Theorem (Local Gauss-Bonet) \( \bar{x}: U \to S \) orthogonal parameterization, \( S \) oriented. \( R \subset S \) simple region,

\( \bar{x}: [0,1] \to S \) simple, closed, piecewise regular, arc-length parameterized curve,

\( \bar{x}(0,1) = \partial R \), positively oriented. Let \( \bar{x}(s_0), \ldots, \bar{x}(s_k) \) be the vertices of \( \partial R \),

and \( \theta_0, \ldots, \theta_k \) the external angles. Then \( (s_k + 1 = 1) \)

\[
\sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s) \, ds + \int_{\partial R} K \, ds + \frac{k}{2} \sum_{i=0}^{k} \theta_i = 2\pi.
\]

Proof: Write \( \bar{x}(s) = \bar{x}(u(s), v(s)) \). Recall: \( k_g(s) = \left[ \frac{D\bar{x}'}{ds} \right]' \left( = \left< \frac{D\bar{x}'}{ds}, N \times \bar{x}'(s) \right> \right) \)

\[
= \frac{1}{2\sqrt{EG}} (E_u v' - E_v u') + \phi_i', \quad s \in [s_i, s_{i+1}].
\]

\( \phi_i \) = angle from \( \bar{x}_u \) to \( \bar{x}'(s) \) \( \Rightarrow \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s) \, ds = \frac{k}{2} \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} \frac{1}{2\sqrt{EG}} (-E_v u' + E_u v') \, ds + \sum_{i=0}^{k} \left( \phi_i(s_{i+1}) - \phi_i(s_i) \right) \)

Recall Green's thm: \( \int_{\partial R} F \, dx + G \, dy = \iint_{R} \left( G_x - F_y \right) \, dx \, dy \)

\[
\Rightarrow \int_{\partial R} \frac{1}{2\sqrt{EG}} \left( -E_v \frac{du}{ds} + E_u \frac{dv}{ds} \right) \, ds = \iint_{R} \frac{1}{2\sqrt{EG}} \left( -E_v du + E_u dv \right) = \iint_{R} \left[ \left( \frac{G_u}{2\sqrt{EG}} \right)_u + \left( \frac{G_v}{2\sqrt{EG}} \right)_v \right] \, du \, dv
\]

\[
= -\iint_{\bar{x}'(R)} K \sqrt{EG} \, du \, dv = -\iint_{\bar{x}'(R)} K \sqrt{EG} + 2 \, du \, dv = -\iint_{R} K \, ds
\]

Plug in: \( \sum_{i=0}^{k} \int_{s_i}^{s_{i+1}} k_g(s) \, ds + \int_{\partial R} K \, ds = \frac{k}{2} \left( \phi_i(s_{i+1}) - \phi_i(s_i) \right) = 2\pi - \sum_{i=0}^{k} \theta_i \) (Thm. of turning tangents) \( \square \)
Ex. let \( \bar{x} : [0, l] \to \mathbb{R} \) parameterize \( \mathcal{D} \), \( R = \) simple region, as above. Let \( \bar{p} = \bar{x}(0) = \bar{x}(l) \), \( \omega_0 \in T_p \mathcal{D} \).

Let \( \omega(s) \) = parallel transport of \( \omega(0) = \omega_0 \) along \( \bar{x} \). Let's assume \( \bar{x} \) is regular.

Then \( 0 = \int_0^l \left[ \frac{d\omega}{ds} \right] ds = \int_0^l \frac{1}{2iE_0} \left( C_n \omega'(s) - E_v \omega'(s) \right) ds + \int_0^l \varphi'(s) ds, \)

\( \varphi(s) = \) angle from \( \bar{x}_t \) to \( \omega(\bar{x}) \). This \( \pi = \int \int_R K \, ds + \int_0^l \varphi'(s) \, ds \)

\( \Rightarrow \varphi(l) - \varphi(0) = \int \int_R K \, ds \) how much \( \omega \) turns

Corollary \( K(p) = \lim \frac{\varphi(R) - \varphi(0)}{\text{area}(R)} \). (E.g. take \( R = \bar{x}(\varepsilon\text{-ball around } \bar{x}'(p)) \), \( \varepsilon \to 0 \).

