Def. Let \( \tilde{x} : I \to S \) be a parameterized curve and \( w_0 \in T_{\tilde{x}(t_0)} S \), \( t_0 \in I \).

Let \( w \) be the parallel vector field along \( \tilde{x} \) with \( w(t_0) = w_0 \).

Then \( w(t_1) \in T_{\tilde{x}(t_1)} S \) is called the parallel transport of \( w_0 \) along \( \tilde{x} \) from \( \tilde{x}(t_0) \) to \( \tilde{x}(t_1) \).

Recall: \( w \) is parallel \( \iff \frac{Dw}{dt} = 0 \iff \frac{dw}{dt} \) is normal to \( T_{\tilde{x}(t)} S \).

Rmk. (1) If \( v_0 \in T_{\tilde{x}(t_0)} S \), and if \( v = \) parallel vector field along \( \tilde{x} \), then \( \langle v(t), w(t) \rangle = \text{const.} \)

Therefore, parallel transport along \( \tilde{x} \) is an isometry \( T_{\tilde{x}(t_0)} S \to T_{\tilde{x}(t_1)} S \).

(Linear)

(2) Parallel transport only depends on the curve joining start & end points, not on its parameterization.

Let \( \tilde{x}, \tilde{y} \) be two regular parameterizations, \( \tilde{y}(s) = \tilde{x}(t(s)) \).

Then \( \frac{d}{dt} w(\tilde{x}(t)) = dw \cdot \tilde{x}'(t) \) and \( \frac{d}{ds} w(\tilde{y}(s)) = dw \cdot \tilde{y}'(s) = (dw \cdot \tilde{x}'(t(s))) \cdot t'(s) \).

Therefore, \( \frac{d}{dt} w(\tilde{y}(s)) \perp T_{\tilde{y}(s)} S \) iff \( \frac{d}{dt} w(\tilde{x}(t)) \perp T_{\tilde{x}(t)} S \) \( (t = t(s)) \).
One can define parallel transport along parameterized piecewise regular curves $\tilde{x} : [0, L] \rightarrow S$, i.e., $\exists \, t_0 < t_1 < \cdots < t_k < t_{k+1} = L \, s.t.

$\tilde{x} \mid_{[t_i, t_{i+1}]}$ is a regular curve, and $\tilde{x}$ is continuous.

Parallel transport along $\tilde{x} = \text{parallel transport along } \tilde{x} \mid_{[t_0, t_1]}$, then transport the result $(\in T_{\tilde{x}(t_1)} S)$ along $\tilde{x} \mid_{[t_1, t_2]}$, etc.

Examples (1) Parallel transport from $p \in S^2$ to $q \in S^2$ does depend on the curve connecting $p$ and $q$.

Let's take $p=q=$ north pole.

(2) If $S=$ plane, then parallel transport does not depend on the curve.
(3) \( S^2 \), parameterize \( \tilde{x}(u,v) = (\sin u \cos v, \sin u \sin v, \cos u) \).

Fix a parallel \( u = u_0 \in (0, \pi) \), parameterize by

\[
\tilde{x}(t) = \tilde{x}(u(t), v(t)), \quad \begin{cases} u(t) = u_0 \\ v(t) = t. \end{cases}
\]

Compute parallel transport along \( \tilde{x} \). We recall the Christoffel symbols

\[
\Gamma^1_{11} = 0, \quad \Gamma^1_{12} = 0, \quad \Gamma^2_{22} = -\cos u \sin u,
\]

\[
\Gamma^1_{12} = 0, \quad \Gamma^2_{12} = \cot u, \quad \Gamma^2_{22} = 0.
\]

If \( w(t) = a(t) \tilde{x}_u + b(t) \tilde{x}_v \), the equation \( \frac{dw}{dt} = 0 \) reads

\[
\begin{cases}
   a'(t) - (\cos u_0 \sin u_0) b(t) = 0, \\
   b'(t) + (\cot u_0) a(t) = 0.
\end{cases}
\]

Take another derivative \( \Rightarrow \)

\[
\begin{cases}
   a'' = \cos u_0 \sin u_0, \\
   b'' = -\cot u_0, \\
   a'(t) = -\cos^2 u_0 b.
\end{cases}
\]

- For \( (10) = (1) \), get \( (a(t), b(t)) = (\cos(t \cos u_0), -\frac{1}{\sin u_0} \sin(t \cos u_0)) \).

- For \( (01) = (1) \), get \( (a(t), b(t)) = (\frac{\sin(u_0)}{\sin(t \cos u_0)}, \cos(t \cos u_0)) \).
Geodesics. In the plane, for any 2 points \( p, q \in \mathbb{R}^2 \), the curve connecting \( p \) and \( q \) of minimal length \( \ell \) is the straight line \( \bar{\alpha}: [0,1] \to \mathbb{R}^2 \) (parameterized by arc length). This \( \bar{\alpha} \) is characterized by \( \bar{\alpha}'' = 0 \) (and \( \|\bar{\alpha}'(0)\| = 1 \)). (Thus, \( \bar{\alpha} \) has vanishing curvature: no detours.)

