(3) (Jacob) Let \( \vec{x} : I \rightarrow \mathbb{R}^3 \) be a closed, regular, parameterized curve with everywhere nonzero curvature. Assume the curve \( \vec{x}(s) \) of normal vectors is simple.

Then \( \vec{n}(I) \subset S^2 \) subdivides \( S^2 \) into two regions of equal area \((2\pi)\).

**Pf.** WLOG \( \vec{x}(s) \) parameterized by arc length.

Let \( \tilde{S} = \text{arc length of } \vec{x}(s) ; \text{ write } ds \text{ for } \frac{d}{ds}.

\[ \text{Geodesic curvature of } \vec{n} \text{ is } \kappa_g = \langle \tilde{\vec{n}}, \tilde{\vec{n}} \times \tilde{\vec{n}} \rangle. \]

Have \( \vec{n} = \frac{d\vec{s}}{d\tilde{s}} = \frac{d\tilde{\vec{s}}}{d\tilde{s}} \frac{d\tilde{\vec{s}}}{d\tilde{s}} = (-k\vec{t} + \tau \vec{b}) \frac{d\tilde{\vec{s}}}{d\tilde{s}}, \]

\[ \vec{n} = (-k\vec{t} + \tau \vec{b}) \frac{d\tilde{\vec{s}}}{d\tilde{s}} + (-k\vec{t} + \tau \vec{b}) \frac{d\tilde{\vec{s}}}{d\tilde{s}} \]

\[ = (\frac{d\tilde{\vec{s}}}{d\tilde{s}}) \left( -k\vec{t} + \tau \vec{b} \right) \]

\[ = (\frac{d\tilde{\vec{s}}}{d\tilde{s}}) \left( -k\vec{x}' + \tau \vec{b} \right) = \frac{d}{d\tilde{s}} \text{ arctan}(\frac{\tau}{k}) \cdot \frac{d\tilde{\vec{s}}}{d\tilde{s}} = -\frac{d}{d\tilde{s}} \text{ arctan}(\frac{\tau}{k}). \]

Apply Gauss-Bonnet to a region \( R = S^2 \) bounded by \( \vec{x}(I) : \)

\[ 2\pi = \int_R K d\sigma + \int_{\partial R} k_g(s) d\tilde{s} = \text{area}(R). \]

\[ = 0. \]

**IV. 5. Exponential map.**

We already encountered the idea: given \( p \in S \), we can "shoot out" geodesics from \( p \).

This gives natural ways to parameterize a neighborhood of \( p \), as we will see.

Given \( p \in S, \vec{v} \in T_p S, \exists! \) parameterized geodesic \( g : (-\varepsilon, \varepsilon) \rightarrow S, g(0) = p, g'(0) = \vec{v}. \)

Write \( g(\vec{v}, t) = g(\vec{v}) \) to stress dependence on \( \vec{v}. \)

**Lemma.** If the geodesic \( g(t, \vec{v}) \) is defined for \( t \in (-\varepsilon, \varepsilon) \), then \( g(t, \lambda \vec{v}) \) with \( \lambda > 0 \)

is defined for \( t \in (-\varepsilon, \varepsilon) \), and \( g(t, \lambda \vec{v}) = g(\lambda t, \vec{v}). \)

**Proof.** Let \( g(t) = g(\vec{v}, t), \alpha(t) = g(\lambda t), \) then \( \alpha(0) = g(0), \alpha'(0) = \lambda g'(0) = \lambda \vec{v}, \) and

\( D\alpha(t) = \lambda^2 Dg(\vec{v}) \big|_{t = \lambda t} = 0 \Rightarrow \alpha \) is a geodesic by uniqueness, \( \alpha(t) = g(t, \lambda \vec{v}). \square \)

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Def. If \( v \in T_p S, \|v\| \neq 0 \) is st. \( y(\|v\|, \frac{v}{\|v\|}) = y(1, v) \) is defined, we set:
\[
\exp_p(v) := y(1, v).
\]

Ex. (1) \( S = \text{plane}, \ p \in S, \ v \in T_p S \Rightarrow \exp_p(v) = p + v. \)
(2) \( S = S^2, \ p = (0,0,1), \ v = (\rho \cos \theta, \rho \sin \theta, 0) \)
\[
\Rightarrow \exp_p(v) = (\cos \rho \cos \theta, \cos \rho \sin \theta, \sin \rho).
\]
(So: \( p, \theta \) are polar coordinates (usually denoted \( \theta, r \) in this order).)

