The sign of the geodesic curvature of a curve $C \subset S$ depends on the orientation of $S$ and the orientation of $C$.

**Def.** Let $w$ be a smooth field of unit tangent vectors along $\alpha: I \to S$, $S$ an oriented surface. Then

$$\frac{dw}{dt} \perp w \implies \frac{dw}{dt} \text{ is orthogonal to } N \text{ and to } w \quad (\langle \frac{dw}{dt}, w \rangle = 0).$$

(0)$\frac{d}{dt} (\|w(t)\|^2)$

Define $\left[ \frac{dw}{dt} \right] := \langle \frac{dw}{dt}, N \times w(t) \rangle$ is the algebraic value of the covariant derivative.

(=) $\frac{dw}{dt} = \left[ \frac{dw}{dt} \right] \cdot (N \times w(t)).$

**Ex.** $\dot{x} = \text{arc length parameterized curve on } S \implies k_g(s) = \left[ \frac{d\dot{x}}{ds} \right]$. 

$N \times w(t)$
Lemma. Let \( w_1 \) and \( w_2 \) be two smooth vector fields along \( \vec{x} : I \to S \) with \( \| w_1(t) \| = \| w_2(t) \| = 1 \) \( \forall t \in I \).

Then \( \begin{bmatrix} \frac{Dw_2}{dt} \\ \frac{Dw_1}{dt} \end{bmatrix} = \psi'(t) \) where \( \psi(t) = \vec{x}(w_1(t), w_2(t)) \).

Proof. \( w_1^* = N \times w_1, \ w_2^* = N \times w_2 \). Then
\[
\begin{align*}
  w_2(t) &= (\cos \psi(t)) w_1(t) + (\sin \psi(t)) w_1^*(t) \\
  w_2^*(t) &= - (\sin \psi(t)) w_1(t) + (\cos \psi(t)) w_1^*(t).
\end{align*}
\]

Compute \( w_2'(t) = -\psi' \sin \psi \ w_1 + \psi' \cos \psi \ w_1^* \\
+ \cos \psi \ w_1^* + \sin \psi \ (w_1^*)' = -\left< w_1^*, w_1^* \right> \\
\Rightarrow \begin{bmatrix} \frac{Dw_2}{dt} \\ \frac{Dw_1}{dt} \end{bmatrix} = \left< w_2', w_1^* \right> \\
= \psi' + \left< w_1', w_1^* \right> = \psi' + \left[ \frac{Dw_1}{dt} \right].
\]

In particular, if \( w_1 \) is parallel along \( \vec{x}(t) \), then \( \frac{Dw_2}{dt} = \psi'(t) = \text{rate of change of } \vec{x}(w_1, w_2) \).

E.g. \( w_1 = \vec{y}'(s) \), \( \vec{y} \) = unit speed geodesic.
Prop. Let \( \vec{x}(u,v) \) be an orthogonal parameterization of an oriented surface \( S \) (so that \( N = \frac{\vec{x}_u \times \vec{x}_v}{|\vec{x}_u \times \vec{x}_v|} \)), and let \( \mathbf{w}(t) \) be a smooth unit vector field along \( \vec{x}(u(t), v(t)) \). Then

\[
\left[ \frac{\mathbf{w}}{dt} \right] = \frac{1}{2 \text{EG}} \left( G_u \mathbf{v} - E_v \mathbf{u} \right)
\]

where \( \varphi = \text{angle between } \vec{x}_u \text{ and } \mathbf{w} \).

Proof. \( e_1 = \frac{\vec{x}_u}{\sqrt{E}} \), \( e_2 = \frac{\vec{x}_v}{\sqrt{G}} \) are orthonormal tangent vectors to \( S \), and \( e_1 \times e_2 = N \). \( \Rightarrow \) \[ \left[ \frac{\mathbf{w}}{dt} \right] - \left[ \frac{de_1}{dt} \right] = \varphi' \] and \( \left[ \frac{de_2}{dt} \right] = \left\langle \frac{de_2}{dt}, N \times e_1 \right\rangle = \left\langle \frac{de_1}{dt}, e_2 \right\rangle. \]

Since \( T = 0 \), \( \left\langle \mathbf{x}_{uu}, \mathbf{x}_u \right\rangle = 2_u \left\langle \mathbf{x}_u, \mathbf{x}_v \right\rangle - \left\langle \mathbf{x}_u, \mathbf{x}_{uv} \right\rangle = E_u - \frac{1}{2} E_v \mathbf{x}_u \mathbf{x}_v = -\frac{1}{2} E_v. \)

\[ \Rightarrow \left\langle (e_1)_u, e_2 \right\rangle = \left\langle \left( \frac{\mathbf{x}_u}{\sqrt{E}}, \frac{\mathbf{x}_v}{\sqrt{G}} \right) \right\rangle = -\frac{1}{2 \text{EG}} E_v. \]

Likewise, \( \left\langle (e_1)_v, e_2 \right\rangle = \frac{1}{2 \text{EG}} G_u. \)

Example. If \( \mathbf{w}(t) = \mathbf{x}(u(t), v(t)) \), \( \mathbf{x} \) orthogonal parameterization, then \( w = \text{unit vector field along } \mathbf{x} \) is parallel iff

\[ \varphi'(t) = -\frac{1}{2 \text{EG}} (G_u \mathbf{v} - E_v \mathbf{u}) = \mathbf{B}(t), \] \[ \varphi(t) = \gamma(x_u, w). \]

Therefore, \( \varphi(t) = \varphi(t_0) + \int_{t_0}^t B(s) \, ds, \) \( \varphi(t_0) = \gamma(x_u(u(t_0), v(t_0)), w(t_0)). \)
IV. 4 Gauss–Bonnet Theorem. \( S \) = regular, oriented surface.

