Proof of Bonnet's theorem \((K > 0 \Rightarrow \text{diam}(S) \leq \frac{\pi}{\sqrt{K}})\)

Given any 2 points \(p, q \in S\), there exists (by Hopf-Rinow) a minimal geodesic \(\gamma\) on \(S\) joining \(p\) to \(q\). We shall prove that \(L = d(p, q) \leq \frac{\pi}{\sqrt{K}}\).

Assuming that \(L > \frac{\pi}{\sqrt{K}}\), we consider a variation of the geodesic \(\gamma: [0, 1] \to S\) defined as follows: let \(w_0 \in T_{\gamma_0}S\) be a unit vector, \(L \gamma'(0)\), and let \(w(s), s \in [0, 1]\), be the parallel transport of \(w_0\) along \(\gamma\). (In particular, \(\|w(s)\| = 1, \langle w(s), \gamma'(s) \rangle = 0 \forall s\).)

Consider the vector field \(V(s) = f(s)w(s)\), with \(f\) to be chosen with \(f(0) = 0, f'(0) = 0\). For a proper variation \(\gamma^r\) of \(\gamma\) with variational vector field \(V\), we then have

\[
L''(r) = \int_0^1 \left( \frac{d}{dr} V^2 - K(s) \|V(s)\|^2 \right) ds = \int_0^1 f'(s)^2 - K(s)f(s)^2 ds.
\]

We claim that for \(f(s) = \sin\left(\frac{\pi}{L} s\right)\), this is negative; indeed,

\[
L''(0) = \int_0^1 \left( \frac{\pi^2}{L^2} \cos^2\left(\frac{\pi}{L} s\right) - K \sin^2\left(\frac{\pi}{L} s\right) \right) ds
\]

\[
K > \frac{\pi^2}{L^2} \quad \Rightarrow 
= \frac{\pi^2}{L^2} \int_0^1 \cos^2\left(\frac{\pi}{L} s\right) ds = \frac{\pi^2}{2L} \cdot \frac{L}{\pi} \cdot \sin\left(\frac{\pi}{L} \right) \bigg|_0^1 = 0.
\]

On the other hand, any curve joining \(p\) to \(q\) has length \(L = L(0)\), hence \(L(0)\) must be a local minimum for all variations \(\Rightarrow L''(0) > 0\).

Contradiction!

\[
\Rightarrow \ L = \frac{\pi}{\sqrt{K}}, \text{ finishing the proof.} \quad \square
\]

Remark regarding the choice of \(f\): using the same ideas and techniques that we used in the discussion of geodesics on \(H\), convince yourself that the functional \(I(f) = \int_0^1 f(s)^2 - K(s)f(s)^2 ds\) has \(f\) as a critical point (for all variations of \(f\)) if \(f''(s) + K(s)f(s) = 0\); this suggests trying \(\sin\) & \(\cos\) for \(f\). Needing to satisfy \(f(0) = 0, f'(0) = 0\) gives the \(f\) above. (Note it is not necessary to find the “best” \(f\); any \(f\) for which \(L''(0) < 0\) serves our purpose in the proof.)
End of semester fun:

VI. Lorentzian geometry in 2 dimensions.

Def. A (2-dimensional) spacetime is an abstract surface \( S \) together with a bilinear form \( \langle \cdot, \cdot \rangle : T_p S \times T_p S \to \mathbb{R} \) of signature \((1,1)\) which depends smoothly on \( p \); its coefficients \( E(u,v), F(u,v), G(u,v) \) in local coordinates \((u,v)\) depend smoothly on \( u,v \).

Example: Minkowski space, \( S = \mathbb{R}^2 \), coordinates \( tx \):
\[
E = -1, \quad F = 0, \quad G = 1. \quad \text{So} \quad \langle a \frac{\partial}{\partial t} + b \frac{\partial}{\partial x}, a \frac{\partial}{\partial t} + b \frac{\partial}{\partial x} \rangle = -a^2 + b^2.
\]

Def. \( S = \) spacetime, \( p \in S \).

(i) \( v \in T_p S \neq 0 \) is called

- \text{timelike} if \( \langle v, v \rangle_p < 0 \)
- \text{lightlike} if \( \langle v, v \rangle_p = 0 \)
- \text{spacelike} if \( \langle v, v \rangle_p > 0 \).

(ii) A regular curve \( x: I \to S \) is called

- \text{timelike}... if \( x'(t) \) is timelike,... for all \( t \in I \).

Physics: light travels along lightlike curves, massive objects along timelike curves.

Example: Minkowski space, \( v = a \frac{\partial}{\partial t} + b \frac{\partial}{\partial x} \):
\[
\begin{array}{c}
\text{Timelike:} \quad \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial t} + \sqrt{1-v^2} (\text{Speed of light: } 1) \\
\text{Lightlike:} \quad \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \\
\text{Spacelike:} \quad \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial t} + 2 \frac{\partial}{\partial x} (\text{faster-than-light})
\end{array}
\]

Def. \( L_p = \{ v \in T_p S : \langle v, v \rangle_p < 0 \} \to T_p S \) is the light cone at \( p \in S \).

Useful way to visualize properties of spacetime: draw light cones at a few points.