Prop. Given \( p \in S, \exists \varepsilon > 0 \) st. \( \exp_p \) is defined and smooth in \( \{ v \in T_p S : \|v\| < \varepsilon \} \) \( p \in (\cdot) \).

Pf. ODE theory for the geodesic equation. \( \square \)

Generalizing example (1), we can always use \( \exp_p \) to produce parameterizations of \( S \):

Prop. \( \exp_p : B_p(\varepsilon) \subset T_p S \rightarrow S \) is a diffeomorphism \( \exp : U \subset B_p(\varepsilon) \rightarrow \exp(U) \subset S \) in a neighborhood \( U \subset B_p(\varepsilon) \) of \( 0 \in T_p S. \)

Pf. We wish to apply the inverse function theorem. Compute
\[
d(\exp_{p})_0(v) = \frac{d}{dt} \exp_p(0 + tv) \big|_{t=0} = \frac{d}{dt} y(1, tv) \big|_{t=0} \\
= \frac{d}{dt} y(t, v) \big|_{t=0} = v.
\]
\( \Rightarrow d(\exp_p)_0 : T_p S \rightarrow T_p S \) (really \( T_0(T_p S) \rightarrow T_p S \)) is invertible. \( \square \)

Def. If \( U \subset T_p S \) is a neighborhood of \( 0 \), and \( \exp : U \rightarrow \exp(U) \) is a diffeomorphism then \( v \mapsto \exp_p(v) \) is called a normal neighborhood of \( p \).

Coordinate systems on \( V \):
- Normal coordinates corresponding to a system of rectangular coordinates in \( T_p S. \) (That is: \( e_1, e_2 \in T_p S \) orthonormal \( \Rightarrow x(u,v) = \exp_p(ue_1 + ve_2), \)
  with \( (u,v) \) the normal coordinates.)
- Geodesic polar coordinates corresponding to polar coordinates in \( T_p S. \)
Remark. In normal coordinates \((u,v)\), the geodesics through \(p\) are precisely the images under \(\exp_p\) of straight lines \((u(t),v(t)) = (at,bt)\).

We have \(E(p) = 1, F(p) = 0, G(p) = 1\). Hence \(E_u - E_v = F_u - F_v = G_u - G_v = 0\) at \(p\).

We shall study geodesic polar coordinates in some detail.

Choose in \(T_pS\), \(p \in S\), a system of polar coordinates \((r, \Theta)\) where \(r\) is the radius \((r = ||\cdot||_{p})\) and \(\Theta \in (0,2\pi)\) is the polar angle relative to a closed half line \(l \subset T_pS\).

\[\exp_p : T_pS \to S\]

Set \(L = \exp_p(L)\). With \(V = \exp_p(U)\) a normal neighborhood of \(p\), \(\exp_p : V \cap L \to U \cap L\) is still a diffeomorphism.

Def. The images under \(\exp_p\) of circles in \(U\) centered at \(O\) are geodesic circles of \(V\).

\(\{r = \text{const.}\}\) lines in \(U\) through \(O\)

\(\{\Theta = \text{constant}\}\) radial geodesics.

(In \(V \cap L\), there are curves \(\{r = \text{const.}\}\).)

Proposition. Let \(\mathbf{x} : U \cap L \to V \cap L\) be a system of geodesic polar coordinates \((p, \Theta)\).