Def. A non-constant, smooth parameterized curve \( \bar{\gamma}: I \to S \) is called a geodesic if \( \bar{\gamma}'(t) \) is parallel along \( \bar{\gamma}(t) \), i.e., \( \frac{D\bar{\gamma}'}{dt} = 0 \).

A regular, connected curve \( \gamma \subset S \) is a geodesic if for every \( p \in \gamma \), the arc length parameterization \( \bar{\gamma}(t) \) of \( \gamma \) near \( p \) is a geodesic (\( \frac{D\bar{\gamma}}{dt} = 0 \)).

Rmk. \( \bar{\gamma} \) geodesic \( \Rightarrow \|\bar{\gamma}''(t)\| = \text{const.} \neq 0 \).

\( \frac{D\bar{\gamma}'}{dt} = 0 \) does not imply \( \frac{D\bar{\gamma}'}{dt} = \bar{\gamma}''(t) = 0 \) (unlike in the planar case).

Rather, \( \frac{D\bar{\gamma}'}{dt} = k_n(t)N(t) \). (\#) Conversely, (\#) is an extrinsic characterization of geodesics.

\( k_n \) normal curvature of \( \bar{\gamma} \).
Prop. Given a point \( p \in S, \mathbf{w} \in \mathbf{T}_p S, \mathbf{w} \neq \mathbf{0} \), there exist \( \varepsilon > 0 \) and a unique parameterized geodesic \( \bar{\gamma}: (-\varepsilon, \varepsilon) \to S \) with \( \bar{\gamma}(0) = p, \bar{\gamma}'(0) = \mathbf{w} \).

Proof. Write \( \bar{\gamma}(t) = \bar{x}(u(t), v(t)) \), then \( \bar{\gamma}'(t) = u'(t) \bar{x}_u(u(t), v(t)) + v'(t) \bar{x}_v(u(t), v(t)) \), and

\[
\frac{d\bar{\gamma}'}{dt} = \bar{x}_u \left( u'' + (u')^2 \Gamma^1_{11} + 2u'v' \Gamma^1_{12} + (v')^2 \Gamma^1_{22} \right) \\
+ \bar{x}_v \left( v'' + (u')^2 \Gamma^2_{11} + 2u'v' \Gamma^2_{12} + (v')^2 \Gamma^2_{22} \right) = 0
\]

is equivalent to

\[
\frac{d^2}{dt^2} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \text{expressions involving} \\ u', v', \Gamma^k_{ij}(u, v) \end{pmatrix} - \text{a 2nd order (nonlinear) ODE,}
\]

w/ unique solution given initial data \( \left( u(0), v(0) \right) \leftarrow p, \left( u'(0), v'(0) \right) \leftarrow w \). \( \square \)

Examples (1) Sphere \( S^2 \): every great circle is a geodesic, and vice versa.

Indeed, \( P \cap S^2 = C \)

The normals of \( C = \bar{\gamma}''(s) \) coincide with the normal vector of \( S^2 \).
(2) Right cylinder.

What else?

Note: Definition of geodesics is intrinsic (only depends on 1st f.f.).

Consider \( \mathbb{R}^2 \) and \( S \).

\[(u,v) \mapsto (au,\sin bu)\]

is a local isometry.

\( \Rightarrow \) All geodesics on the right cylinder are helices.
(3) **Claim:** If all geodesics of a connected surface $S \subset \mathbb{R}^3$ are planar, then $S$ is contained in a plane or a sphere.

**Proof:** Let $p \in S$, $v \in T_p S$, $\|v\|=1$. The geodesic $\tilde{\gamma} : (-\varepsilon, \varepsilon) \to S$ with $\tilde{\gamma}(0) = p$, $\tilde{\gamma}'(0) = v$ lies in a plane $P \subset \mathbb{R}^3$.

Since $\tilde{\gamma}$ is a geodesic, \[ \frac{D^2 \tilde{\gamma}}{dt^2} = 0 \Rightarrow \tilde{\gamma}''(t) = k_n(t) N(\tilde{\gamma}(t)). \]

If $k_n(0) \neq 0$, then, viewing $\tilde{\gamma}$ as a curve in $\mathbb{R}^3$, we have $\tilde{\gamma}'(t) = \tilde{t}(t)$, $\tilde{\gamma}'' = k_n \tilde{n}$, therefore, WLOG \[
\tilde{n}(t) = N(\tilde{\gamma}(t)), k(t) = k_n(t); \text{ and }\]
\[
\tilde{n}' = -k \tilde{t} - \tilde{t} \tilde{b} = -k \tilde{t}. \Rightarrow dN_p(v) = dN_p(\tilde{\gamma}'(0)) = \tilde{n}'(0) = -k \tilde{t}(0) = -k_n \tilde{t}(0) \]
\[= -k_n v. \]

$dN_p$ is a linear map. Therefore (adding some details), every $v \in T_p S$ is an eigenvector of $dN_p$. \[\Rightarrow dN_p(v) = \lambda(p)v \quad \forall v \in T_p S \text{ for some } \lambda(p) \in \mathbb{R}, \lambda \text{ is smooth function } S \to \mathbb{R}. \]
\[ \Rightarrow S \text{ is umbilic} \Rightarrow S \subset \text{plane or sphere.} \]