so $k_{1,2} = H \pm \sqrt{H^2 - K}$.

In particular, putting $k_1 = H + \sqrt{H^2 - K}$, $k_2 = H - \sqrt{H^2 - K}$, the functions $k_1, k_2$ are continuous, and smooth except possibly at the points where $H^2 = K$.

(Ex.: These are precisely the umbilical points)

03/10/2020

Example. Surfaces of revolution $S$.

$\mathbf{x}(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, \psi(v)), \quad 0 < u < 2\pi, \quad a < v < b, \quad \varphi(v) > 0.$

- First fundamental form: $\mathbf{x}_u = (\varphi'(v) \cos u, \varphi'(v) \sin u, 0), \quad \mathbf{x}_v = (\varphi'(v) \cos u, \varphi'(v) \sin u, \psi'(v))$.

$E = \varphi'^2, \quad F = 0, \quad G = (\psi')^2 + (\psi')^2.$

For convenience parameterize generating curve by arc length so $G = 1$

- $N(u, v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\sqrt{\mathbf{x}_u \cdot \mathbf{x}_u \mathbf{x}_v \cdot \mathbf{x}_v}} = \frac{(\varphi'(v) \cos u, \varphi'(v) \sin u, -\psi'(v))}{\sqrt{(\varphi'(v))^2 + (\psi'(v))^2}}$

- $\mathbf{x}_{uu} = (-\varphi'' \cos u, -\varphi'' \sin u, 0), \quad e = \langle N, \mathbf{x}_{uu} \rangle = -\varphi''$

- $\mathbf{x}_{uv} = (-\varphi' \sin u, \varphi' \cos u, 0), \quad f = \langle N, \mathbf{x}_{uv} \rangle = 0$

- $\mathbf{x}_{vv} = (\varphi'' \cos u, \varphi'' \sin u, \psi''), \quad g = \langle N, \mathbf{x}_{vv} \rangle = \varphi'' \psi' - \varphi' \psi''$

$\Rightarrow K = \frac{eg - f^2}{Eg - f^2} = \frac{\varphi''(\varphi'' \psi' - \varphi' \psi'' \psi'' - \varphi'' \psi')}{(\varphi'(v))^2} = -\frac{(\varphi')^2 \varphi''^2 + (\varphi'')^2 \varphi'' - \varphi''}{(\varphi')^2}$

- $\varphi'' \psi' = -\varphi' \psi''$

Rule $K = 0 \iff \varphi'' = 0 \iff \varphi$ linear, thus $\psi' = \text{const} \Rightarrow \psi' \neq \text{const} \Rightarrow \psi$ linear.

(i) $\varphi = \text{const.} \quad \psi = \text{const.} \quad \psi = \text{const.} \quad \varphi = \text{const.} \quad \psi$ (planar) (cylinder) (sphere) (cone)

- Mean curvature $a_{\mu} = \frac{e}{E} = \frac{\varphi'}{\varphi}, \quad a_{\nu} = -\frac{f}{g} = \frac{\varphi'}{\varphi} \psi' - \psi'' \psi' = -\frac{e}{g}$

- Principal curvatures $k_1, k_2 = H \pm \sqrt{H^2 - K} = -a_{\mu}, -a_{\nu}$
Example. Let $S$ be a regular surface, $p \in S$. After rotation and translation, we may assume $\mathbf{p} = \mathbf{0} \in \mathbb{R}^3$, $T_p S = \text{xy-plane } \{z=0\}$

New $p$, $S$ is a graph, with local parameterization

$$\mathbf{r}(u,v) = (u,v,h(u,v)),$$

where $h(0,0) = 0$, $h_u(0,0) = 0$, $h_v(0,0) = 0$

$$\Rightarrow \mathbf{N}(0,0) = (0,0,1). \text{ Compute at } p.$$  

1st fund. form: $E = 1$, $F = 0$, $G = 1$.

2nd fund. form: $e = \langle \mathbf{N}, \mathbf{r}_u \rangle = h_{uu}$, $f = \langle \mathbf{N}, \mathbf{r}_v \rangle = h_{uv}$, $g = h_{vv}$.

$$\Rightarrow \text{Gauss curvature } K = \frac{eg-f^2}{EG-F^2} = h_{uu}h_{vv} - h_{uv}^2,$$

mean curvature $H = \frac{(E+G)/2}{EG-F^2} = -h_{uu} + h_{vv}$.

