Theorem (Bonnet). Given smooth $E,F,G,e,f,g: \mathbb{R}^3 \to \mathbb{R}$ such that $E>0, G>0, E G-F^2>0$, and such that the Gauss and Codazzi equations are satisfied formally (i.e. replacing $T^i_j$ by their expressions in terms of $E,F,G$). Then for all $g \in \mathbb{R}^3$, there exists a neighborhood $U \supset V$ of $g$, and a diffeomorphism $\tilde{x}: U \to \tilde{x}(U) \subset \mathbb{R}^3$ such that the regular surface $\tilde{x}(U)$ has $E,F,G$ and $e,f,g$ as the coefficients of its $1^{\text{st}}$ & $2^{\text{nd}}$ fundamental form, respectively.

- If $U$ is connected, and if $\tilde{y}: U \to \tilde{y}(U) \subset \mathbb{R}^3$ is another diffeomorphism satisfying the same conditions, then there exists a rigid motion $\gamma$ of $\mathbb{R}^3$ (translation & rotation) s.t. $\tilde{y} = \gamma \circ \tilde{x}$.


Uniqueness: similar to uniqueness proof in fundamental theorem of curves (how does $\tilde{x}_u, \tilde{x}_v, N$ evolve along some curve?).

04/07/2020

Theorem (Liebmann). Let $S$ be a compact, connected, regular surface with constant Gauss curvature $K$. Then $S$ is a sphere.

The proof proceeds in several steps.

**Step 1.** Suppose $S \subset \mathbb{R}^3$ is a compact regular surface. Then $\exists p \in S$ s.t. $K(p)>0$.

Proof. Let $p \in S$ be the point where the function $f(q)=\|q-p\|^2$, $q \in S$, attains its maximum. Consider $\tilde{x}(\cdot e, e) \to S$ parameterized by arc-length, with $\tilde{x}(0)=p$. The normal curvature of $\tilde{x}$ at $p$ is $\geq \frac{1}{\|\tilde{x}'(0)\|}$. (Homework 5, ex. 1) $\Rightarrow$ All normal curvatures at $p$ are $\geq \frac{1}{\|\tilde{x}'(0)\|} \Rightarrow K(p) \geq \frac{1}{\|\tilde{x}'(0)\|^2}$.

**Step 2.** Suppose every point of $S$ is umbilic, i.e. $dN_p(w)=\gamma(p)w$, $\forall p \in S, w \in T_p S$.

Then the conclusion holds.

Proof. We already proved that $S$ must be contained in a plane or sphere.
Since $K > 0$ on $S$, $S$ must be contained in a sphere $\Sigma$. Since $S$ is a regular surface, $S$ is open in $\Sigma$; since $S$ is compact, $S$ is also closed in $\Sigma$. Since $\Sigma$ is connected, we must have $S = \Sigma$.

(2 principal curvatures)

Step 3 Suppose $\exists p \in S$ which is not umbilic. Then $k_1(p) \neq \sqrt{K}$, therefore $k_2(p) < \sqrt{K} < k_1(p)$. The continuous function $k_1$ attains a maximum at some point $q \in S$. Since $k_2 = \frac{k_1}{k_1}$, $k_2$ attains a minimum at $q$, and $k_2(q) = k_1(q)$. The following lemma shows that this contradicts $K > 0$, hence we are done $\square$

Lemma (Hilbert) Suppose $k_1(p) > k_2(p)$, and $p$ is a local maximum of $k_1$. Then $K(p) \leq 0$.

Proof: There exists a neighborhood $V$ of $p$ s.t. $k_1 > k_2$ on $U$; therefore, $k_1, k_2$ are smooth on $V$, and so are the principal directions $X_1, X_2$.

$\Rightarrow$ The vector fields $X_1, X_2$ on $V$ are smooth, and $X_1(p), X_2(p)$ are linearly independent.

$\Rightarrow$ $\exists$ parameterization $\bar{x}: U \to S$ of a neighborhood of $p$ s.t.

$\bar{x}_u = aX_1, \quad \bar{x}_v = bX_2$ for some smooth $a, b > 0$.

But then $F = \langle \bar{x}_u, \bar{x}_v \rangle = ab \langle X_1, X_2 \rangle = 0$

and $f = \langle \bar{x}_u, N_v \rangle = \langle \bar{x}_u, dN(\bar{x}_v) \rangle = \langle \bar{x}_u, -k_1 \bar{x}_v \rangle = 0$;

hence $k_1 = \frac{e}{a}, \quad k_2 = \frac{g}{a} \quad$ (might need to exchange $u,v$ to make this so).

Codazzi equations become

$e_v = \frac{E}{2} (k_1 + k_2)$

$g_u = \frac{G}{2} (k_1 + k_2)$.

(Use $k_1 > k_2$)

Differentiate $e = E k_1$ in $v$:

$\frac{E}{2} (k_1 + k_2) = E_v k_1 + E(k_1) \downarrow$

$\iff E(k_1)_v = \frac{E}{2} (-k_1 + k_2) \iff E_v = -\frac{2E(k_1)_v}{k_1 - k_2}$.