For global Gauss-Bonnet, need some topological facts:

Def. A triangle is a simple region with three vertices and external angles \( \Theta_i \neq 0 \).

Def. A connected region \( R \) is regular if \( R \) is compact and its boundary \( \partial R \) is a finite union of simple closed curves which do not intersect.

Def. A triangulation of a regular region \( R \subset S \) is a finite family

\( J = \{T_1, \ldots, T_n\} \) of triangles \( T_i \subset S \) s.t.
(i) \( \bigcup_{i=1}^{n} T_i = \mathbb{R} \), (ii) If \( T_i \cap T_j \neq \emptyset, i \neq j \), then \( T_i \cap T_j \) is either a common edge or a common vertex of \( T_i \) and \( T_j \).

Given a triangulation \( T \) of \( \mathbb{R} \), let \( F, E, V \) denote the number of triangles (\( F \), "faces"), edges (\( E \), counted once), vertices (\( V \), counted once). \( \chi(\mathbb{R}) := F - E + V \) is called the Euler characteristic of \( \mathbb{R} \).

Facts: (i) Every regular domain admits a triangulation.

(ii) \( \chi(\mathbb{R}) \) is independent of the triangulation.

(iii) Let \( S \) be an oriented regular surface covered by coordinate charts \( \tilde{x}_i : U_i \to S \) compatible with the orientation \( S \). Let \( R \subset S \) be a regular region. Then there exists a triangulation \( T \) of \( R \) s.t. every \( T \in T \) is contained in a single chart \( \tilde{x}_i(U_i) \). Moreover, if the boundary of each \( T \) is oriented positively, then adjacent triangles determine along their common edge opposite orientations.
EX. (1) \( \begin{align*} F = 1, \ E = 3, \ V = 3 \implies \chi(R) = 1. \end{align*} \)

(2) \( \chi(R) = 1. \)

(3) \( \chi(\Delta) = 0. \)

(4) \( \n \text{ holes } \implies \chi = 1-n. \)

(5) Regard a compact surface as a regular region with empty boundary. \( S^2: \)
\[ F = 8, \ E = 12, \ V = 6 \implies \chi(S^2) = 2. \]

(6) Let \( S = \text{torus with g holes} = \text{genus g surface (compact)} \)
\[ \implies \chi(S) = 2 - 2g. \]
\[ \chi(\bigcirc) = 0. \]

Example (6) (and (5)) gives all regular compact surfaces up to homeomorphism. (orientable, connected.) If \( S \) is compact, \( \chi(S) > 0 \implies S \simeq \text{sphere.} \)
**Global Gauss-Bonnet.** Let $R \subset S$ be a regular region inside of an oriented surface $C$. Denote by $C_1, \ldots, C_p$ the closed, simple, positively oriented, piecewise regular curves comprising $\partial R$, and denote by $\Theta_1, \ldots, \Theta_q$ the collection of exterior angles of the $C_i$. Then

$$\sum_{i=1}^{p} \int_{C_i} \log \Theta \, ds + \iint_{R} K \, d\sigma + \sum_{j=1}^{q} \Theta_j = 2\pi \chi(R),$$

(sum of integrals, arc-length as variable,)
(over the regular arcs of $C_i$)

**Proof.** From local Gauss-Bonnet + counting.

**Cor. (Improved "local" Gauss-Bonnet.)** If $R \subset S$ is a simple region ($\Rightarrow \chi(R) = 1$), then

$$\sum_{i=0}^{k} \int_{s_i} \log \Theta \, ds + \iint_{R} K \, d\sigma + \sum_{i=0}^{k} \Theta_i = 2\pi.$$  \quad \text{genus} = \# holes

**Cor.** Let $S$ be orientable compact surface. Then $\iint_{S} K \, d\sigma = 2\pi \chi(S) = 2\pi (2 - 2g).$  \quad \text{genus} \geq 0

$$\int_{T_2} K \, d\sigma = 0.$$
Applications. 1) S = connected, compact, \( k > 0 \) \( \Rightarrow \) S is homeomorphic to a sphere.