Goal:

\[ \alpha + \beta + \gamma = \pi + \int T K \, ds. \]

Will consider a more situation: let \( \vec{z} : [0, l] \rightarrow S \) be a parameterized curve which

- is simple: \( \vec{z}(t) \neq \vec{z}(s), \ 0 \leq t, s \leq l, \ t \neq s, \)

- is closed: \( \vec{z}(l) = \vec{z}(0), \)

- is piecewise regular: \( \vec{z} \big|_{[t_i, t_{i+1}]} \) is regular for some partition \( 0 = t_0 < t_1 < \cdots < t_k = t_{k+1} = l. \)

At the vertices \( \vec{z}(t_i), \) define \( \Theta_i = \) (oriented) exterior angle \( \in [-\pi, \pi]. \)

Intuition for \( \Theta_i \) (and their signs):

- Sharp turn:
  - geodesic curvature is positive.
Suppose \( \vec{z}(t) \in \mathbb{R}^2 \) for some parameterization \( \vec{z} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2 \).

Let \( \psi_i : [0, 2\pi] \rightarrow \mathbb{R} \) denote the angle from \( \vec{z}_u \) to \( \vec{z}(t) \) (chosen to be smooth in \( t \)).

**Theorem of Turning Tangents.** \[
\sum_{i=0}^{k} (\psi_i(t_{i+1}) - \psi_i(t_i)) + \sum_{i=0}^{k} \theta_i = \pm 2\pi, \quad \text{where } \pm \text{ depends on the orientation of } \vec{z}.
\]

(Note: \[
\sum_{i=0}^{k} (\psi_i(t_{i+1}) - \psi_i(t_i)) = \sum_{i=0}^{k} (\psi_i(t_{i+1}) - \psi_{i+1}(t_{i+1})) + (\psi_k(t_{k+1}) - \psi_0(t_0))
\]

indef. of the choice of \( \vec{z}_u \).)

**Rem.** Compare with regular planar curve \( \vec{z}(s) \), \( \vec{z}(s) = (\cos \psi(s), \sin \psi(s)) \), and \[\int_0^l \psi'(s) \, ds = 2\pi I, \quad I \in \mathbb{Z} .\]

If in the Theorem, \( \vec{z} \) has no vertices, \[\pm 2\pi = \sum_{i=0}^{k} (\psi_i(t_{i+1}) - \psi_i(t_i)) = \sum_{i=0}^{k} \int_{t_i}^{t_{i+1}} \psi_i'(t) \, dt
\]

\[= \int_0^l \psi'(s) \, ds.
\]

So \( I = \pm 1 \); this comes from \( \vec{z} \) being a simple curve.
Orientation. Let \( R \subset S \) be the union of a connected open subset of \( S \) with its boundary.

\( R \) is called simple if \( R \) is homeomorphic to a disk, and if the boundary \( \partial R \) of \( R \) is the image of a simple, closed, piecewise regular, parameterized curve \( \bar{x}: I \rightarrow S, \| \bar{x}'(t) \| = 1. \)

\( \bar{x} \) is positively oriented if for each \( t \in I = [0, L] \), there exists an orthogonal basis \( \{ \bar{x}'(t), w(t) \} \) of \( T_{\bar{x}(t)}S \) s.t.:

\[
\begin{align*}
\bar{x}'(t) \times w(t) &= N \\
w(t) &\text{ points towards the region } R.
\end{align*}
\]

(i.e. \( \bar{x} \) rotates ccw around \( R \)).

Surface integrals. Let \( R \subset \bar{x}(U) \subset S \) be a region contained in a coordinate chart on \( S \); let \( f: S \rightarrow R \) be a smooth function. Then

\[
\int_R f \, ds := \int_{\bar{x}^{-1}(R)} f(\bar{x}(u,v)) \sqrt{EG-F^2} \, du \, dv.
\]

(Cf. area: \( \text{area}(R) = \int_R 1 \, ds \).) This is independent of the choice of local coordinates.
Theorem (Local Gauss-Bonnet Theorem.) Let \( \vec{x}: U \to S\) be an orthogonal parameterization of \( S\)-oriented surface. Let \( R = \vec{x}(U) \) be a simple region, with positively oriented boundary curve \( \vec{x}: [\eta, \zeta] \to S\).

Let \( \vec{x}(s_0), \ldots, \vec{x}(s_k) \) and \( \theta_0, \ldots, \theta_k \) be the vertices and exterior angles; \( 0 = s_0 < s_1 < \cdots < s_k < s_{k+1} = \zeta \).

Then
\[
\sum_{i=0}^{k} \int_{S_i} k_g(s) \, ds + \sum_{i=0}^{k} \theta_i + \int_{R} K \, ds = 2\pi.
\]

Ex. \( R \) = geodesic triangle:

Gauss-Bonnet

\[
\Rightarrow \sum_{i=0}^{k} \theta_i + \int_{R} K \, ds = 2\pi
\]

\[
\sum_{i=0}^{k} (\pi - \psi_i) = 3\pi - \sum_{i=0}^{k} \psi_i
\]

\[
\Rightarrow \pi + \int_{R} K \, ds = \sum_{i=0}^{k} \psi_i.
\]