Differentiate $g = G k_2$ in $u$:

$G(k_2)_u = \frac{6G}{2} (k_1 - k_2) \iff G_u = \frac{2G(k_2)_u}{k_1 - k_2}$.

At $p$, this implies $E_v = 0, G_u = 0$. 

Since $F=0$, the intrinsic formula for $K$ reads (exercise!)

$$K = -\frac{1}{\sqrt{|E|}} \left( \frac{E_v}{E_s} v^2 + \frac{6u}{E_s} u^2 \right)$$

$$\Rightarrow -2EG \cdot K = E\dot{w} + 6G_{uu} \ \ (\because)E\dot{v} + (\because)G_{uv} = E\dot{w} + G_{uu} \ \ \text{at } p.$$ 

But $E\dot{w}(p) = \frac{2E}{k_1 k_2} (k_1)_{uv}$, $G_{uu}(p) = \frac{\partial G}{\partial u u} (k_2)_{uv}$.

Therefore,

$$-2EG \cdot K = \frac{2E}{k_1 k_2} \left( E (k_1)_{uv} + G (k_2)_{uv} \right) \leq 0 \ \ \text{at } p.$$ 

$$\Rightarrow K(p) \leq 0.$$ 

IV.3 Parallel transport, geodesics

How to compare tangent vectors at different points on a regular surface $S$ differ.

differentiate vector fields.

**Def.** Let $w$ be a smooth vector field on $U \subset S \subset \mathbb{R}^3$. Let $p \in U$ and $\gamma \in T_p S$.

Let $\alpha: (-1,1) \to S$ be a regular parameterized curve with $\alpha(0) = p$, $\dot{\alpha}(0) = \gamma$.

Let $w \circ \alpha: (-1,1) \to \mathbb{R}^3$ is the restriction of $w$ to $\alpha$.

Then the covariant derivative of $w$ at $p$ in the direction $\gamma$ is defined by

$$\frac{Dw}{dt}(0) = D_{\gamma}w(p) := T^*_{\gamma}(\frac{dw}{dt})|_0 \in T_p S$$

where $T^*_p \mathbb{R}^3 \to T_p S$ is the orthogonal projection, defined by

$$N \cdot T^*_p(v) = v - \langle v, N(p) \rangle N(p).$$
Claim: \( Dw \) only depends on \( u, v, \) and \( 1\text{st} \) fund. form on \( S. \)

Proof. Write \( \dot{x}(t) = \dot{x}(u(t), v(t)) \) and \( w(t) = a(u(t), v(t)) \dot{x}_u + b(u(t), v(t)) \dot{x}_v \)

\[ = a(t) \dot{x}_u (u(t), v(t)) + b(t) \dot{x}_v (u(t), v(t)). \]

Then \( \frac{dw}{dt} = a' (\ddot{x}_u u' + \dot{x}_w v') + b (\ddot{x}_u u' + \dot{x}_w v') + a' \dot{x}_u + b' \dot{x}_v \)

\[ \Rightarrow \frac{Dw}{dt} = (a' + a (T_{11}^1 u' + T_{12}^1 v') + b (T_{11}^1 u' + T_{12}^1 v')) \dot{x}_u + \]

\[ + (b' + a (T_{11}^2 u' + T_{12}^2 v') + b (T_{11}^2 u' + T_{12}^2 v')) \dot{x}_v. \] (X)

\( \text{(using } \dot{x}(u,v) = \dot{x}_u u + \dot{x}_v v, \text{ w/ } u, v, \text{ orthornormal}) \)

Rem. If \( S \) is a plane, parameterized so that \( E=1, F=0, G=0, \) then \( T_{1j}^k = 0 \ \forall i,j,k=1,2, \)
and thus (X) implies \( \frac{Dw}{dt} = a' \dot{x}_u + b' \dot{x}_v = \frac{dw}{dt}. \)

\( \Rightarrow \) \( \frac{Dw}{dt} \) generalizes the usual derivative of vector fields on \( \mathbb{R}^2 \) (or planes in \( \mathbb{R}^3 \))
to curved surfaces.

For a vector field \( w \) along a smooth parameterized curve \( \alpha: I \to S \) (i.e. \( w(t) \in T_{\alpha(t)} S \)),
we can define \( \frac{Dw}{dt} \) in the same way.

Ex. \( w(t) = \dot{\alpha}(t) \Rightarrow \frac{Dw}{dt} = \text{ tangential component of } \ddot{\alpha} \text{ "intrinsic acceleration" of } \dot{\alpha}. \)

Def. A vector field \( w \) along a parameterized curve \( \dot{\alpha}: I \to S \) is parallel if \( \frac{Dw}{dt} = 0 \ \forall t \in I. \)

Ex.

(1) \( S= \text{ plane} \)

(2) \( S= \text{ sphere} \)

(3) \( S= \text{ sphere} \)