Then \(E = E(p, \Theta), F, G\) satisfy \(E = 1, F = 0, \lim_{r \to 0} G = 0, E_m(TG)_{\Theta} = 1\).

\((\text{Example}\, \, S = \{r = 0\} \subset \mathbb{R}^2, p = (0,0), L = \{(x,0)\}, x(p,\Theta) = (g \cos \Theta, g \sin \Theta, 0), E = 1, F = 0, G = g^2)\)

Proof. We have \(\mathbf{x}(p, \Theta) = \exp_p(g \ast \mathbf{w}_\Theta), \mathbf{w}_\Theta = \text{unit vector in } T_pS, \mathbf{x}(1, \mathbf{w}_\Theta) = \Theta\).

\(E = \langle \delta, \mathbf{x}_{\Theta} \rangle, \text{ and } p \to \mathbf{x}(p, \Theta)\) is an arc length parametrized geodesic for all \(\Theta_0\) \(\Rightarrow E = 1\).

\(F\) this is the content of Gauss' Lemma, which we state separately.
Lemma (Gauss). Suppose $\exp_p(w)$ is defined. Then
\[
\langle (d \exp_p)_v(a), (d \exp_p)_v(b) \rangle_{\exp_p(w)} = \langle v, w \rangle_p \quad \forall v, w \in T_p S.
\]
Cor. For $\vec{v} \perp \vec{w}$, this implies that geodesic circles are orthogonal to radial geodesics:

$\langle (d \exp_p)_v(w), (d \exp_p)_v(v) \rangle = $ tangent vector of radial geodesic; $\langle (d \exp_p)_v(w), (d \exp_p)_v(v) \rangle = $ tangent vector of geodesic circle $\Rightarrow t = 0$ in the Prop.

Proof of Lemma. Nothing to do if $w = 0$.

For $w \neq 0$, write $\vec{v} = \vec{v}_r + \vec{v}_t$, $\vec{v}_t = c \vec{w}$, $c \in \mathbb{R}$, $\langle \vec{v}_t, \vec{w} \rangle = 0$.

Already know

$\langle (d \exp_p)_v(w), (d \exp_p)_v(v) \rangle = c \| \vec{w} \|^2 = \langle w, v \rangle_p$, so need to consider only the case $\vec{v} = \vec{v}_r \perp \vec{w}$.

Since $\exp_p$ is defined on an open set (ODE theory), $\exists \varepsilon > 0$ s.t. $\exp_p$ is defined for $t = 0, 1$, $\vec{w}(s) = \vec{w} + s \vec{v}$, $s \in (-\varepsilon, \varepsilon)$.

Let $\vec{f}(t, s) = \exp_p(t \vec{w}(s))$; for fixed $s$, $t \mapsto \vec{f}(t, s)$ is a geodesic.

Compute:

(i) $\frac{\partial}{\partial t} \vec{f}(1, s) = \frac{\partial}{\partial t} \exp_p(t \vec{w}(s)) \bigg|_{t=1} = (d \exp_p)_{\vec{w}(s)}(\vec{w}(s))$.

(ii) $\frac{\partial}{\partial s} \vec{f}(1, s) = (d \exp_p)_{\vec{w}(s)}(\vec{w}(s)) \Rightarrow \frac{\partial}{\partial s} \vec{f}(1, 0) = (d \exp_p)_{\vec{w}}(\vec{w}).$

Furthermore,

$\frac{\partial}{\partial s} \langle \frac{\partial}{\partial t} \vec{f}(t, s), \frac{\partial}{\partial s} \vec{f}(t, s) \rangle = \langle \frac{\partial^2}{\partial t^2} \vec{f}(t, s) \frac{\partial}{\partial s} \vec{f}(t, s) + \frac{\partial}{\partial t} \vec{f}(t, s) \frac{\partial}{\partial s} \vec{f}(t, s), \frac{\partial}{\partial s} \vec{f}(t, s) \frac{\partial}{\partial s} \vec{f}(t, s) \rangle$

vector fields along the curve $t \mapsto \vec{f}(t, s)$, $t \mapsto \vec{f}(t, s)$ is a geodesic.

