Intrinsic geometry of surfaces ("local geometry using the first fundamental form")

IV.1. Isometries, conformal maps

Def.: A diffeomorphism \( \phi : S_1 \rightarrow S_2 \) between regular surfaces is an isometry if for all \( p \in S_1 \) and all \( \vec{v} \in T_p S_1 \) we have \( I_p(\vec{v}) = I_{\phi(p)}(d\phi_p(\vec{v})) \).

\[ \langle \vec{v}_1, \vec{v}_2 \rangle_p = \langle d\phi_p(\vec{v}_1), d\phi_p(\vec{v}_2) \rangle_{\phi(p)} \] \( S_1 \) and \( S_2 \) are then said to be isometric.

A map \( \phi : V \subset S_1 \rightarrow S_2 \) is a local isometry at \( p \in V \) if there exist neighborhoods \( V_1 \subset V \) and \( V_2 \subset S_2 \) so that \( \phi : V \rightarrow V_2 \) is an isometry.

Example: \( S_1 = \{ y = a^2 \} \) (x-z plane), \( S_2 = \{ x^2 + y^2 = 1 \} \) (right cylinder over the unit circle).

Consider the map \( \phi : (x, y, z) \in S_1 \rightarrow (\cos x, \sin x, z) \in S_2 \).

Take \( \vec{v} = (a, b, 0) \in T_p S_1, p = (\alpha, \beta, 0) \in S_2 \). Then

\[ I_p(\vec{v}) = \| \vec{v} \|^2 = a^2 + b^2, \text{ and } d\phi_p(\vec{v}) = \left( \begin{array}{c} -\sin \alpha \cos \beta \\ \cos \alpha \cos \beta \\ 0 \end{array} \right), \text{ so } \]

\[ I_{\phi(p)}(d\phi_p(\vec{v})) = \| d\phi_p(\vec{v}) \|^2 = a^2 + b^2 \]

\( \Rightarrow \phi \) is a local isometry.

Note generally. Assume there are parameterizations \( \vec{x}_1 : U \rightarrow S_1, \vec{x}_2 : U \rightarrow S_2 \)

such that \( E_1 = \| (\vec{x}_1)_u \|^2 = E_2, \quad F_1 = \langle (\vec{x}_1)_u, (\vec{x}_1)_v \rangle = F_2, \quad G_1 = G_2 \) (as function on \( U \)).

Then \( \vec{x}_2 \circ \vec{x}_1^{-1} : \vec{x}_1(U) \rightarrow S_2 \) is a local isometry.

Proof: Let \( p \in \vec{x}_1(U), \vec{w} \in T_p S_1 \), then

\[ \vec{w} = (\vec{x}_1 \circ \vec{x}_1^{-1})(\vec{u}(t), \vec{v}(t)) \]

Thus, \( \vec{w} = \vec{u}'(0)(\vec{x}_1)_u(\vec{u}(0)) + \vec{v}'(0)(\vec{x}_2)_v(\vec{v}(0)) \).

By definition, \( d\phi_p(\vec{w}) = (\phi \circ (\vec{x}_1 \circ \vec{x}_1^{-1}))(\vec{u}'(0) + \vec{v}'(0)) = (\vec{x}_2 \circ \vec{x}_1^{-1})'(0) \).

Since \( E_2(\vec{u}(0)) = \| (\vec{x}_2)_u(\vec{u}(0)) \|^2 = \| (\vec{x}_1)_u(\vec{u}(0)) \|^2 = E_1(\vec{u}(0)) \), etc., we get

\[ I_p(\vec{w}) = E_1(\vec{u})^2 + 2F_1 \vec{u} \vec{v} + G_1(\vec{v})^2 = E_2(\vec{u})^2 + 2F_2 \vec{u} \vec{v} + G_2(\vec{v})^2 = I_{\phi(p)}(d\phi_p(\vec{w})) \). \]
Using this, we can also define:

**Def.** Let \( S \) be a connected regular surface. Then the (intrinsic) distance between \( p \) and \( q \in S \) is

\[
d(p, q) := \inf \ell(\tilde{x}),
\]

where the inf is taken over all piecewise smooth curves \( \tilde{x} \) starting at \( p \) and ending at \( q \).

