mathbf{z}(s), \mathbf{N}(\mathbf{z}(s)) \rangle \right|_{s=0} - \langle \mathbf{z}(0), dN_p(\mathbf{z}(0)) \rangle_{\mathbb{R}^3} \]

\[ = 0 \quad \text{(always)} \]

\[ = -\langle dN_p(\mathbf{z}(0)), \mathbf{z}(0) \rangle_{\mathbb{R}^3}. \]
Prop. The differential $dN_p: T_p S \rightarrow T_p S$ is self-adjoint. That is,

\[(\ast) \quad \langle dN_p(v), \omega \rangle = \langle v, dN_p(\omega) \rangle, \quad v, \omega \in T_p S.\]

**Corollary.** There exists an orthonormal basis $e_1, e_2$ of $T_p S$ and real numbers $k_1, k_2 \in \mathbb{R}$ such that $dN_p(e_1) = -k_1 e_1$, $dN_p(e_2) = -k_2 e_2$.

**Definition.** $k_1, k_2$ are called the principal curvatures of $S$ at $p$.

A vector $v \in T_p S$ is called a principal direction at $p$ if $k_1 v = dN_p(v)$ or $k_2 v = dN_p(v)$ is called a principal direction at $p$. (If $k_1 \neq k_2$, the principal directions are $\pm e_1, \pm e_2$.)

**Proof of Proposition.** It suffices to check (\ast) for a basis $v, \omega$ of $T_p S$.

Let $\mathbf{u} : U \subseteq \mathbb{R}^2 \rightarrow S$ be a parameterization of $S$ near $p = \mathbf{u}(u_0, v_0)$, and let $N : U \rightarrow \mathbb{R}^3$ denote the Gauss map in local coordinates (i.e., $N(u, v) = \text{unit normal}$ vector at $\mathbf{u}(u, v)$), then $dN_p(x, u, v) = \frac{\partial}{\partial s} N(u_0 + s, v_0)|_{s = 0} = N_u(u_0, v_0)$.

Taking $\bar{v} = x_u$, $\bar{\omega} = x_v$, we need to show:

\[\langle N_u, x_v \rangle = \langle x_u, N_v \rangle \quad (\#)\]

Now, we have

\[\langle N, x_v \rangle = 0 \quad \Rightarrow \quad \langle N_u, x_v \rangle + \langle N, x_u \rangle = 0. \quad (1)\]

\[\langle x_u, N \rangle = 0 \quad \Rightarrow \quad \langle x_u, N_v \rangle + \langle x_v, N \rangle = 0. \quad (2)\]

Since $x_v = x_u^*$, subtracting the 2nd from the 1st eqn proves (\#). \(\square\)

**Def.** The quadratic form $\mathcal{II}_p$ on $T_p S$, defined by

\[\mathcal{II}_p(v) := -\langle dN_p(v), v \rangle, \quad v \in T_p S,\]

is called the second fundamental form of $S$ at $p$.

Also write $\mathcal{II}(\bar{v}, \bar{\omega}) := -\langle dN_p(\bar{v}), \bar{\omega} \rangle$, $\bar{v}, \bar{\omega} \in T_p S$ (symmetric bilinear form).

In particular, $k_1 = \max_{v \in T_p S} \mathcal{II}_p(v)$, $k_2 = \min_{v \in T_p S} \mathcal{II}_p(v)$. (Indeed, if $e_1, e_2$ are
orthogonal principal directions, then \( T_p(\cos \theta e_i + \sin \theta e_j) = k_1 \cos \theta e_i + k_2 \sin^2 \theta e_j \).

Maximal for \( \theta = 0, \pi \), minimal for \( \theta = \frac{\pi}{2}, \frac{3\pi}{2} \).

**Definition** Let \( p \in S \). Then the **Gauss curvature** of \( S \) at \( p \) is
\[
K(p) = \det(dN_p) = k_1 k_2.
\]

and the **mean curvature** is
\[
H(p) = -\frac{1}{2} \text{tr}(dN_p) = \frac{k_1 + k_2}{2}.
\]

- **Recall**: \[
\begin{align*}
\text{area}(A_2) & \approx \|\alpha_x \times \alpha_y\| \text{area}(A_1) \\
& = \frac{1}{E G - F^2} \text{area}(A_1).
\end{align*}
\]

Now:
\[
\begin{align*}
\text{area}(A_3) & \approx \|dN_p(e_1) \times dN_p(e_2)\| \text{area}(A_2) \\
& = |k_1 k_2| \|\alpha_x \times \alpha_y\| \text{area}(A_2) \\
& = |K| \text{area}(A_2).
\end{align*}
\]

(Gauss' original interpretation.)

- **Examples**
  - Plane: \( k_1 = 0, k_2 = 0 \Rightarrow K = 0, H = 0 \).
  - Cylinder: \( k_1 = 0, k_2 = -1 \Rightarrow K = 0, H = -\frac{1}{2} \).
  - Unit sphere: \( k_1 = k_2 = -1 \Rightarrow K = 1, H = -1 \).
  - Hyperbolic paraboloid, at \( p = (0,0,0) \): \( k_1 = 2, k_2 = -2 \Rightarrow K = -4, H = 0 \).

**Definition** A point \( p \in S \) is called
- **elliptic** if \( K > 0 \),
- **hyperbolic** if \( K < 0 \),
- **parabolic** if \( K = 0 \) but \( dN_p \neq 0 \),
- **planar** if \( K = 0 \) and \( dN_p = 0 \).