Examples:
1. Minkowski: \( x^+ \times \times \times \) (Physics: light (or observer) traveling into \( r < r_0 \) or traversing cannot enter \( r > r_0 \); \( r_0 \): event horizon.}

(2) Toy black hole: \( x^+ \times \times \times \times \)
Rmk. Specifying the light cone at \( p \in \mathbb{S} \) determines \( \langle \cdot, \cdot \rangle_p \) up to a conformal factor (i.e. if \( \langle w, v \rangle_p = 0 \) iff \( \langle \lambda w, \lambda v \rangle_p = 0 \) for \( \lambda \in \mathbb{R}^+ \)), then \( \exists \lambda \in \mathbb{R}, \lambda \neq 0, \) s.t. \( \langle w, v \rangle_p = \lambda \langle w, v \rangle_p \) for \( \forall w, v \in T_p \mathbb{S} \).

- Intrinsic geometry notions apply: parallel transport, geodesics, Gauss curvature, etc.
- "Freely falling observer" moves through \( S \) along timelike geodesics --- geodesics \( y : I \rightarrow S \) (thus with vanishing acceleration \( \dot{y}''(s) = 0 \)) s.t. \( y'(s) \) is timelike.

**Ex. Minkowski**

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"Length" of curves?

**Def.** If \( x : [0,1] \rightarrow \mathbb{S} \) is a smooth timelike curve, then the proper-time along \( x \) is

\[
\tau(x) = \int_0^1 \sqrt{-\langle x'(t), x'(t) \rangle} \, dt \quad (= \text{time elapsed on watch}).
\]

**Fact:** If in a normal neighborhood \( V \subseteq \mathbb{S} \) of \( p \), \( y : [0,1] \rightarrow \mathbb{S} \) is a timelike geodesic connecting \( p \) and \( q \in V \), then \( \tau(y) \geq \tau(x) \) for any other timelike curve \( x : [0,1] \rightarrow \mathbb{S} \), \( x(0) = p \), \( x(1) = q \).

\[
\begin{align*}
\tau(y) & > \tau(x) \quad \text{when } x \text{ and } y \\
& \text{meet again at } q, \text{ less time has elapsed for } x \\
& \text{than for } y. \quad \text{(More generally, the twin paradox)}
\end{align*}
\]

Rmk. The funny signs are due to the negative sign of the "squared length" of timelike vectors.
Def. A time orientation on \( S \) is a smooth vector field \( T \) on \( S \) st \( T(p) \in T_p S \) is timelike for all \( p \in S \). Given \( T \), we call a timelike vector \( v \in T_p S \) future-directed timelike if \( \langle v, T \rangle_p > 0 \).

\[ \text{(2)} \quad \text{Not time orientable.} \]

Ex. (1) Minkowski:

\[ \text{future timelike vectors} \quad \text{light cone} \]

\[ \text{time orientation} \quad \uparrow \]

\[ \text{Examples (expanding universe, de Sitter space): } S = \mathbb{R}^2, \text{ coordinates } (t, x), \]
\[ E = -1, \quad F = 0, \quad G = e^{2t}, \quad \text{(Gaussian curvature } K = -1). \]

\[ \text{Examples of timelike geodesics: } \gamma(t) = (t, x(t)), \quad t \in \mathbb{R}. \]

\[ \text{A lightlike geodesic: } \gamma(t) = (t, x_0 + e^{2t}) \]

\[ \text{Let } p = (t_0, 0). \text{ Then the causal past: } J^-(p) = \{ q \in S : \text{ future timelike or lightlike curve from } q \text{ to } p \} \]

\[ \text{can be computed explicitly: } \]
\[ J^-(p) = \{ (t, x) : t \leq t_0, \quad 1 \leq e^{-t} - e^{-t_0} \} \]

\[ \text{As } t_0 \to \infty, \text{ this converges to the set: } \]
\[ R = \{ (t, x) : 1 \leq e^{-t} \} \]

Consider \( \gamma_1(s) = (s, 0) \) and \( \gamma_2(s) = (s, \frac{1}{s}). \)

Both are geodesics. However, if \( s_0 > \log 2 \) and \( s \in R \) is arbitrary,
there does not exist a future timelike or lightlike curve from $\gamma_2(s)$ to $\gamma_1(s)$! "A stellar explosion at $\gamma_2(s)$ will never be detected by the observer $\gamma_1$!" (This is a perfectly sensible statement often made mysterious by claiming that the spacetime expands faster than the speed of light.)

(2) Big bang: FLRW spacetime, $E(t, x) = -1$, $F(t, x) = 0$, $G(t, x) = t^{2/3}$, $S = \{(t, x) : t > 0\}$.

Gauss curvature: $K(t, x) = \frac{2e}{t^2} \to \infty$ as $t \to 0$. $S$ is past geodesically incomplete; in fact, any past timelike curve starting at some $p \in S$ has finite proper time.

General relativity: 4-dim. spacetimes, with coefficients of metric satisfying a second order partial differential equation (Einstein field equation).

"Einstein vacuum equation": $\forall p \in S$, $\nabla^2_p S, v = 0$;

$R_{q(p,v)}$ := average of the Gauss curvatures $K(p)$ of all surfaces

$\left\{\exp_t(-tv+sw) \right\} \in S$ (where $w \to 0, \omega \perp v$)

$= 0$. 