Proof. \( \int_S k \, d\sigma > 0 \Rightarrow \chi(S) > 0 \Rightarrow S \cong \mathbb{S}^2. \quad \square \)

2) S = connected, orientable, compact, \( k > 0 \). \( \gamma_1, \gamma_2 \) = two distinct simple closed geodesics

\( \Rightarrow \gamma_1 \cap \gamma_2 \neq \emptyset \).

Proof. S is homeomorphic to a sphere.

If \( \gamma_1 \cap \gamma_2 = \emptyset \),
then \( \gamma_1 \cup \gamma_2 = \partial R \) for a region R
with \( \chi(R) = 0 \). By Gauss-Bonnet,
\( 2\pi \chi(R) = \int_R k \, d\sigma > 0 \). \( \square \)
Let \( \mathbf{x} : I \to \mathbb{R}^3 \) be a closed, regular, parameterized curve with everywhere nonzero curvature.

Assume the curve \( \hat{n}(s) \) of normal vectors is simple. Then \( \hat{n}(I) = S^2 \) subdivides \( S^2 \) into two regions of equal area (2π).

Pf: WLOG \( \mathbf{x}(s) \) is parameterized by arclength.

Let \( \hat{s} = \text{arclength of } \hat{n}(s) \); write \( \cdot \) for \( \frac{d}{ds} \), \( \cdot' = \frac{d}{ds} \).

Geodesic curvature of \( \hat{n}(I) \) is \( \overline{k}_g = \langle \hat{n}'', \hat{n} \times \hat{n} \rangle \).

Have \( \hat{n}' = \frac{d\hat{n}}{ds} = \frac{d\hat{n}}{ds} \frac{ds}{d\hat{s}} = (-k\hat{t} - t\hat{b}) \frac{ds}{d\hat{s}} \),

\[
\hat{n}'' = (-k\hat{t} - t\hat{b}) \left( \frac{d^2s}{d\hat{s}^2} \right) + (-k'\hat{t} - t'\hat{b}) \left( \frac{ds}{d\hat{s}} \right)^2
- (k^2 + t^2) \hat{n} \left( \frac{ds}{d\hat{s}} \right)^2,
\]

\[
1 = ||\hat{n}'||^2 = \left( \frac{ds}{d\hat{s}} \right)^2 \quad ||\hat{n}''||^2 = \left( \frac{ds}{d\hat{s}} \right)^2 (k^2 + t^2) \implies \left( \frac{ds}{d\hat{s}} \right)^2 = \frac{1}{k^2 + t^2}.
\]

\[
\overline{k}_g = \frac{ds}{d\hat{s}} \langle \hat{n}'', \hat{n} \times \hat{n} \rangle = \left( \frac{ds}{d\hat{s}} \right)^3 (Tk' - t'k) = \frac{ds}{d\hat{s}} \frac{tk' - t'k}{t^2 + k^2}
= -\frac{ds}{d\hat{s}} \frac{d}{d\hat{s}} \left( \arctan \left( \frac{t}{k} \right) \right) = -\frac{d}{d\hat{s}} \left( \arctan \left( \frac{t}{k} \right) \right).
\]

Apply Gauss-Bonnet: \( 2\pi = 2\pi \chi(R) = \int_{\partial R} k \, d\sigma + \int_{\partial R} \overline{k}_g \, d\sigma = \int_{\partial R} k \, d\sigma = \text{area}(R). \)