**Prop.** Let \( \phi : S_1 \to S_2 \) be an isometry between connected regular surfaces.

Then \( d(p, q) = d(\phi(p), \phi(q)) \) for all \( p, q \in S_1 \).

**Proof.** Given \( p, q \in S_1 \), \( \varepsilon > 0 \), and a piecewise smooth curve \( \tilde{x} : [0, 1] \to S_1 \),

\[
\ell(\tilde{x}) = \int_0^1 \sqrt{\sum (\dot{x}(t))^2} \, dt \leq d(p, q) + \varepsilon,
\]

we have

\[
\ell(\phi \circ \tilde{x}) = \int_0^1 \sqrt{\sum (\phi \circ \tilde{x}(t))^2} \, dt = \int_0^1 \sqrt{\sum (\dot{x}(t))^2} \, dt = \ell(\tilde{x}),
\]

therefore \( d(\phi(p), \phi(q)) \leq d(p, q) + \varepsilon \) (since \( \phi \circ \tilde{x} \) connects \( \tilde{x} \) and \( \phi(q) \)). Since \( \varepsilon > 0 \) was arbitrary, this shows that

\[
d(\phi(p), \phi(q)) \leq d(p, q).
\]

The converse inequality follows by applying the same reasoning to \( \phi^{-1} \). \( \square \)

**Def.** A diffeomorphism \( \phi : S_1 \to S_2 \) is called a conformal map if

\[
\langle d\phi_p(v), d\phi_p(w) \rangle = \lambda(p)^2 \langle v, w \rangle_p \quad \forall p \in S_1, \ v, w \in T_p S_1,
\]

where \( \lambda : S_1 \to \mathbb{R} \) is a smooth nonzero function. In this case, \( S_1 \) and \( S_2 \) are conformal.

A smooth map \( \phi : V \subset S_1 \to S_2 \) is a local conformal map at \( p \in V \) if there exist neighborhoods \( V_1 \subset S_1 \) of \( p \) and \( V_2 \subset S_2 \) of \( \phi(p) \) s.t.

\( \phi : V_1 \to V_2 \) is a conformal map.

**Geometric meaning:** \( \phi \) preserves angles between tangent vectors (but not necessarily lengths).

Indeed, if \( \tilde{x} : (-1, 1) \to S_1 \) and \( \tilde{y} : (-1, 1) \to S_1 \) are smooth curves, \( \tilde{x}(0) = \tilde{y}(0) = p \).
\[
\begin{align*}
\cos \left( \theta \left( d\theta(x^0), d\phi(x^0) \right) \right) &= \frac{\langle d\phi(x^0), d\phi(x^0) \rangle}{||d\phi(x^0)|| \cdot ||d\phi(x^0)||} = \frac{\lambda}{\lambda x^1 \cdot \lambda x^1} \\
&= \frac{\lambda}{\lambda x^1 \cdot \lambda x^1} = \cos \left( \theta \left( x^1(x^0), x^1(x^0) \right) \right).
\end{align*}
\]

This property characterizes local conformal maps.

**Theorem.** Any two regular surfaces are locally conformal.

**Proof.** This boils down to finding local coordinates on any given regular surface in which \( E(u,v) = x'(u,v) \cdot x'(u,v) \geq 0 \), \( F(u,v) = 0 \), \( G(u,v) = x^2(u,v) \).

\[\gets \text{PDE class!} \quad \square\]

**Example (Mercator projection).** Parameterize the unit sphere, \( S^2 \), by

\[ x(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta), \quad (\theta, \phi) \in U = (0, \pi) \times (0, 2\pi). \]

Reparameterize using

\[ u = \log \tan \left( \frac{\theta}{2} \right) \in (-\infty, \infty), \quad v = \phi \in (0, 2\pi) \]

Then \( y(u,v) = (\tanh u \cos v, \tanh u \sin v, \tanh u) \) (chain rule).

(Indeed, \( \sin \theta = \sin(2 \arctan e^v) = 2 \sin(\arctan(e^v)) \cos(\arctan(e^v)) \)

\[ = 2 \frac{e^v}{1 + e^{2v}} \cdot \frac{1}{1 + e^{2v}} = \frac{2 e^v}{1 + e^{2v}} = \tanh u. \]

\[ \cos \theta = \cos(2 \arctan e^v) = 1 - 2 \sin^2(\arctan(e^v)) = 1 - \frac{2 e^{2v}}{1 + e^{2v}} = \tanh u. \]

Compute \( E = ||y_u||^2 = \text{sech}^2 u \)

\( F = \langle y_u, y_v \rangle = 0 \)

\( G = ||y_v||^2 = \text{sech}^2 u \)

\[ \Rightarrow y^{-1}: \quad x(u) \subset S^2 \rightarrow \mathbb{R}^2 \text{ is a conformal map taking parallels } (\theta = \text{cont}) \]

and meridians \( (\phi = \text{cont}) \) into horizontal and vertical lines on the plane.

**Medieval use for navigation:** Sail in fixed compass direction = travel along curve

\[ x(t) = (\theta(t), \phi(t)) \text{ which has constant angle } \beta \text{ with the meridians } \psi = \text{cont.} \]

\[ \begin{align*}
\cos \beta &= \frac{\langle x'(t), x'(t) \rangle}{||x'(t)|| \cdot ||x'(t)||} = \frac{0}{0}, \quad (\phi)' \sin^2 \beta + \sin^2 \theta (\psi)'^2 \cos^2 \beta = 0
\end{align*} \]
\( \Rightarrow \quad \Theta = \pm \frac{\psi'}{\tan \beta} \quad \Rightarrow \quad \log \tan \left( \frac{\Theta}{2} \right) = \pm (\nu + \omega) \cot \beta. \)

\( \Rightarrow \quad \psi^{-1}(x(x+1)) \in \{ (u,v) : u = \pm (\nu + \omega) \cot \beta \} \) is a line!

(These curves are called *loxodromes* (λοξός = oblique, σπείρα = running).

(2) Stereographic projection \( \psi : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2. \)

This is a conformal map. (Homework 2, ex. 1)

(3) A diffeomorphism \( \varphi : S_1 \to S_2 \) is area-preserving, if the area of any region \( R \subset S_1 \) is equal to the area of \( \varphi(R) \). Prove that if \( \varphi \) is area-preserving and conformal, then \( \varphi \) is an isometry.

**Proof.** \( \chi : U \subset \mathbb{R}^2 \to S_1 \) regular parameterization \( \Rightarrow \chi \circ \varphi : U \to S_2 \) is a regular parameterization. Consider a region \( R = \chi(Q) \subset S_1 \), then

\[
\text{Area}(\varphi(R)) = \int_Q \| (\varphi \circ \chi)_u \times (\varphi \circ \chi)_v \| \, du \, dv
\]

\[
\text{Area}(R) = \int_Q \| \chi_u \times \chi_v \| \, du \, dv.
\]

\( \varphi \) is conformal, so

\[
\langle d\varphi_1(w_1), d\varphi_2(w_2) \rangle = \lambda(p)^2 \langle w_1, w_2 \rangle
\]

\( \Rightarrow E_2 G_2 - F_2^2 = \lambda(p)^2 (E_1 G_1 - F_1). \)

Plug into (x): \( \int \sqrt{E_2 G_2 - F_2^2} \, du \, dv = \int \lambda(p)^2 \sqrt{E_1 G_1 - F_1^2} \, du \, dv \)

\[\| (\varphi \circ \chi)_u \times (\varphi \circ \chi)_v \| = \lambda(p)^2 \sqrt{E_1 G_1 - F_1^2} \]

Since \( Q \) is arbitrary, this forces \( \lambda(p)^2 = 1. \)

(4) Let \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) be given by \( \varphi(u,v) = (u \cos \omega - v \sin \omega, v \cos \omega + u \sin \omega) \), where \( x,y : \mathbb{R}^2 \to \mathbb{R} \) are smooth functions satisfying the Cauchy-Riemann equations

\[
u_x = \psi_y, \quad \nu_y = -\psi_x \quad (\Rightarrow z = x + iy \to u + iv \text{ is complex differentiable).} \)