Note. Only 2nd order Taylor series of $h$ matters, the second fund. form of $S$ at $p$ is the same as that of the paraboloid

$$S_0 = \{(u,v,\frac{1}{2}au^2 + buv + cv^2)\}, a = h_{uu}, b = h_{uv}, c = h_{vv}.$$

Recall: every quadratic form can be diagonalized. Thus, by a rotation in the xy-plane, $S_0 = \{(u,v,Au^2 + Bu^2)\}, A, B \in \mathbb{R}$.

$$S_0 \text{ principal curvature: } -A, -B$$

Gauss curvature: $AB$,

mean curvature: $-\frac{A+B}{2}$,

principal direction: $(-1,0,0), (0,1,0)$

Remark. The reduction to $S_0$ demonstrates that $d\mathbf{N}_p$ has an orthonormal basis of eigenvectors (using the diagonalizability of quadratic forms (spectral theorem)) with real eigenvalues. This implies that $d\mathbf{N}_p$ is self-adjoint.
III 2. Vector fields

(i) In the plane. A vector field is a smooth map \( w: U \subset \mathbb{R}^2 \to \mathbb{R}^2 \).

Examples:
1. \( w(x,y) = (x,y) \) (i.e. \( w(x) = (x,0,x) \))

\[
\begin{array}{c}
\gamma \\
\downarrow \\
\alpha \\
\downarrow \\
\delta
\end{array}
\]

2. \( w(x) = \left( \frac{1}{x}, \frac{1}{x^2} \right) \)

\[
\begin{array}{c}
\gamma \\
\downarrow \\
\alpha \\
\downarrow \\
\delta
\end{array}
\]

A trajectory of \( w \) is a curve \( \alpha(t), t \in I \), satisfying \( \alpha'(t) = w(\alpha(t)) \)

(writing \( \alpha(t) = (x(t), y(t)), w(x,y) = (a(x,y), b(x,y)) \), this means)

\[
\begin{cases}
x'(t) = a(x(t), y(t)) \\
y'(t) = b(x(t), y(t)).
\end{cases}
\]

Then, let \( w : U \subset \mathbb{R}^2 \to \mathbb{R}^2 \) be a smooth vector field. For each \( p \in U \), there exist a neighborhood \( V \subset U \) of \( p \) and an open interval \( I \subset \mathbb{R} \),

and a map \( \tilde{\alpha} : V \times I \to U \) st.

(i) for \( q \in V \), the curve \( \tilde{\alpha}(q, \cdot) : I \to U \) is the (unique) trajectory of \( w \) passing through \( q \), that is,

\[
\tilde{\alpha}(q,0) = q, \quad \frac{\partial}{\partial t}(q,t) = w(\tilde{\alpha}(q,t)).
\]

(ii) \( \tilde{\alpha} \) is smooth.

(The map \( \tilde{\alpha} : I \times V \to U \) is called the (local) flow of \( w \) (near \( p \)).)
Lemma. Let \( w: U \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a smooth vector field, and let \( p \in U \) be such that \( w(p) \neq 0 \). Then \( \exists \) neighborhood \( V \subseteq U \) of \( p \) and a smooth function \( f: V \rightarrow \mathbb{R} \) such that

(i) \( f \) is constant along each trajectory of \( w \),
(ii) \( \frac{\partial f}{\partial y} \neq 0 \) for all \( q \in V \).

Proof. Without loss of generality, \( w(p) = (a_1, 0), a_1 \neq 0 \).

Let \( \tilde{x} : V \times I \rightarrow U \) be the local flow of \( w \) at \( p \), \( V \subseteq U \). Consider

\[ \tilde{x}(y, t) := \tilde{x}(0, y, t) \text{ for } (0, y) \in V, t \in I. \]

(Idea: parameterize space of trajectories near \( p \) by the \( y \)-intercept.)

Then

\[ \frac{d}{dt} \tilde{x}(0, y)(t) = \frac{\partial}{\partial x} \tilde{x}(0, y, 0) = w(\tilde{x})(0, y) = (a_1, 0). \]

\[ \Rightarrow \quad \frac{d}{dt} \tilde{x}(0, y) \text{ is invertible.} \]

Inverses function theorem \( \Rightarrow \tilde{x} \) is a local diffeomorphism near \( (y, t) = (0, 0) \).

Write \( \tilde{x}^{-1}(x, y) = (f(x, y), t(x, y)) \), then \( f(x, y) \) is constant along the trajectories of \( w \), and \( \frac{\partial f}{\partial y}(0, y) = y' \Rightarrow \frac{df}{dq} \neq 0 \) for \( q \neq (0, 0) \), hence \( \frac{df}{dq} \neq 0 \) for \( q \in V \) near \( p \). \( \square \)

Terminology. \( f \) is a (local) first integral of \( w \) near \( p \).

Examples. (1) \( w(x, y) = (x, y) \), \( p = (x_0, y_0) \), \( x_0, y_0 > 0 \).

\[ \Rightarrow \quad E.g. \quad f(x, y) = \frac{y}{x}. \]

(2) \( w(x, y) = (-y, x) \), \( p \neq (0, 0) \).

\[ \Rightarrow \quad E.g. \quad f(x, y) = x^2 + y^2. \]

(iii) Vector fields on surfaces.

Definition. A vector field \( w \) in an open subset \( U \subseteq S \) of a regular surface \( S \) is a map \( w: S \rightarrow \mathbb{R}^3 \) assigning to each \( p \in U \) a vector \( w(p) \in T_p S \).