$= 0$ since
\[
\begin{align*}
\langle \frac{\partial^2}{\partial t^2} T(t, s), \frac{\partial^2}{\partial t^2} T(t, s) \rangle &= \langle \frac{\partial^2}{\partial t^2} T(t, s), \frac{\partial^2}{\partial s^2} T(t, s) \rangle, \\
&= \langle \frac{\partial}{\partial t} T(t, s), \frac{\partial}{\partial s} T(t, s) \rangle = \frac{1}{2} \frac{\partial}{\partial s} \langle \frac{\partial}{\partial t} T(t, s), \frac{\partial}{\partial t} T(t, s) \rangle = \langle \omega(s), \nu(s) \rangle.
\end{align*}
\]

For \( s = 0 \), \( \langle \omega(0), \nu(0) \rangle = \langle \omega, \nu \rangle = 0 \). Therefore, \( \langle \frac{\partial^2}{\partial t^2} T(t, 0), \frac{\partial^2}{\partial s^2} T(t, 0) \rangle \) is constant \( t \).

\[
\Rightarrow \langle \frac{\partial^2}{\partial t^2} (0, 1), \frac{\partial^2}{\partial s^2} (0, 1) \rangle = \lim_{t \to 0} \langle \frac{\partial^2}{\partial t^2} (t, 0), \frac{\partial^2}{\partial s^2} (t, 0) \rangle
\]

\[
= \lim_{t \to 0} \langle \exp(t \nu), \exp(t \nu) \rangle = 0.
\]

Back to the Proposition. In geodesic polar coordinates, have proved \( E=1, F=0 \).

Define normal coordinates \( \hat{u} = \rho \cos \Theta, \hat{v} = \rho \sin \Theta \). (These are smooth coordinates near \( p \) via \( g(\hat{u}, \hat{v}) \) \to \( \exp(\hat{u} e_1 + \hat{v} e_2) \), \( e_1, e_2 \) = unit vector in direction of \( l_1, l_2 \), \( e_1 \) = orthogonal unit vector.)

We have \( \sqrt{E} - F^2 = \| \hat{x}_r \times \hat{x}_\beta \| = \left| \det \begin{pmatrix} \frac{\partial \hat{r}}{\partial \alpha} & \frac{\partial \hat{r}}{\partial \beta} \\ \frac{\partial \hat{\beta}}{\partial \alpha} & \frac{\partial \hat{\beta}}{\partial \beta} \end{pmatrix} \right| \| \hat{y}_r \times \hat{y}_\beta \| \)

\[
\Rightarrow \sqrt{G} = \sqrt{E} - F^2. \quad \text{But at } p, \quad E = \hat{r} - 1, \quad F = 0. \quad \text{1st fund. form in } \hat{u}, \hat{v} \text{ coordinates.}
\]

\[
\Rightarrow \lim_{s \to 0} \sqrt{G} = 0, \quad \lim_{s \to 0} \left. \sqrt{G} \right|_p = \sqrt{E} - F^2 \left|_p = 1. \right.
\]

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Remark. In a coordinate system with \( E=1, F=0 \), we have.

\[
K = -\frac{1}{\sqrt{E}} \left( \frac{E_p}{\sqrt{E}} + \left( \frac{G_p}{\sqrt{E}} \right)_\beta \right) = -\frac{1}{\sqrt{G}} \left( \frac{G_p}{2s} \right) = -\frac{1}{\sqrt{E}} \left( \frac{E_p}{\sqrt{E}} \right)_\beta.
\]

\[
K = -\left( \frac{G_p}{E} \right)_\beta.
\]

Now, in general, knowledge of the Gauss curvature (as a function of 2 variables in local coordinates) does not determine the 1st fund. form of a surface. However: