# INTRODUCTION TO MICROLOCAL ANALYSIS

PETER HINTZ

## Contents

1. Introduction .................................................. 2

2. Schwartz functions and tempered distributions .............. 4
   2.1. Fourier transform and its inverse ...................... 6
   2.2. Sobolev spaces and the Schwartz representation theorem .... 7
   2.3. The Schwartz kernel theorem ............................ 8
   2.4. Differential operators ................................ 8
   2.5. Exercises .............................................. 10

3. Symbols .................................................... 11
   3.1. Ellipticity ............................................. 13
   3.2. Classical symbols ..................................... 14
   3.3. Asymptotic summation .................................. 14
   3.4. Exercises ............................................... 15

4. Pseudodifferential operators ................................ 16
   4.1. Left/right reduction, adjoints ......................... 19
   4.2. Topology on spaces of pseudodifferential operators ....... 22
   4.3. Composition ............................................ 22
   4.4. Principal symbols ..................................... 25
   4.5. Classical operators .................................... 26
   4.6. Elliptic parametrix .................................... 26
   4.7. Boundedness on Sobolev spaces ......................... 27
   4.8. Coordinate invariance, pseudodifferential operators on manifolds, vector bundles ......................... 27
   4.9. Elliptic operators on closed manifolds, Fredholm theory .... 28
   4.10. Exercises ............................................... 28

5. Microlocalization ........................................... 29
   5.1. Wave front set ......................................... 29
   5.2. Microlocal ellipticity .................................. 29
   5.3. Exercises ............................................... 29

---

*Date:* February 14, 2019.

I am grateful to everybody who sends me corrections and comments via email!
1. Introduction

Microlocal analysis is a paradigm for the study of distributions and their singularities. Interesting distributions mostly arise in two ways:

1) as solutions of partial differential equations (PDE), and
2) as integral kernels of operators used to localize, transform, or otherwise ‘test’ a partial differential operator.

In these notes, we explicitly mostly focus on the first kind, and prove very general results about solutions of linear PDE. The second kind will be present throughout, starting in §4, though mostly not explicitly so.

Following a quick reminder on Schwartz functions and tempered distributions in §2, the notes can be roughly divided into two parts. The first part (§§3–4) introduces pseudodifferential operators (ps.d.o.s) on $\mathbb{R}^n$ and their basic properties. Consider for example the Laplacian

$$\Delta = \sum_{j=1}^{n} D_{x_j}^2, \quad D_{x_j} := \frac{1}{i} \partial_{x_j},$$

which is a differential operator of order 2:

$$\Delta \in \text{Diff}^2(\mathbb{R}^n).$$

Consider the operator $L \in \text{Diff}^2(\mathbb{R}^n)$ defined by

$$L := \Delta + 1.$$  \hspace{1cm} (1.3)

Then $L: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is invertible (see Exercise 2.1); what kind of object is its inverse $L^{-1}$? Morally, it should be an operator of order $-2$, since composing it with $L$ gives the identity operator, which has order 0. And indeed, $L^{-1}$ is a pseudodifferential operator of order $-2$,

$$L^{-1} \in \Psi^{-2}(\mathbb{R}^n).$$  \hspace{1cm} (1.4)

By means of the Fourier transform and its inverse (see §2.1), we can write

$$(L^{-1}u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} (1 + |\xi|^2)^{-1} u(y) dy d\xi$$  \hspace{1cm} (1.5)

More generally, we shall define spaces of operators

$$\Psi^m(\mathbb{R}^n), \quad m \in \mathbb{R},$$  \hspace{1cm} (1.6)
acting on Schwartz functions (and much larger function spaces too, such as tempered distributions), with \( \text{Diff}^m(\mathbb{R}^n) \subset \Psi^n(\mathbb{R}^n) \) for \( m = 0, 1, 2, \ldots \), and forming a graded algebra:

\[
\Psi^m(\mathbb{R}^n) \circ \Psi^{m'}(\mathbb{R}^n) \subset \Psi^{m+m'}(\mathbb{R}^n).
\] (1.7)

Roughly speaking, a typical element \( A \in \Psi^m(\mathbb{R}^n) \) is defined similarly to (1.5), but with \( (1 + |\xi|^2)^{-1} \) replaced by a more general symbol \( a(x, \xi) \) with \( |a(x, \xi)| \lesssim (1 + |\xi|^{2m/2}) \); see §3 for the definition of symbols. In §4, we will define \( \Psi^m(\mathbb{R}^n) \) precisely, prove (1.7), as well as the boundedness of ps.d.o.s on a variety of useful function spaces. We will also discuss generalizations of (1.4) for elliptic (pseudo)differential operators. (Ellipticity is a notion concerning only the principal symbol of \( A \); the latter is, roughly speaking, the leading order part of \( a \), i.e. a modulo symbols of order \( m - 1 \), and ellipticity is the requirement that the principal symbol be invertible.) In particular, we shall prove that on closed manifolds (compact without boundary) \( M \), every elliptic operator \( L \in \Psi^m(M) \) is Fredholm as a map on \( C^\infty(M) \), or as a map \( L: H^s(M) \to H^{s-m}(M) \) (\( s \in \mathbb{R} \)); thus, we can solve the equation \( Lu = f \) provided \( f \) satisfies a finite number of linear constraints, and then \( u \) is unique modulo elements of the finite-dimensional space \( \ker L \).

While there are many more interesting things one can say about linear elliptic operators (index theory, Weyl law, degenerate or non-compact problems, etc.), we will switch gears in the second part (§§5–6) of the notes and study non-elliptic phenomena. We begin in §5 by defining the wave front set of a distribution \( u \in \mathcal{S}'(\mathbb{R}^n) \), which is a subset

\[
\text{WF}(u) \subset T^*\mathbb{R}^n \setminus o = \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}),
\] (1.8)

conic in the second factor. (Here, \( o \) is the zero section of the cotangent bundle \( T^*\mathbb{R}^n \).) Its projection onto \( \mathbb{R}^n \) coincides with the singular support, sing \text{supp} \( u \); roughly speaking, \( \text{WF}(u) \) measures where and in what (co)directions \( u \) is singular. As a basic example, see Exercise 5.1, the wave front set of the characteristic function 1_\( \Omega \) of a smooth domain \( \Omega \subset \mathbb{R}^n \) is given by the conormal bundle of the boundary (minus the zero section)

\[
\text{WF}(1_\Omega) = N^*\partial\Omega \setminus o.
\] (1.9)

Elliptic regularity can then be microlocalized: if \( L \in \Psi^m(\mathbb{R}^n) \) has principal symbol \( \ell \), and if \( u \in \mathcal{S}'(\mathbb{R}^n) \) is such that \( Lu \) is smooth, then \( \text{WF}(u) \) is contained in the characteristic set \( \text{Char}(L) \) of \( L \): roughly speaking, the set of those \( (x, \xi) \) where \( \ell \) is not elliptic. For example, the wave operator

\[
\Box = -D_t^2 + \sum_{j=1}^n D_{x_j}^2
\] (1.10)

on \( \mathbb{R}^{1+n} \) has (principal) symbol \( \ell = -\sigma^2 + |\xi|^2 \), \( |\xi|^2 = \sum_{j=1}^n \xi_j^2 \), where we write \( (\sigma, \xi) \) for the momentum variables (dual under the Fourier transform) to \( (t, x) \). Thus,

\[
\text{Char}(\Box) = \{(t, x, \sigma, \xi) \in T^*\mathbb{R}^{1+n} \setminus o : \sigma^2 = |\xi|^2 \}.
\] (1.11)

As a very concrete example, note that

\[
u = H(t - x_1) \implies \Box u = 0,
\] (1.12)

and indeed \( \text{WF}(u) \subset \text{Char}(\Box) \) by (1.9).

The theorem on the propagation of singularities, proved in §6, gives a complete description of the structure of \( \text{WF}(u) \) for \( u \) a distributional solution of an equation \( Lu = f \in C^\infty \); it states that \( \text{WF}(u) \subset \text{Char}(L) \) is invariant under the flow along the Hamilton vector field
of the principal symbol of $L$. In the case of $\Box$, this flow, for time $s \in \mathbb{R}$, maps $(t, x, \sigma, \xi)$ to $(t - 2s\sigma, x + 2s\xi, \sigma, \xi)$; use this to verify the theorem for (1.12)!

We shall prove this using the method of \textit{positive commutators}, which showcases the utility of ps.d.o.s as \textit{tools}, rather than as interesting operators in their own right as in (1.4), and exploits the link between symplectic geometry and ps.d.o.s (a form of the ‘classical–quantum correspondence’). More importantly, this is a very flexible method, which allows one to control solutions of PDE also in more degenerate situations—which arise frequently in applications. We give one example concerning \textit{radial points} in \S 6.3.

These notes draw material from Richard Melrose’s lecture notes [Mel07], available under www-math.mit.edu/~rbm/iml90.pdf, the textbook \textit{Microlocal Analysis for Differential Operators: an Introduction} by Grigis and Sjöstrand [GS94], lecture notes by Jared Wunsch [Wun13], as well as from notes for lectures by András Vasy at Stanford University and Ingo Witt at the University of Göttingen.

2. \textbf{Schwartz functions and tempered distributions}

Let $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$. For an open set $\Omega \subset \mathbb{R}^n$, we denote by $C^k(\Omega)$ the space of $k$ times continuously differentiable functions (with no growth restrictions), and $C^\infty(\Omega) = \bigcap_{k \in \mathbb{N}} C^k(\Omega)$. By $C^k_c(\Omega) \subset C^k(\Omega)$ we denote the space of functions which are bounded, together with their derivatives up to order $k$. We denote by $C^k_c(\Omega)$ the space of compactly supported elements of $C^k(\Omega)$. Unless otherwise noted, all functions will be complex-valued.

We use standard multiindex notation: for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_n^0$, we set

$$x^\alpha := \prod_{j=1}^n x_j^{\alpha_j}, \quad \partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad D_x^\alpha := D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad D = \frac{1}{i} \partial. \tag{2.1}$$

When the context is clear, we shall often simply write $D^\alpha := D_x^\alpha$, and $D_j := D_{x_j}$. We also put

$$|\alpha| := \sum_{j=1}^n \alpha_j, \quad \alpha! := \prod_{j=1}^n \alpha_j!. \tag{2.2}$$

We will moreover use the \textit{Japanese bracket}, defined for $x \in \mathbb{R}^n$ by

$$\langle x \rangle = (1 + |x|^2)^{1/2}.$$

\textbf{Definition 2.1.} The space $\mathcal{S}(\mathbb{R}^n)$ of \textit{Schwartz functions} consists of all $\phi \in C^\infty(\mathbb{R}^n)$ such that for all $k \in \mathbb{N}_0$,

$$\|\phi\|_k := \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \phi(x)| < \infty.$$

\textbf{Example 2.2.} We have $\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n)$. Moreover, we have a (continuous) inclusion $C^\infty_c(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$ with dense range. Recall that there are lots of smooth functions with compact support; indeed, when $K \subset U \subset \mathbb{R}^n$ with $K$ compact and $U$ open and bounded, there exists $\phi \in C^\infty_c(\mathbb{R}^n)$ with $\phi \equiv 1$ on $K$ and $\phi \equiv 0$ on $\mathbb{R}^n \setminus U$. 

Equipped with the countably many seminorms $\| \cdot \|_k$, $\mathcal{S}^0(R^n)$ is a Fréchet space. Directly from the definition, we have continuous maps
\begin{align*}
x_j : \mathcal{S}(R^n) &\rightarrow \mathcal{S}(R^n) \\
D_j : \mathcal{S}(R^n) &\rightarrow \mathcal{S}(R^n)
\end{align*}
(2.3)
Given $a \in \mathcal{C}_0^\infty (R^n)$, pointwise multiplication by $a$ is also continuous. Furthermore, integration is a continuous map
\[ \int : \mathcal{S}(R^n) \rightarrow C. \tag{2.4} \]
Indeed, this follows from
\[ \left| \int_{\mathbb{R}^n} \phi(x) \, dx \right| = \left| \int_{\mathbb{R}^n} \langle x \rangle^{-n} \langle x \rangle^{n+1} \phi(x) \, dx \right| \leq C_n \| \phi \|_{n+1}. \tag{2.5} \]
Other useful operations are the pointwise product
\[ \mathcal{S}(R^n) \times \mathcal{S}(R^n) \ni (\phi, \psi) \rightarrow \phi \psi \in \mathcal{S}(R^n), \quad (\phi \psi)(x) = \phi(x)\psi(x), \tag{2.6} \]
and the exterior product
\[ \mathcal{S}(R^n) \times \mathcal{S}(R^n) \ni (\phi, \psi) \rightarrow \phi \otimes \psi \in \mathcal{S}(R^{2n}), \quad (\phi \otimes \psi)(x, y) = \phi(x)\psi(y). \tag{2.7} \]

**Definition 2.3.** The space $\mathcal{S}'(R^n)$ of tempered distributions is the space of all continuous linear functionals $u : \mathcal{S}(R^n) \rightarrow \mathbb{C}$, equipped with the weak topology: the seminorms are $|u|_\phi := |u(\phi)|$ for $\phi \in \mathcal{S}(R^n)$. We shall usually write $\langle u, \phi \rangle := u(\phi)$.

**Example 2.4.** The $\delta$-distribution is defined by $\langle \delta, \phi \rangle := \phi(0)$. We have $\delta \in \mathcal{S}'(R^n)$ since $|\langle \delta, \phi \rangle| \leq \| \phi \|_0$.

Combining (2.4) and (2.6), we can define a continuous map
\[ \mathcal{S}(R^n) \ni \phi \rightarrow T_\phi \in \mathcal{S}'(R^n), \quad T_\phi(\psi) = \int_{\mathbb{R}^n} \phi(x)\psi(x) \, dx. \tag{2.8} \]

**Proposition 2.5.** The map $\phi \mapsto T_\phi$ is injective, and has dense range in the weak topology.

**Proof.** Regarding injectivity: $T_\phi(\overline{\phi}) = \int_{\mathbb{R}^n} |\phi(x)|^2 \, dx = 0$ implies $\phi = 0$. To prove the density, it suffices to show that, given $u \in \mathcal{S}'(R^n)$ and $\phi_1, \ldots, \phi_N \in \mathcal{S}(R^n)$ as well as any $\epsilon > 0$, there exists $\phi \in \mathcal{S}(R^n)$ such that $|\langle u, \phi_j \rangle - \langle \phi, \phi_j \rangle| < \epsilon$ for all $j = 1, \ldots, N$. Assuming, as one may, that the $\phi_j$ are orthonormal with respect to the $L^2(R^n)$ inner product, this holds (with $' < \epsilon$ replaced by $' = 0'$) for $\phi = \sum_{j=1}^N \langle u, \phi_j \rangle \overline{\phi_j}$. \hfill $\square$

This allows us to extend the maps (2.3) by continuity and duality to $\mathcal{S}'(R^n)$: indeed, for $u, \phi \in \mathcal{S}(R^n)$, we have
\[ \langle x_j u, \phi \rangle = \langle u, x_j \phi \rangle, \quad \langle D_j u, \phi \rangle = \langle u, -D_j \phi \rangle, \]
and the right hand sides now make sense also for $u \in \mathcal{S}'(R^n)$. Similarly, by duality and starting from (2.7), pointwise multiplication by a Schwartz function extends to a continuous map on $\mathcal{S}'(R^n)$; more generally, this is true for multiplication by a function in $\mathcal{C}_0^\infty(R^n)$.

Other notions, which will be significantly refined later, are:
Definition 2.6. Let \( u \in \mathcal{S}'(\mathbb{R}^n) \). Then the support, \( \text{supp} \, u \), is the complement of the set of \( x \in \mathbb{R}^n \) such that there exists \( \chi \in C_0^\infty(\mathbb{R}^n), \chi(x) \neq 1 \), such that \( \chi u = 0 \).

The singular support, \( \text{sing supp} \, u \), is the complement of the set of \( x \in \mathbb{R}^n \) such that there exists \( \chi \in C_c^\infty(\mathbb{R}^n), \chi(x) \neq 1 \), such that \( \chi u \) is smooth, i.e. \( \chi u = T_\phi, \phi \in \mathcal{S}(\mathbb{R}^n) \).

2.1. Fourier transform and its inverse. We define the Fourier transform of \( \phi \in \mathcal{S}(\mathbb{R}^n) \) by

\[
(\mathcal{F}\phi)(\xi) = \hat{\phi}(\xi) := \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \phi(x) \, dx, \quad \xi \in \mathbb{R}^n,
\]

and the inverse Fourier transform of \( \psi \in \mathcal{S}(\mathbb{R}^n) \) by

\[
(\mathcal{F}^{-1}\psi)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \psi(\xi) \, d\xi, \quad x \in \mathbb{R}^n.
\]

As in (2.5), one finds \( \|\mathcal{F}\phi\|_0 \leq C_\delta \|\phi\|_{n+1} \) and \( \|\mathcal{F}^{-1}\phi\|_0 \leq C_\delta \|\phi\|_{n+1} \). Moreover, we have

\[
\mathcal{F}(D_{xj}\phi) = \xi_j \mathcal{F}\phi, \quad \mathcal{F}(x_j\phi) = -D_{xj} \mathcal{F}\phi,
\]

\[
\mathcal{F}^{-1}(D_{xj}\phi) = -x_j \mathcal{F}^{-1}\phi, \quad \mathcal{F}^{-1}(x_j\phi) = D_{xj} \mathcal{F}^{-1}\phi,
\]

using integration by parts for the first and third statement; reading these from right to left shows that

\[
\|\mathcal{F}\phi\|_k \leq C_\delta \|\phi\|_{k+n+1} \quad \forall \, k \in \mathbb{N}_0,
\]

hence the (inverse) Fourier transform preserves the Schwartz space:

\[
\mathcal{F}, \mathcal{F}^{-1}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).
\]

Example 2.7. The Fourier transform of \( \delta \) is calculated by \( \langle \mathcal{F}\delta, \psi \rangle = \langle \delta, \mathcal{F}\psi \rangle = \hat{\psi}(0) = \int_{\mathbb{R}^n} \psi(x) \, dx \), so \( \mathcal{F}\delta = 1 \).

We recall the proof that \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) are indeed inverses to each other.

Theorem 2.8. We have \( \mathcal{F} \circ \mathcal{F}^{-1} = \mathcal{F}^{-1} \circ \mathcal{F} = I \) on \( \mathcal{S}(\mathbb{R}^n) \) and \( \mathcal{S}'(\mathbb{R}^n) \).

Proof. By (2.11), we have \( \mathcal{F}^{-1} \mathcal{F} \mathcal{F} = \mathcal{F}^{-1} \mathcal{F} \mathcal{F} = \mathcal{F}^{-1} \mathcal{F} \mathcal{F} \), i.e. the map \( A := \mathcal{F}^{-1} \mathcal{F}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \) commutes with differentiation along and multiplication by coordinates. Given \( \phi \in \mathcal{S}(\mathbb{R}^n) \) and \( x_0 \in \mathbb{R}^n \), we can write

\[
\phi(x) = \phi(x_0) + \sum_{j=1}^n \phi_j(x)(x_j - (x_0)_j), \quad \phi_j(x) = \int_0^1 (\partial_j \phi)(x_0 + t(x - x_0)) \, dt.
\]
The fact that \( \phi_j \) is in general not Schwartz is remedied by fixing a cutoff \( \chi \in C_\infty_c(\mathbb{R}^n) \), identically 1 near \( x_0 \), and writing \( \phi(x) = \chi(x)\phi(x) + (1 - \chi(x))\phi(x) \), so
\[
\phi(x) = \chi(x)\phi(x_0) + \sum_{j=1}^n \tilde{\phi}_j(x)(x_j - (x_0)_j),
\]
\[
\tilde{\phi}_j(x) = \chi(x)\phi(x_j) + \frac{(1 - \chi(x))\phi(x)}{|x - x_0|^2}(x_j - (x_0)_j).
\]
Since \( A \) annihilates every term in the sum, we have \( (A\phi)(x_0) = \phi(x_0) (A\chi)(x_0) \); note that the constant \( (A\chi)(x_0) \) here does not depend on \( \phi \), and not on the cutoff \( \chi \) either (since the left hand side does not involve \( \chi \) at all).

The same cutoff \( \chi \) can be used to evaluate \( A\phi \) at points \( x \) close to \( x_0 \); but
\[
D_{x_j}(A\chi)(x) = 0
\]
for \( x \in \chi^{-1}(1) \). We conclude that \( A = cI \) for some constant \( c \in \mathbb{C} \). One can find \( c \) by explicitly evaluating
\[
\mathcal{F}(e^{-|x|^2})(\xi) = \pi^{n/2}e^{-|\xi|^2/4}, \quad \mathcal{F}^{-1}(e^{-|\xi|^2/4})(x) = \pi^{n/2}e^{-|x|^2},
\]
so \( c = 1 \) indeed. The proof that \( \mathcal{F}\mathcal{F}^{-1} = I \) is completely analogous.

We also recall that \( \mathcal{F} \) is an isomorphism on \( L^2(\mathbb{R}^n) \); this follows from the density of \( \mathcal{S}(\mathbb{R}^n) \) in \( L^2(\mathbb{R}^n) \) and the following fact:

**Proposition 2.9.** For \( \phi \in \mathcal{S}(\mathbb{R}^n) \), we have
\[
\|\mathcal{F}\phi\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2}\|\phi\|_{L^2(\mathbb{R}^n)}.
\]

**Proof.** Analogously to (2.14), we have
\[
\int (\mathcal{F}\phi)(\xi)\tilde{\psi}(\xi) \, d\xi = (2\pi)^n \int \phi(x)\mathcal{F}^{-1}\overline{\psi}(x) \, dx, \quad \phi, \psi \in \mathcal{S}(\mathbb{R}^n).
\]
Plugging in \( \psi = \mathcal{F}\phi \) proves the proposition. \( \square \)

### 2.2. Sobolev spaces and the Schwartz representation theorem

Using the Fourier transform, we can define operators which differentiate a ‘fractional number of times’:

**Definition 2.10.** For \( s \in \mathbb{R} \) (or \( s \in \mathbb{C} \)), we let
\[
\langle D \rangle^s = (1 + |D|^2)^{s/2} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), \quad \langle D \rangle^s = \mathcal{F}^{-1}(\xi)^s \mathcal{F}.
\]

This agrees for \( s \in 2\mathbb{N}_0 \) with the usual definition, and for \( s = -2 \) with the operator (1.4). What is implicitly used here is that multiplication by \( (1 + |\xi|^2)^s \) is continuous on \( \mathcal{S}(\mathbb{R}^n) \).

**Definition 2.11.** For \( s \in \mathbb{R} \), the Sobolev space of order \( s \) is defined by
\[
H^s(\mathbb{R}^n) := \{ u \in \mathcal{S}'(\mathbb{R}^n) : \langle D \rangle^s u \in L^2(\mathbb{R}^n) \}.
\]

With the norm
\[
\|u\|_{H^s} := \|\langle D \rangle^s u\|_{L^2} = (2\pi)^{-n/2}\|\langle \xi \rangle^s \mathcal{F} u\|_{L^2},
\]

it is a Hilbert space.

**Example 2.12.** The \( \delta \)-distribution at \( 0 \in \mathbb{R}^n \) satisfies \( \delta \in H^s(\mathbb{R}^n) \) for all \( s < -n/2 \).
Since multiplication by $\langle x \rangle^r$ is continuous on $\mathcal{S}'(\mathbb{R}^n)$ for any $r \in \mathbb{R}$, we can more generally define \textit{weighted Sobolev spaces},

$$\langle x \rangle^r H^s(\mathbb{R}^n) := \{ u \in \mathcal{S}'(\mathbb{R}^n) : \langle x \rangle^{-r} u \in H^s(\mathbb{R}^n) \}.$$ (2.25)

The second part of the following is (a version of) the \textit{Schwartz representation theorem}:

\textbf{Theorem 2.13.} \textit{We have}

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_{s, r \in \mathbb{R}} \langle x \rangle^r H^s(\mathbb{R}^n), \quad \mathcal{S}'(\mathbb{R}^n) = \bigcup_{s, r \in \mathbb{R}} \langle x \rangle^r H^s(\mathbb{R}^n).$$ (2.26)

\textit{Proof.} See Exercises 2.2 and 2.3. \hfill \Box

It easily implies (using Sobolev embedding, Exercise 2.2) that every tempered distribution is a sum of terms of the form $x^a D^{\beta} a$, $a \in C_0^\infty(\mathbb{R}^n)$.

2.3. \textbf{The Schwartz kernel theorem.} The Schwartz kernel theorem is a philosophically useful fact, establishing a 1–1 correspondence between the ‘most general’ operators in the present context—mapping Schwartz functions to tempered distributions—and distributional integral kernels, also called \textit{Schwartz kernels}. To state this, we note that any distribution $K \in \mathcal{S}'(\mathbb{R}^{n+m})$ induces a bounded linear operator $\mathcal{S}(\mathbb{R}^m) \to \mathcal{S}'(\mathbb{R}^n)$ by integration along the $\mathbb{R}^m$ factor, to wit

$$(O_K \phi)(\psi) := \langle K, \psi \boxtimes \phi \rangle = \int \left( \int_{\mathbb{R}^m} K(x, y) \phi(y) \, dy \right) \psi(x) \, dx, \quad \phi \in \mathcal{S}(\mathbb{R}^m), \, \psi \in \mathcal{S}(\mathbb{R}^n).$$ (2.27)

Formally, one usually writes

$$(O_K \phi)(x) = \int_{\mathbb{R}^m} K(x, y) \phi(y) \, dy.$$ (2.28)

\textbf{Theorem 2.14.} \textit{The map $K \mapsto O_K$ is a bijection between $\mathcal{S}'(\mathbb{R}^{n+m})$ and the space of continuous linear operators $\mathcal{S}(\mathbb{R}^m) \to \mathcal{S}'(\mathbb{R}^n)$}. \hfill \Box

\textit{Example 2.15.} The Schwartz kernel of the identity operator $I$ on $\mathcal{S}(\mathbb{R}^n)$ is given by

$$K(x, y) = \delta(x - y), \quad x, y \in \mathbb{R}^n.$$ (2.29)

2.4. \textbf{Differential operators.} Given $a_\alpha \in C_0^\infty(\mathbb{R}^n)$ for $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq m$, we can define the $m$-th order differential operator

$$A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha.$$ (2.30)

Since multiplication by $a_\alpha$ is continuous on $\mathcal{S}(\mathbb{R}^n)$, $A$ defines a continuous linear operator on $\mathcal{S}(\mathbb{R}^n)$. By duality, $A$ extends (uniquely) to an continuous linear operator on $\mathcal{S}'(\mathbb{R}^n)$.

\textbf{Definition 2.16.} By $\text{Diff}^m(\mathbb{R}^n)$, we denote the space of all operators $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ of the form (2.30).
Given $A$ as in (2.30), let us define the full symbol of $A$ to be

$$\sigma(A)(x, \xi) := \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha. \quad (2.31)$$

Then, in view of (2.11), we can write

$$(Au)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot \xi} a(x, \xi) \hat{u}(\xi) \, d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} a(x, \xi) u(y) \, dy \right) \, d\xi,$$

which we read as an iterated integral. On the other hand, the Schwartz kernel $K$ of $A$ is easily verified to be

$$K(x, y) = \sum_{|\alpha| \leq m} a_\alpha(x)(D^\alpha \delta)(x - y), \quad (2.33)$$

so (formally) we have

$$K(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} a(x, \xi) \, d\xi,$$

which is indeed (rigorously) correct if one reads this as the Fourier transform of $a$ in $\xi$.

**Proposition 2.17.** Let $A \in \text{Diff}^m(\mathbb{R}^n)$. Then $A$ is local, that is,

$$\text{supp} \, Au \subset \text{supp} \, u, \quad u \in \mathscr{S}'(\mathbb{R}^n),$$

and $A$ is pseudolocal, that is,

$$\text{sing supp} \, Au \subset \text{sing supp} \, u, \quad u \in \mathscr{S}'(\mathbb{R}^n). \quad (2.35)$$

The proof is straightforward. From the perspective of the Schwartz kernel $K$ of $A$, (2.35) is really due to the fact that $K(x, y)$ is supported on the diagonal $x = y$, while (2.36) is really due to the fact that $K(x, y)$ is smooth away from $x = y$. (That is, adding to $K$ an element of $\mathscr{S}(\mathbb{R}^{2n})$ preserves (2.36), but destroys (2.35).) Since as microlocal analysts we are interested in singularities, it is the property (2.36) which we care about most; and this will persist when $A$ is a pseudodifferential operator. On the other hand, the only continuous linear operators $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ satisfying condition (2.35) are differential operators, see Exercise 2.6.

We mention three features of differential operators concerning their principal symbol.

**Definition 2.18.** Given $m \in \mathbb{N}_0$ and a differential operator $A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$, its principal symbol is defined as

$$\sigma_m(A)(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha, \quad (2.37)$$

i.e. keeping only the top order terms.

**Proposition 2.19.** Let $A \in \text{Diff}^m(\mathbb{R}^n)$.

1. Define the adjoint $A^*$ of $A$ by $\int_{\mathbb{R}^n} (A^* u)(x) \overline{v}(x) \, dx = \int_{\mathbb{R}^n} u(x) \overline{(Av)(x)} \, dx$, $u, v \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Then $A^* \in \text{Diff}^m(\mathbb{R}^n)$, and the principal symbol is

$$\sigma_m(A^*)(x, \xi) = \overline{\sigma_m(A)(x, \xi)}. \quad (2.38)$$

2. Let $B \in \text{Diff}^{m'}(\mathbb{R}^n)$. Then $A \circ B \in \text{Diff}^{m+m'}(\mathbb{R}^n)$, and

$$\sigma_{m+m'}(A \circ B)(x, \xi) = \sigma_m(A)(x, \xi) \sigma_{m'}(B)(x, \xi). \quad (2.39)$$
(3) Let \( \kappa : \mathbb{R}^n \to \mathbb{R}^n \) be a diffeomorphism which is the identity outside of a compact set. Define \( A_{\kappa} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \) by \((A_{\kappa} u)(y) := (A(u \circ \kappa^{-1}))(\kappa(y))\). Then \( A_{\kappa} \in \text{Diff}^m(\mathbb{R}^n) \), and the principal symbols are related via
\[
\sigma_m(A_{\kappa})(y, \eta) = \sigma_m(A)(\kappa(y), (\kappa'(y)^T)^{-1}\eta).
\] (2.40)

Proof. Exercise 2.8.

Thus, the principal symbol is well-defined as a function on \( T^*\mathbb{R}^n \), and is a homomorphism from the (non-commutative) algebra \( \text{Diff}(\mathbb{R}^n) = \bigcup_{m \in \mathbb{N}_0} \text{Diff}^m(\mathbb{R}^n) \) into the commutative algebra of functions \( a(x, \xi) \) which are homogeneous polynomials in \( \xi \) with coefficients in \( C_0^\infty(\mathbb{R}^n) \).

2.5. Exercises.

Exercise 2.1. 
(1) Show that \( \Delta + 1 : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \) is an isomorphism.

(2) Find a non-trivial solution \( u \in C^\infty(\mathbb{R}^n) \) of \( (\Delta + 1)u = 0 \). Why does this not contradict the first part?

Exercise 2.2. (Sobolev embedding.) Let \( s > n/2 \).

(1) Prove that there exists a constant \( C_s < \infty \) such that for \( \phi \in \mathcal{S}(\mathbb{R}^n) \), the estimate
\[
\|\phi\|_{L^\infty(\mathbb{R}^n)} \leq C_s \|\phi\|_{H^s(\mathbb{R}^n)}.
\] (2.41)
holds. (Hint. Pass to the Fourier transform.) Deduce that \( H^s(\mathbb{R}^n) \subset C_b^0(\mathbb{R}^n) \).

(2) Show more generally that \( H^s(\mathbb{R}^n) \subset C_b^k(\mathbb{R}^n) \) for \( s > n/2 + k \).

(3) Prove the first equality in Theorem 2.13.

Exercise 2.3. (Schwartz representation theorem.) Prove the second equality in Theorem 2.13 as follows.

(1) Given \( u \in \mathcal{S}'(\mathbb{R}^n) \), there exist \( C, k \) such that \( |u(\phi)| \leq C\|\phi\|_k \).

(2) Let \( R_q = (x)^{-q}(D)^{-q} \). Then \( R_q \) is an isomorphism on \( \mathcal{S}(\mathbb{R}^n) \) and \( \mathcal{S}'(\mathbb{R}^n) \). Moreover, for sufficiently large \( q \), we have \( \|R_q \phi\|_k \leq C\|\phi\|_{L^2(\mathbb{R}^n)} \) (for some other constant \( C \)). (Hint. Use the previous exercise. It may be convenient to take \( s \) there and \( q \) here to be even integers.)

(3) Denoting \( R_q^* = (D)^{-q}(x)^{-q} \), deduce that \( R_q^* u \in L^2(\mathbb{R}^n) \), and conclude that \( u \in \langle x \rangle^q H^{-q}(\mathbb{R}^n) \).

Exercise 2.4. (Schwartz kernel theorem I.) Prove the injectivity claim of Theorem 2.14. (Hint. Let \( K \in \mathcal{S}'(\mathbb{R}^{n+m}) \) be given with \( O_K = 0 \). Given \( \phi \in \mathcal{S}(\mathbb{R}^{n+m}) \), you need to show that \( \langle K, \phi \rangle = 0 \). You know that this is true when \( \phi \) is a finite linear combination of exterior products \( \psi_1 \boxtimes \psi_2, \psi_1 \in \mathcal{S}(\mathbb{R}^n), \psi_2 \in \mathcal{S}'(\mathbb{R}^m) \). Try to use the Fourier transform, or Fourier series, to approximate \( \phi \) by such linear combinations. It may help to first reduce to the case that \( \text{supp} K \) is compact.)

Exercise 2.5. (Schwartz kernel theorem II.) Let \( A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^m) \) be continuous. Prove the surjectivity claim of Theorem 2.14 as follows.

(1) The continuity of \( A \) is equivalent to the statement that for all \( \psi \in \mathcal{S}(\mathbb{R}^n) \) there exists \( N > 1 \) such that \( |\langle A\phi, \psi \rangle| \leq N\|\phi\|_N \) for all \( \phi \in \mathcal{S}(\mathbb{R}^m) \).
(2) There exist $N, M \in \mathbb{R}$ such that $A$ extends by continuity to a bounded operator

$$A: \langle x \rangle^{-M} H^M(\mathbb{R}^m) \to \langle x \rangle^N H^{-N}(\mathbb{R}^n).$$

(3) The operator

$$A':= \langle D \rangle^{-N-n/2-1} \langle x \rangle^{-N} A \langle D \rangle^{-M-m/2-1} \langle x \rangle^{-M}$$

is bounded from $H^{-m/2-1}(\mathbb{R}^m)$ to $C^0_b(\mathbb{R}^n)$.

(4) Evaluate $A' \delta_y$ for $y \in \mathbb{R}^m$ and deduce that $A'$ has a Schwartz kernel $K' \in C^0(\mathbb{R}^{n+m})$.

(5) By relating the Schwartz kernels of $A'$ and $A$, prove that $A = O_K$ for some $K \in \mathcal{S}(\mathbb{R}^{n+m})$.

Exercise 2.6. Let $A: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a continuous linear operator, and suppose for all $u \in \mathcal{S}(\mathbb{R}^n)$, we have $\text{supp } Au \subset \text{supp } u$. Prove that $A$ is a differential operator. (Hint. Show that the Schwartz kernel $K$ of $A$ has support in the diagonal \{x = y\}. Then show/recall that distributions with support on a submanifold $S$ are locally finite linear combinations of (differentiated) $\delta$-distributions (with coefficients in $C^\infty(S)$) at $S$. Cf. (2.33). To prove that $A$ is a differential operator of finite order, exploit that $K$ is a tempered distribution.)

Exercise 2.7. Show that the principal symbol $\sigma_m(A)$ of $A \in \text{Diff}^m(\mathbb{R}^n)$ captures the `high frequency behavior' of $A$ in the following sense: for $x_0, \xi_0 \in \mathbb{R}^n$, we have

$$\sigma_m(A)(x_0, \xi_0) = \lim_{\lambda \to \infty} \lambda^{-m} (e^{-i\lambda \xi_0} A e^{i\lambda \xi_0})(x_0),$$

where $e^{i\xi_0 \cdot x}$ is the function $x \mapsto e^{i\xi_0 \cdot x}$.


3. Symbols

As a first step towards the definition of pseudodifferential operators, we generalize the class of symbols $a(x, \xi)$ from polynomials in $\xi$ to more general functions:

**Definition 3.1.** Let $m \in \mathbb{R}$, $n, N \in \mathbb{N}$. Then the space of (uniform) symbols of order $m$

$$S^m(\mathbb{R}^n; \mathbb{R}^N) \subset C^\infty(\mathbb{R}^n \times \mathbb{R}^N)$$

(3.1)

consists of all functions $a = a(x, \xi)$ which for all $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}_0^N$ satisfy the estimate

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|}.$$  

(3.2)

for some constants $C_{\alpha\beta}$. We also write

$$S^m(\mathbb{R}^N) := S^m(\mathbb{R}^0; \mathbb{R}^N)$$

(3.3)

for symbols only depending on the symbolic variable $\xi$.

The gain of decay upon differentiation in the $\xi$-variables is often called symbolic behavior (in $\xi$).

**Remark 3.2.** Sometimes these symbol classes are denoted $S^m_\infty(\mathbb{R}^n; \mathbb{R}^N)$, the subscript `$\infty$' indicating the uniform boundedness in $C^\infty$ of the `coefficients', i.e. the $x$-variables. There exist many generalizations and variants of the class $S^m(\mathbb{R}^n; \mathbb{R}^N)$, such as: symbols of type $\rho, \delta$; symbols which in addition have symbolic behavior in $x$ (these are symbols of scattering (pseudo)differential operators); or symbols with joint symbolic behavior in $(x, \xi)$ (symbols of isotropic operators). See [Mel07, §4] and [Hör71, §1.1].
Equipped with the norms given by the best constants in (3.2), or more concisely
\[
\|a\|_{m,k} := \sup_{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^N} \max_{|\alpha| + |\beta| \leq k} \langle \xi \rangle^{-m+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)|,
\]
the space \(S^m(\mathbb{R}^n; \mathbb{R}^N)\) is a Fréchet space. Directly from the definition, we note that differentiations
\[
D_\alpha^m : S^m(\mathbb{R}^n; \mathbb{R}^N) \to S^m(\mathbb{R}^n; \mathbb{R}^N), \\
D_\xi^\beta : S^m(\mathbb{R}^n; \mathbb{R}^N) \to S^{m-|\beta|}(\mathbb{R}^n; \mathbb{R}^N)
\]
are continuous.

**Example 3.3.** Full symbols of differential operators of order \(m\) on \(\mathbb{R}^n\), see (2.31), lie in \(S^m(\mathbb{R}^n; \mathbb{R}^n)\). A special case of this is: given \(a \in C^\infty_0(\mathbb{R}^n)\), the function \((x, \xi) \mapsto a(x)\) lies in \(S^0(\mathbb{R}^n; \mathbb{R}^n)\) (for any \(N\)).

**Example 3.4.** Let \(m \in \mathbb{R}\). Then \(\langle \xi \rangle^m \in S^m(\mathbb{R}^n; \mathbb{R}^n)\). (See Exercise 3.1.)

**Proposition 3.5.** Pointwise multiplication of symbols is a continuous bilinear map
\[
S^m(\mathbb{R}^n; \mathbb{R}^N) \times S^{m'}(\mathbb{R}^n; \mathbb{R}^N) \to S^{m+m'}(\mathbb{R}^n; \mathbb{R}^N).
\]

**Proof.** This follows from the Leibniz rule: for \(a \in S^m(\mathbb{R}^n; \mathbb{R}^N)\), \(b \in S^{m'}(\mathbb{R}^n; \mathbb{R}^N)\), and \(\alpha, \beta \in \mathbb{N}_0^n\), \(\beta \in \mathbb{N}_0^n\), we have
\[
|\partial_\alpha^\alpha \partial_\xi^\beta (a \cdot b)| = \sum_{\alpha', \alpha'' = \alpha} \binom{\alpha}{\alpha'} (\partial_\xi^\beta \partial_\xi^\beta a)(\partial_\xi^\beta \partial_\xi^\beta b)
\]
\[
\leq \sum_{\alpha' + \alpha'' = \alpha} C_{\alpha' \beta} C_{\alpha'' \beta'} \langle \xi \rangle^{m + m' - |\beta| - |\beta'|}
\]
\[
\leq C_{\alpha \beta} \langle \xi \rangle^{m + m' - |\beta|}.
\]

We note the trivial continuous inclusion
\[
m \leq m' \implies S^m(\mathbb{R}^n; \mathbb{R}^N) \subseteq S^{m'}(\mathbb{R}^n; \mathbb{R}^N),
\]
hence the \(S^m(\mathbb{R}^n; \mathbb{R}^N)\) give a filtration of the space of all symbols \(\bigcup_{m \in \mathbb{R}} S^m(\mathbb{R}^n; \mathbb{R}^N)\). In the other direction, we define the space of residual symbols by
\[
S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N) := \bigcap_{m \in \mathbb{R}} S^m(\mathbb{R}^n; \mathbb{R}^N).
\]
Equipped with the norms \(\|\cdot\|_{m,k}\), \(m, k \in \mathbb{N}\), this is again a Fréchet space.

**Example 3.6.** We have \(\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^N) \subseteq S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N)\). Moreover, given a cutoff \(\chi \in C^\infty_0(\mathbb{R}^N)\), the pullback of \((x, \xi) \mapsto \chi(\xi)\) to \(\mathbb{R}^n \times \mathbb{R}^N\) defines a residual symbol.

While the inclusion (3.7) never has dense range for \(m < m'\), there is a satisfying replacement:

**Proposition 3.7.** Let \(m < m'\). Then \(S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N)\) is a dense subspace of \(S^m(\mathbb{R}^n; \mathbb{R}^N)\) in the topology of \(S^{m'}(\mathbb{R}^n; \mathbb{R}^N)\). More precisely, for any \(a \in S^m(\mathbb{R}^n; \mathbb{R}^N)\) there exists a sequence \(a_j \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N)\) which is uniformly bounded in \(S^m(\mathbb{R}^n; \mathbb{R}^N)\) and converges to \(a\) in the topology of \(S^{m'}(\mathbb{R}^n; \mathbb{R}^N)\).
Proof. Fix a cutoff function \( \chi \in C_c^\infty(\mathbb{R}^N) \subset S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N) \) (see Example 3.6) which is identically 1 in \(|\xi| \leq 1\) and identically 0 when \(|\xi| \geq 2\). By Proposition 3.5, we have

\[
a_j(x, \xi) := a(x, \xi) \chi(\xi/j) \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N).
\]

To prove the proposition, it suffices to show, in view of Proposition 3.5, that

\[
\chi_j(\xi) := \chi(\xi/j)
\]

is bounded in \(S^0(\mathbb{R}^N)\) and converges to 1 in the topology of \(S^\epsilon(\mathbb{R}^N)\) for all \(\epsilon > 0\). Regarding the former, we have \(|\chi_j(\xi)| \leq ||\chi||_{0,0}\) for all \(j\), while for \(|\beta| \geq 1\) we have \(\partial_\xi^\beta \chi_j(\xi) \equiv 0\) for \(|\xi| \leq 1\), and

\[
|\xi|^{|\beta|} \partial_\xi^\beta \chi_j(\xi) = \chi(\xi/j), \quad \chi(\xi) = |\xi|^{|\beta|} \partial_\xi^\beta \chi(\xi) \in C_c^\infty(\mathbb{R}^N).
\]

Regarding the latter, we note that supp(\(\chi_j - 1\)) \(\subset \{j \leq |\xi| \leq 2j\}\), hence

\[
|\chi(\xi/j) - 1| \leq j^{-\epsilon} |\xi|^\epsilon.
\]

For derivatives, we note that the support observation and (3.11) give

\[
|\xi|^{|\beta| - \epsilon} \partial_\xi^\beta \chi_j(\xi) \leq j^{-\epsilon} |\chi(\xi/j)|.
\]

Thus, \(\|\chi_j\|_{\epsilon, k} \leq C_{k, \epsilon} j^{-\epsilon} \to 0, j \to \infty\), as desired. \(\square\)

3.1. Ellipticity. We now generalize the key property of the symbol of the operator \(L = \Delta + 1\) in (1.3).

**Definition 3.8.** Let \(m \in \mathbb{R}\). A symbol \(a \in S^m(\mathbb{R}^n; \mathbb{R}^N)\) is (uniformly) elliptic if there exists a symbol \(b \in S^{-m}(\mathbb{R}^n; \mathbb{R}^N)\) such that \(ab - 1 \in S^{-1}(\mathbb{R}^n; \mathbb{R}^N)\).

**Proposition 3.9.** Let \(m \in \mathbb{R}\), and \(a \in S^m(\mathbb{R}^n; \mathbb{R}^N)\). Then the following are equivalent:

1. \(a\) is elliptic.
2. There exist constants \(C, c > 0\) such that

\[
|\xi| \geq C \implies |a(x, \xi)| \geq c|\xi|^m.
\]

3. There exist constants \(C, c > 0\) such that

\[
|a(x, \xi)| \geq c|\xi|^m - C|\xi|^{m-1}, \quad |\xi| \geq 1.
\]

**Proof.** If \(a\) is elliptic, then in the notation of Definition 3.8, we have

\[
1 - C\langle \xi \rangle^{-1} \leq |a(x, \xi)||b(x, \xi)| \leq C|a(x, \xi)|\langle \xi \rangle^{-m},
\]

for some constant \(C > 0\), that is,

\[
|a(x, \xi)| \geq c\langle \xi \rangle^m - \langle \xi \rangle^{m-1}.
\]

This proves (3.15). This in turn implies (3.14) since for all \(c > 0\), there exists \(C > 0\) such that \(|\xi|^{m-1} \leq c|\xi|^m\) for \(|\xi| \geq C\) (indeed, this holds for \(C = c^{-1}\)).

Conversely, if (3.14) holds, choose a cutoff \(\chi \in C^\infty(\mathbb{R}^n)\), \(\chi(\xi) = 0\) for \(|\xi| \leq 2C\), \(\chi(\xi) = 1\) for \(|\xi| \geq 3C\), then (see Exercise 3.2)

\[
b(x, \xi) := \chi(\xi)/a(x, \xi) \in S^{-m}(\mathbb{R}^n; \mathbb{R}^N),
\]

and \(a(x, \xi)b(x, \xi) = \chi(\xi) \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N)\). \(\square\)
Note that if \( a \in S^m(\mathbb{R}^n; \mathbb{R}^N) \) is elliptic, then so is \( a + a' \) for any \( a' \in S^{m-1}(\mathbb{R}^n; \mathbb{R}^N) \).

Thus, ellipticity is only a condition on the equivalence class
\[
[a] \in S^m(\mathbb{R}^n; \mathbb{R}^N)/S^{m-1}(\mathbb{R}^n; \mathbb{R}^N).
\]

(3.19)

For full symbols of differential operators, we can identify \([a]\) with the leading order, homogeneous of degree \( m \), part of \( a \). Compare with Definition 2.18 and Proposition 2.19.

### 3.2. Classical symbols

An important subclass of symbols mimics those of differential operators: they are sums of homogeneous (in \( \xi \)) functions. More precisely, we call a function \( a(x, \xi) \), defined for \( \xi \neq 0 \), (positively) homogeneous of order \( m \in \mathbb{C} \) if
\[
a(x, \lambda \xi) = \lambda^m a(x, \xi), \quad \lambda > 0.
\]

(3.20)

**Definition 3.10.** Let \( m \in \mathbb{C} \). Then \( S^m_{\text{hom}}(\mathbb{R}^n; \mathbb{R}^N \setminus \{0\}) \) is the space of all functions \( a(x, \xi) \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^N \setminus \{0\})) \), positively homogeneous of order \( m \) in \( \xi \), such that for all \( \alpha, \beta \in \mathbb{N}^n \)
\[
|\partial^n_x \partial^\beta_\xi a(x, \xi)| \leq C_{\alpha, \beta} |\xi|^{m-|\beta|}, \quad \xi \neq 0.
\]

(3.21)

**Definition 3.11.** Let \( m \in \mathbb{C} \), and fix a cutoff \( \chi \in C^\infty_c(\mathbb{R}^N) \) which is identically 1 near 0. A symbol \( a \in S^{\text{Re} m}(\mathbb{R}^n; \mathbb{R}^N) \) is called a **classical symbol of order \( m \)** if there exist functions \( a_{m-j} \in S^m_{\text{hom}}(\mathbb{R}^n \times (\mathbb{R}^N \setminus \{0\})) \) such that for all \( J \in \mathbb{N} \), we have
\[
a - \sum_{j=0}^{J-1} (1 - \chi)a_{m-j} \in S^{\text{Re} m-J}(\mathbb{R}^n; \mathbb{R}^N).
\]

(3.22)

The space of classical symbols of order \( m \) is denoted \( S^m_{\text{cl}}(\mathbb{R}^n; \mathbb{R}^N) \). Finally, we put
\[
S^\infty_{\text{cl}}(\mathbb{R}^n; \mathbb{R}^N) := S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N).
\]

(3.23)

Equipped with the seminorms of \( a_{m-j} \) and the remainders \( a - \sum_{j=0}^{J-1} (1 - \chi)a_{m-j} \) in the respective spaces, \( S^m_{\text{cl}}(\mathbb{R}^n; \mathbb{R}^N) \) is a Fréchet space. Proposition 3.7 fails dramatically for classical symbols: indeed (Exercise 3.4),
\[
S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N) \subset S^m_{\text{cl}}(\mathbb{R}^n; \mathbb{R}^N)
\]

is closed for any \( m \in \mathbb{C} \).

(3.24)

We have the following straightforward lemma (Exercise 3.5):

**Lemma 3.12.** The homogeneous terms \( a_{m-j} \) in (3.22) are uniquely determined by \( a \).

For \( a \in S^m(\mathbb{R}^n; \mathbb{R}^N) \) as in Definition 3.11, we can thus identify the equivalence class \([a]\) with the leading order homogeneous part \( a_m \), or even more simply with the function \( \mathbb{R}^n \times S^{N-1} \ni (x, \xi) \mapsto a_m(x, \xi) \), where \( S^{N-1} = \{ \xi \in \mathbb{R}^N : |\xi| = 1 \} \) is the unit sphere. Cf. (2.37).

### 3.3. Asymptotic summation

There is a (general) ‘converse’ to (3.22) which is very useful when performing iterative constructions which yield lower order corrections:

**Proposition 3.13.** Let \( a_j \in S^{m_j}(\mathbb{R}^n; \mathbb{R}^N) \), \( j \geq 0 \), and suppose \( \lim \sup_{j \to \infty} m_j = -\infty \). Let \( \bar{m}_j := \sup_{j' \geq j} m_{j'} \), and \( m = \bar{m}_0 \). Then there exists a symbol \( a \in S^m(\mathbb{R}^n; \mathbb{R}^N) \) such that for all \( J \in \mathbb{N} \)
\[
a - \sum_{j=0}^{J-1} a_j \in S^{m_J}(\mathbb{R}^n; \mathbb{R}^N).
\]

(3.25)
Moreover, $a$ is unique modulo $S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N)$.

We call $a$ `the' asymptotic sum of the $a_j$, and write

$$a \sim \sum_{j=0}^{\infty} a_j. \quad (3.26)$$

**Proof of Proposition 3.13.** This is similar to Borel’s theorem about the existence of a smooth function with prescribed Taylor series at 0. Uniqueness is clear, since any two asymptotic sums $a, a'$ satisfy $a - a' \in S^{\infty}(\mathbb{R}^n; \mathbb{R}^N)$, with $\bar{m}_j = -\infty$, hence $a - a'$ is residual indeed.

For existence, we may partially sum finitely many of the $a_j$ and thereby reduce to the case that $a_j \in S^{m-j}(\mathbb{R}^n; \mathbb{R}^N)$, $j \geq 0$, and $\bar{m}_j = m - j$. Fix a cutoff $\chi \in C^\infty(\mathbb{R}^n)$, identically 0 in $|\xi| \leq 1$ and equal to 1 for $|\xi| \geq 2$. With $\epsilon_j > 0$, $\epsilon_j \to 0$, to be determined, we wish to set

$$a(x, \xi) := \sum_{j=0}^{\infty} \chi(\epsilon_j \xi) a_j(x, \xi). \quad (3.27)$$

This sum is locally finite, hence $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N)$. Choosing $\epsilon_j$ more precisely, we can arrange that

$$\|\chi(\epsilon_j \cdot) a_j\|_{m-j, j'} \leq 2^{-j'}, \quad j > j' \geq 0. \quad (3.28)$$

Indeed, for fixed $j, j'$, we can choose $\epsilon_j > 0$ such that this holds since $(1 - \chi(\epsilon_j \cdot)) a_j \to 0$ in $S^{m-j'}(\mathbb{R}^n; \mathbb{R}^N)$ as $\epsilon_j \to 0$, as in the proof of Proposition 3.7; but for any fixed $j$, (3.28) gives a finite number of conditions on $\epsilon_j$, one for each $0 \leq j' < j$.

But then $\chi(\epsilon_j \xi) a_j(x, \xi) + \sum_{j=j+1}^{\infty} \chi(\epsilon_j \xi) a_j(x, \xi)$ converges in $S^{m-j'}(\mathbb{R}^n; \mathbb{R}^N)$. Thus, (3.27) converges in $S^m(\mathbb{R}^n; \mathbb{R}^N)$, and we have

$$a(x, \xi) - \sum_{j=0}^{J-1} a_j(x, \xi) = \sum_{j=0}^{J-1} (1 - \chi(\epsilon_j \xi)) a_j(x, \xi) + \sum_{j=J}^{\infty} \chi(\epsilon_j \xi) a_j(x, \xi) \in S^{m-j}(\mathbb{R}^n; \mathbb{R}^N), \quad (3.29)$$

as desired. \hfill \Box

The space $S^m_{cl}(\mathbb{R}^n; \mathbb{R}^N)$ can then be characterized as the space of symbols in $S^{Re \: m}(\mathbb{R}^n; \mathbb{R}^N)$ which are asymptotic sums of symbols which in $|\xi| \geq 1$ are positively homogeneous of degree $m - j$, $j \in \mathbb{N}_0$.

### 3.4. Exercises.

**Exercise 3.1.** (1) Let $m \in \mathbb{R}$. Show that $(\xi)^m \in S^m(\mathbb{R}^N)$. By expanding into Taylor series in $1/|\xi|$, show that indeed $(\xi)^m \in S^m_{cl}(\mathbb{R}^N)$.

(2) More generally, let $\mu \in \mathbb{C}$. Show that $(\xi)^\mu \in S^m_{cl}(\mathbb{R}^N)$.

**Exercise 3.2.** (1) Show that if $a \in S^m(\mathbb{R}^n; \mathbb{R}^N)$ satisfies (3.14), and $\chi \in S^0(\mathbb{R}^N)$ vanishes for $|\xi| \leq 2C$, then $\chi/a \in S^{-m}(\mathbb{R}^n; \mathbb{R}^N)$.

(2) If in addition $a$ and $\chi$ are classical symbols, show that $\chi/a$ is classical as well.

**Exercise 3.3.** (1) Let $f \in C^\infty(\mathbb{R})$. Show that if $a \in S^0(\mathbb{R}^n; \mathbb{R}^N)$, then also $f \circ a \in S^0(\mathbb{R}^n; \mathbb{R}^N)$.

(2) Show that if $a \in S^0(\mathbb{R}^n; \mathbb{R}^N)$ is elliptic, then there exists $b \in S^0(\mathbb{R}^n; \mathbb{R}^N)$ such that $a - b^2 \in S^{-1}(\mathbb{R}^n; \mathbb{R}^N)$. 
Exercise 3.4. Prove (3.24).

Exercise 3.5. Prove Lemma 3.12. (Hint. Use induction on $j$; the case $j = 0$ is the main content.)

4. Pseudodifferential operators

For developing the theory of ps.d.o.s, it is useful to consider slightly more general symbols, in the class

$$\langle x - y \rangle^w S^m(\mathbb{R}_x^n \times \mathbb{R}_y^n; \mathbb{R}_\xi^n) = \{ \langle x - y \rangle^w \tilde{a} : \tilde{a} \in S^m(\mathbb{R}_x^n \times \mathbb{R}_y^n; \mathbb{R}_n^n) \},$$

(4.1)

where $w \in \mathbb{R}$. Our immediate goal will be to make sense of the following definition.

Definition 4.1. Let $m, w \in \mathbb{R}$, and $a \in \langle x - y \rangle^w S^m(\mathbb{R}_x^n \times \mathbb{R}_y^n; \mathbb{R}_\xi^n)$. Then we define its quantization $\operatorname{Op}(a)$ by

$$(\operatorname{Op}(a)u)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} a(x, y, \xi) u(y) \, dy \, d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

(4.2)

Previously, see (2.32), we only considered the special case of the left quantization of a left symbol $a \in S^m(\mathbb{R}_x^n; \mathbb{R}_\xi^n)$, independent of $y$:

$$(\operatorname{Op}_L(a)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} a(x, \xi) u(y) \, dy \, d\xi; \quad u \in \mathcal{S}(\mathbb{R}^n).$$

(4.3)

this immediately makes sense as an iterated integral for $u \in \mathcal{S}(\mathbb{R}^n)$, and should be thought of as ‘differentiate first, then multiply by coefficients’. Dually, we can consider the right quantization of a right symbol $a \in S^m(\mathbb{R}_y^n; \mathbb{R}_\xi^n),

$$(\operatorname{Op}_R(a)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} a(y, \xi) u(y) \, dy \, d\xi,$$

(4.4)

which does not immediately make sense (similarly to (4.2)); this should be thought of as ‘multiply by coefficients, then differentiate’. (Take $a(z, \xi) = \xi^\alpha a_\alpha(z)$ with $a_\alpha \in C_0^\infty(\mathbb{R}^n)$ and evaluate $\operatorname{Op}_L(a)u$ and $\operatorname{Op}_R(a)u$!)

The quantization map (4.2) should be read as ‘multiply $(y)$, then differentiate $(\xi)$, then multiply $(x)$’. (Try this with $a(x, y, \xi) = a_1(x) \xi^\alpha a_2(y)$.) We shall see below that every operator $\operatorname{Op}(a)$ can be written as $\operatorname{Op}(a) = \operatorname{Op}_L(a_L) = \operatorname{Op}_R(a_R)$ for suitable left and right symbols $a_L$ and $a_R$ of the same order as $a$, see §4.1. (You have done most of the work for proving this for differential operators, i.e. in the case that $a$ is a polynomial in $\xi$, in Exercise 2.8.)

Lemma 4.2. Let $w \in \mathbb{R}$, $m < -n$, and let $a = \langle x - y \rangle^w \tilde{a}, \tilde{a} \in S^m(\mathbb{R}_x^n \times \mathbb{R}_y^n; \mathbb{R}_n^n)$. Then the integral (4.2) is absolutely convergent and defines a continuous operator

$$\operatorname{Op}(a) : \mathcal{S}(\mathbb{R}^n) \rightarrow \langle x \rangle^w C^0_c(\mathbb{R}^n).$$

(4.5)

More precisely, for $N > n + |w|$, there exists a constant $C < \infty$ such that

$$\| \operatorname{Op}(a)u \|_{\langle x \rangle^w C^0_c(\mathbb{R}^n)} \leq C \| \tilde{a} \|_{m,0} \| u \|_N, \quad u \in \mathcal{S}(\mathbb{R}^n).$$

(4.6)

For the proof, we need a simple lemma:

Lemma 4.3. Let $w \in \mathbb{R}$. Then $\langle x + y \rangle^w \leq 2^{|w|/2} \langle x \rangle^w \langle y \rangle^{|w|}$. 


Proof. By the triangle and Cauchy–Schwarz inequalities, we have
\[ 1 + |x + y|^2 \leq 1 + 2|x|^2 + 2|y|^2 \leq 2(1 + |x|^2)(1 + |y|^2). \] (4.7)
If \( w > 0 \), then taking this to the power \( w/2 \) proves the lemma. For \( w = 0 \), the lemma is the equality \( 1 = 1 \). For \( w < 0 \), hence \(-w > 0\), we obtain, analogously to (4.7),
\[ (x)^{-w} \leq 2^{-w/2} (x + y)^{-w} (y)^{-w}, \] (4.8)
which upon multiplication by \( (x)^w (x + y)^w \) gives the desired result. \( \square \)

Proof of Lemma 4.2. Since \( u \) is Schwartz, we have \( |u(y)| \leq C_N \|u\|_N \langle y \rangle^{-N} \) for all \( N \in \mathbb{N}_0 \). Therefore, the integrand in (4.2) satisfies
\[ |e^{i(x-y)\cdot \xi}a(x,y,\xi)u(y)| \leq C \langle x-y \rangle^w \|\hat{a}\|_{m,0} \langle \xi \rangle^m \cdot \|u\|_N \langle y \rangle^{-N} \]
\[ \leq C \langle x \rangle^w \cdot \langle \xi \rangle^m \langle y \rangle^{|w|-N} \cdot \|\hat{a}\|_{m,0} \|u\|_N. \] (4.9)
This is integrable in \( (y, \xi) \) provided \( m < -n \) and \( |w| - N < -n \), proving the lemma. \( \square \)

Proposition 4.4. Let \( w \in \mathbb{R} \) and \( a = \langle x - y \rangle^w \hat{a}, \hat{a} \in S^{\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \). Then the quantization \( \text{Op}(a): \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \) is continuous. In fact, for all \( k \in \mathbb{N}_0 \), \( m \in \mathbb{R} \), there exists \( N \in \mathbb{N} \) and a constant \( C \) such that
\[ \| \text{Op}(a)u \|_k \leq C \|\hat{a}\|_{m,N} \|u\|_N. \] (4.10)

Lemma 4.5. Differentiations \( D_x^\alpha \) and \( D_y^\alpha \) are continuous maps \( \langle x - y \rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \to \langle x - y \rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \). More precisely,
\[ \| \langle x - y \rangle^{-w} D_x^\alpha a \|_{m,k} \leq C \|\langle x - y \rangle^{-w} a \|_{m,k+|\alpha|}, \] (4.11)
likewise for \( D_y^\alpha a \).

Proof. It suffices to prove the claim for \( D_{x_1} \). For \( a(x,y,\xi) = \langle x - y \rangle^w \hat{a}(x,y,\xi), \hat{a} \in S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \), we have
\[ \partial_{x_1} a = \langle x - y \rangle^w \partial_{x_1} \hat{a} + w \langle x - y \rangle^{w-2}(x_1 - y_1)\hat{a}. \] (4.12)
The first summand lies in \( \langle x - y \rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \), and the second summand even lies in the smaller space \( \langle x - y \rangle^{w-1} S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \). \( \square \)

Proof of Proposition 4.4. The key is that for \( \xi \neq 0 \), the phase \( (x - y) \cdot \xi \) has no critical points in \( y \). We exploit this by writing
\[ (1 - \xi \cdot D_y)e^{i(x-y)\cdot \xi} = \langle \xi \rangle^2 e^{i(x-y)\cdot \xi}, \] (4.13)
so upon integrating by parts in \( y \), one gains decay in \( \xi \). Concretely, for \( N \in \mathbb{N} \), we have
\[ \text{Op}(a)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} ((1 - \xi \cdot D_y)^N e^{i(x-y)\cdot \xi} \langle \xi \rangle^{-2N} a(x,y,\xi)u(y)) \, dy \, d\xi \]
\[ = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} (1 + \xi \cdot D_y)^N (\langle \xi \rangle^{-2N} a(x,y,\xi)u(y)) \, dy \, d\xi. \] (4.14)
By the Leibniz rule, we have
\[ (1 + \xi \cdot D_y)^N (\langle \xi \rangle^{-2N} a(x,y,\xi)u(y)) = \sum_{|\gamma| \leq N} a_\gamma(x,y,\xi) \cdot D_y^\gamma u, \] (4.15)
where
\[ a_\gamma(x, y, \xi) = \sum_{|\delta|, |\epsilon| \leq N} c_{\gamma\delta\epsilon}(\xi)^{-2N} \xi^\delta D_y^\epsilon a(x, y, \xi) \] (4.16)
for some combinatorial constants \( c_{\gamma\delta\epsilon} \). By Lemma 4.5, \( \tilde{a}_\gamma := (x - y)^{-w} a_\gamma \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \), and for any \( m \in \mathbb{R} \),
\[ \| \tilde{a}_\gamma \|_{m-N,0} \leq C \| \tilde{a} \|_{m,N}. \] (4.17)
Thus, if \( N > m + n \), Lemma 4.2 applies, giving
\[ \| \text{Op}(a_\gamma) D^7 u \|_{(x)^{\nu C_0}(\mathbb{R}^n)} \leq C \| \tilde{a}_\gamma \|_{m-N,0} \| D^7 u \|_M, \quad M > n + |w|, \] (4.18)
and therefore
\[ \| \text{Op}(a) u \|_{(x)^{\nu C_0}(\mathbb{R}^n)} \leq C \| \tilde{a} \|_{m,N} \| u \|_M, \quad M > n + N + |w|. \] (4.19)
To get higher regularity and decay, let now \( \alpha, \beta \in \mathbb{N}^n_0 \), then
\[ x^\alpha D_x^\beta \text{Op}(a) u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (D_\xi + y)^\alpha e^{i(x-y)\cdot \xi}(\xi + D_x)^\beta a(x, y, \xi) u(y) \, dy \, d\xi \]
\[ = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi}(\xi - D_\xi)^\alpha (\xi + D_x)^\beta a(x, y, \xi) u(y) \, dy \, d\xi. \] (4.20)
This can be expanded using the Leibniz rule; note that powers of \( y \) are acceptable since \( u \) is Schwartz. We thus obtain
\[ \| x^\alpha D_x^\beta \text{Op}(a) u \|_{(x)^{\nu C_0}(\mathbb{R}^n)} \leq C \| \tilde{a} \|_{m,N} \| u \|_N \] (4.21)
for \( N \) sufficiently large (depending on \( m, n, \alpha, \beta \)). Thus, \( \text{Op}(a) u \in \mathcal{S}(\mathbb{R}^n) \), finishing the proof. \( \square \)

This shows that the map
\[ \langle x - y \rangle^w S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (a, u) \mapsto \text{Op}(a) u \in \mathcal{S}(\mathbb{R}^n) \] (4.22)
is a continuous bilinear map when putting the topology of \( \langle x - y \rangle^w S^{m'}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \) on the first factor (for any \( m' \in \mathbb{R} \)). By Proposition 3.7, it thus extends by continuity to a continuous bilinear map
\[ \langle x - y \rangle^w S^{m}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (a, u) \mapsto \text{Op}(a) u \in \mathcal{S}(\mathbb{R}^n). \] (4.23)
Identifying \( \text{Op}(a) \) with its Schwartz kernel, we thus get a continuous map
\[ \text{Op}: \langle x - y \rangle^w S^{m}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n), \] (4.24)
which is given (interpreted as a limit along a sequence of residual symbols) by
\[ \text{Op}(a)(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} a(x, y, \xi) \, d\xi. \] (4.25)
(This is of course much weaker than (4.23).)

**Remark 4.6.** Let \( \chi \in C_c^\infty(\mathbb{R}^n) \) be identically 1 near 0. Given \( a \in \langle x - y \rangle^w S^{m}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \), (the proof of) Proposition 3.7 implies that
\[ \text{Op}(a) u(x) = \lim_{j \to \infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} \chi(\xi/j) a(x, y, \xi) u(y) \, dy \, d\xi, \] (4.26)
with convergence in \( \mathcal{S}(\mathbb{R}^n) \).
**Definition 4.7.** Let $m \in \mathbb{R}$. The space of pseudodifferential operators of order $m$,

$$\Psi^m(\mathbb{R}^n),$$

is the space of all operators $\text{Op}(a) : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ for $a \in (x - y)^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. (As we show in the next section, this space is independent of $w \in \mathbb{R}$, hence we do not make $w$ explicit in the notation. See Exercise 4.1 for the case of differential operators.) The space of residual operators is

$$\Psi^{-\infty}(\mathbb{R}^n) = \bigcap_{m \in \mathbb{R}} \Psi^m(\mathbb{R}^n).$$

Note that a priori it is not clear that $\Psi^{-\infty}(\mathbb{R}^n)$ is equal to the space of quantizations of residual symbols (it is certainly contained in the latter); we show this in Proposition 4.10 below.

By duality, we can define the action of $A = \text{Op}(a) \in \Psi^m(\mathbb{R}^n)$ on tempered distributions: for $u, v \in \mathscr{S}(\mathbb{R}^n)$ and $a \in (x - y)^w S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, we have

$$\langle \text{Op}(a)u, v \rangle = \iiint_{\mathbb{R}^{3n}} e^{i(x-y)\cdot \xi} a(x, y, \xi) u(y)v(x) \, dy \, d\xi \, dx$$

$$= \iiint_{\mathbb{R}^{3n}} e^{i(x-y)\cdot \xi} a(y, x, -\xi) v(y)u(x) \, dy \, d\xi \, dx$$

$$= \langle u, \text{Op}(a')v \rangle,$$

where we put

$$a'(x, y, \xi) = a(y, x, -\xi).$$

Since $a \mapsto a'$ is an isomorphism on $(x - y)^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, the equality

$$\text{Op}(a') = \text{Op}(a'), \quad \text{that is, } \langle \text{Op}(a)u, v \rangle = \langle u, \text{Op}(a')v \rangle$$

continuous to hold for $a \in (x - y)^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. By the density $\mathscr{S}(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$, we can thus uniquely extend, by continuity, $\text{Op}(a)$ to an operator on $\mathscr{S}'(\mathbb{R}^n)$ via (4.31).

### 4.1. Left/right reduction, adjoints.

In this section, we shall prove:

**Theorem 4.8.** Let $a \in (x - y)^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. Then there exists a unique left symbol $a_L \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$\text{Op}(a) = \text{Op}(a_L) = \text{Op}_L(a_L),$$

and a unique right symbol $a_R \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$\text{Op}(a) = \text{Op}(a_R) = \text{Op}_R(a_R).$$

The symbols $a_L, a_R$ depend continuously on $a$. Modulo residual symbols, they are given by asymptotic sums

$$a_L(x, \xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \left( \partial^\alpha_x \partial^\alpha_y a(x, y, \xi) \right) |_{y = x},$$

$$a_R(y, \xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{(-1)^{|\alpha|}}{\alpha!} \left( \partial^\alpha_x \partial^\alpha_y a(x, y, \xi) \right) |_{x = y}.$$  

(The summands are ordered by increasing $|\alpha|.$)
Definition 4.9. In the notation of Theorem 4.8, we call $a_L$, resp. $a_R$ the left, resp. right reduction of the full symbol $a$. Writing $A = \text{Op}(a)$, we write

$$a_L =: \sigma_L(A), \quad a_R =: \sigma_R(A). \quad (4.36)$$

We first consider the case $m = -\infty$ of Theorem 4.8 and give a description of kernels of residual operators:

Proposition 4.10. An operator $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a residual operator, $A \in \Psi^{-\infty}(\mathbb{R}^n)$, if and only if its Schwartz kernel $K(x,y)$ is smooth and satisfies

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \leq C_{\alpha\beta N} \langle x-y \rangle^{-N} \quad \forall \alpha, \beta, N. \quad (4.37)$$

Moreover, any such $A$ can be written as $A = \text{Op}_L(a_L) = \text{Op}_R(a_R)$ for unique symbols $a_L, a_R \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$.

Proof. Since $A \in \Psi^N(\mathbb{R}^n)$ for all $N \in \mathbb{R}$, we can write $A = \text{Op}(a_N)$ with $a_N \in (x-y)^w S^{-N}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. Taking $N$ large, the Schwartz kernel $K$ of $A$ is then given by

$$K(x,y) = (\mathcal{F}_3^{-1} a_N)(x,y,x-y), \quad (4.38)$$

with $\mathcal{F}_3$ denoting the Fourier transformation in the third argument ($\xi$). For any fixed $k \in \mathbb{N}_0$, the symbol $a_N(x,y,\xi)$ has Schwartz seminorms $\|a_N(x,y,\xi)\|_N$ bounded by $\langle x-y \rangle^w$ for $N \geq N(k)$; estimates for the (inverse) Fourier transform as in (2.12) thus imply that

$$|\partial_x^\alpha \partial_y^\beta K(x,y)| \leq C_{\alpha\beta N} \langle x-y \rangle^w \langle x-y \rangle^{-k} \quad (4.39)$$

for all $\alpha, \beta, k$, giving (4.37).

We prove the converse momentarily; first, we turn to the second part. Note that if $K$ satisfies (4.37), we define

$$a_L(x,\xi) = \int_{\mathbb{R}^n} e^{-iz\cdot\xi} K(x,z) \, dz. \quad (4.40)$$

Then $\text{Op}(a_L) = K$ by the Fourier inversion formula, and the estimates (4.37) imply $a_L \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$. Similarly, we have $K = \text{Op}(a_R)$ for

$$a_R(y,\xi) = \int_{\mathbb{R}^n} e^{-iz\cdot\xi} K(y+z,y) \, dz. \quad (4.41)$$

Finally, if $K$ satisfies (4.37), then so does $\hat{K}(x,y) = \langle x-y \rangle^{-w} K(x,y)$; by what we have already proved, we can write $\hat{K} = \text{Op}_L(\hat{a}_L)$, $\hat{a}_L \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$, and therefore $K = \text{Op}(a)$ where $a(x,y,\xi) = \langle x-y \rangle^w \hat{a}_L(x,\xi) \in \langle x-y \rangle^w S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, proving that $K \in \Psi^{-\infty}(\mathbb{R}^n)$ in the sense of Definition 4.7. \qed

Corollary 4.11. $\Psi^{-\infty}(\mathbb{R}^n) = \text{Op}(\langle x-y \rangle^w S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)) = \text{Op}_{L/R}(S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n))$.

To handle the case of general orders $m \in \mathbb{R}$, we note that integration by parts in $\xi$ implies the equality of Schwartz kernels

$$\text{Op}((y-x)^\alpha a)(x,y) = (2\pi)^{-n} \int ((-D_\xi)^\alpha e^{i(x-y)\cdot\xi}) a(x,y,\xi) \, d\xi \quad (4.42)$$

$$= (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} D_\xi^\alpha a(x,y,\xi) \, d\xi = \text{Op}(D_\xi^\alpha a)(x,y),$$

$$\text{Op}(\langle x-y \rangle^w S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)) = \text{Op}_{L/R}(S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)).$$
first for \( a \in (x - y)^{-w} S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \), and then for symbols of order \( m \) by density and continuity.

**Proof of Theorem 4.8.** Let \( N \in \mathbb{N} \), then Taylor’s formula states

\[
a(x, y, \xi) = \sum_{|\alpha| < N} \frac{1}{\alpha!} (y - x)^\alpha \partial_y^\alpha a(x, y, \xi) |_{y=x} + R_N(x, y, \xi),
\]

\[
R_N(x, y, \xi) = \sum_{|\alpha| = N} \frac{N}{\alpha!} (y - x)^\alpha \int_0^1 (1 - t)^{N-1} (\partial_y^\alpha a)(x, x + t(y - x), \xi) \, dt.
\]

(4.43)

In view of the symbolic estimates for \( a \), the remainder satisfies the estimate

\[
|\partial_x^\beta \partial_y^\gamma \partial_\xi^\delta R_N(x, y, \xi)| \leq C_{\beta, \gamma, N} (x - y)^{w+N} \langle \xi \rangle^{m-N-|\delta|},
\]

hence

\[
R_N \in (x - y)^{w+N} S^{m-N}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n).
\]

(4.44)

(The additional off-diagonal growth is the reason for working with the more general symbol class (4.1).) Using the identity (4.42), we have

\[
\text{Op} \left( a - \sum_{|\alpha| < N} \frac{1}{\alpha!} (D_\xi^\alpha \partial_y^\alpha a)|_{y=x} \right) = \text{Op}(R_N) \in \Psi^{m-N}(\mathbb{R}^n)
\]

(4.46)

for all \( N \). Note that terms in the sum with \( |\alpha| = k \) are left symbols of order \( m - k \). Thus, we can let \( b \in S^m(\mathbb{R}^n; \mathbb{R}^n) \) be an asymptotic sum

\[
b \sim \sum_\alpha \frac{1}{\alpha!} (D_\xi^\alpha \partial_y^\alpha a)|_{y=x},
\]

(4.47)

and then

\[R := \text{Op}(a - b) \in \bigcap_{N \in \mathbb{N}} \Psi^{m-N}(\mathbb{R}^n) = \Psi^{-\infty}(\mathbb{R}^n).
\]

(4.48)

By Proposition 4.10, we then have \( R = \text{Op}_L(r) \) for some \( r \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n) \). Therefore,

\[A = \text{Op}_L(a_L), \quad a_L := b + r.
\]

(4.49)

The continuous dependence of \( a_L \) on \( a \) is left to the reader; the key ingredient is that the asymptotic summation (4.47) can be performed continuously in \( a \). (In fact, the construction in the proof of Proposition 3.13 does the job here.)

Reduction to a right symbol is proved analogously. Instead of going through the argument, one can instead use duality as in (4.29), the idea being that the adjoint of a left quantization is a right quantization (and vice versa). Namely, using (4.30), we write the adjoint of \( \text{Op}(a) \) as \( \text{Op}(a)^\dagger = \text{Op}(a^\dagger_L) \) for \( a^\dagger_L \in S^m(\mathbb{R}^n; \mathbb{R}^n) \), and then

\[
\text{Op}(a) = \text{Op}(a^\dagger)^\dagger = \text{Op}(a_L^\dagger)^\dagger = \text{Op}((a_L^\dagger)^\dagger) = \text{Op}_R(a_R),
\]

(4.50)

where \( a_R(y, \xi) = a^\dagger_L(y, -\xi) \). The formula for right reductions gives

\[
a_L^\dagger(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} ((-\partial_\xi)^\alpha D_\xi^\alpha a)(y, x, -\xi)|_{y=x},
\]

yielding the asymptotic description (4.35) of \( a_R \).
It remains to prove the uniqueness of \( a_L, a_R \). For this, note that a left symbol \( a_L \) can be viewed as an element \( a_L \in C^\infty(\mathbb{R}^n_+; \mathcal{S}(\mathbb{R}^n_+)) \), and the Schwartz kernel of \( \text{Op}(a_L) \) is

\[
\text{Op}(a_L)(x, x - z) = (F_2^{-1} a_L)(x, z).
\]

(4.52)

Since \( F_2 \) is an isomorphism of \( C^\infty(\mathbb{R}^n; \mathcal{S}(\mathbb{R}^n)) \), \( \text{Op}(a_L) = 0 \) implies \( a_L = 0 \). The proof for \( a_R \) is similar. \( \square \)

**Corollary 4.12.** Let \( m \in \mathbb{R} \) or \( m = -\infty \). Then \( \Psi^m(\mathbb{R}^n) = \text{Op}_{L/R}(S^m(\mathbb{R}^n; \mathbb{R}^n)) \).

A slight variant of (4.29) gives the first part of the following corollary; the second part is an immediate application of Theorem 4.8.

**Corollary 4.13.** Let \( A \in \Psi^m(\mathbb{R}^n) \), then

\[
\int_{\mathbb{R}^n} (A^* u)(x) \tilde{v}(x) \, dx = \int_{\mathbb{R}^n} u(x) (\text{Op}(A)v)(x) \, dx, \quad u, v \in \mathcal{S}(\mathbb{R}^n).
\]

(4.53)

defines an operator \( A^* \in \Psi^m(\mathbb{R}^n) \). If \( A = \text{Op}(a) \), then \( A^* = \text{Op}(a^*) \), \( a^*(x, y, \xi) = \bar{a}(y, x, \xi) \).

If \( A = \text{Op}_L(a_L) \), then \( A^* = \text{Op}_L(a_L^*) \) with

\[
a_L^*(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial^\alpha_x D_\xi^\alpha \pi(x, \xi)
\]

(4.54)

Moreover, \( \Psi^m(\mathbb{R}^n) \ni A \mapsto A^* \in \Psi^m(\mathbb{R}^n) \) is a continuous conjugate-linear map.

**4.2. Topology on spaces of pseudodifferential operators.** Let \( m \in \mathbb{R} \) or \( m = -\infty \).

Since \( \text{Op}_L: S^m(\mathbb{R}^n; \mathbb{R}^n) \to \Psi^m(\mathbb{R}^n) \) is an isomorphism of vector spaces, it is natural to transport the Fréchet space structure of \( S^m(\mathbb{R}^n; \mathbb{R}^n) \) to \( \Psi^m(\mathbb{R}^n) \) via \( \text{Op}_L \). For instance:

**Lemma 4.14.** Let \( \chi \in C^\infty_c(\mathbb{R}^n_+) \) be identically 1 near 0, and put \( J_\epsilon = \text{Op}(\chi(\epsilon \cdot)) \), \( \epsilon > 0 \). Then \( J_\epsilon \in \Psi^{-\infty}(\mathbb{R}^n) \) is uniformly bounded in \( \Psi^0(\mathbb{R}^n) \) and converges to the identity operator \( I = \text{Op}(1) \) in the topology of \( \Psi^0(\mathbb{R}^n) \) for any \( \eta > 0 \).

**Proof.** This is equivalent to the main part of (the proof of) Proposition 3.7. \( \square \)

It is reassuring to note that one can equally well define the topology on \( \Psi^m(\mathbb{R}^n) \) using the right quantization. This is a consequence of the following result.

**Proposition 4.15.** Let \( m \in \mathbb{R} \) or \( m = -\infty \). Then the isomorphism of vector spaces \( \text{Op}_R: S^m(\mathbb{R}^n; \mathbb{R}^n) \to \Psi^m(\mathbb{R}^n) \) is an isomorphism of Fréchet spaces.

**Proof.** Right reduction \( \sigma_R \) is the inverse of \( \text{Op}_R \). By definition of the Fréchet space structure of \( \Psi^m(\mathbb{R}^n) \), the proposition is thus equivalent to the continuity of \( \sigma_R \circ \text{Op}_L \), which is part of Theorem 4.8. \( \square \)

**4.3. Composition.** Proving that composition of ps.d.o.s produces another ps.d.o. is now straightforward:

**Theorem 4.16.** Let \( A \in \Psi^m(\mathbb{R}^n) \), \( B \in \Psi^m'(\mathbb{R}^n) \). Then \( A \circ B: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \) is a pseudodifferential operator,

\[
A \circ B \in \Psi^{m+m'}(\mathbb{R}^n),
\]

(4.55)
and its left symbol is given as an asymptotic sum

$$\sigma_L(A \circ B) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial^\alpha \sigma_L(A) \cdot D^\alpha_x \sigma_L(B). \quad (4.56)$$

The bilinear map \((A, B) \mapsto A \circ B\) is continuous.

Note that the symbolic expansion (4.56) is local in \((x, \xi)\): the symbols of \(A\) and \(B\) do not ‘interact’ at all, modulo residual terms, at distinct points in phase space \(\mathbb{R}^n_x \times \mathbb{R}^n_\xi\).

**Proof of Theorem 4.16.** Write \(A = \text{Op}_L(a)\) and \(B = \text{Op}_R(b_R)\). Assume first that \(A, B \in \Psi^{-\infty}(\mathbb{R}^n)\), then for \(u, v \in \mathcal{S}(\mathbb{R}^n)\), we have

$$Av(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x, \xi) \hat{v}(\xi) \, d\xi,$$

$$\hat{Bu}(\xi) = \int_{\mathbb{R}^n} e^{-iy \cdot \xi} b_R(y, \xi) u(y) \, dy.$$

Thus,

$$ABu(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} a(x, \xi) b_R(y, \xi) u(y) \, dy \, d\xi,$$

giving \(A \circ B = \text{Op}(c), c(x, y, \xi) = a(x, \xi) b_R(y, \xi)\). (This is one of the reasons for considering such general symbols!) By density and continuity, this continues to hold for \(A, B\) as in the statement of the theorem.

The asymptotic expansion (4.56) follows from\(^1\)

$$\sigma_C(A \circ B)(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial^\alpha (a(x, \xi) D_y b_R(y, \xi) |_{y=x})$$

$$\sim \sum_{\beta, \gamma} \frac{1}{\beta! \gamma!} \partial_{\xi}^\beta a(x, \xi) \cdot \partial_\xi^\gamma D^\beta D^\gamma x \left( \sum_{\delta} \frac{(-1)^{||\delta||}}{\delta!} (\partial_{\xi}^\delta D^\delta x b)(x, \xi) \right)$$

$$\sim \sum_{\beta} \frac{1}{\beta!} \partial_{\xi}^\beta a(x, \xi) \cdot D^\beta x \left( \sum_{\epsilon} \frac{1}{\epsilon!} \partial_{\xi}^\epsilon D^\epsilon x b(x, \xi) \sum_{\gamma + \delta = \epsilon} \frac{\epsilon!}{\gamma! \delta!} (-1)^{||\delta||} \right)$$

and the observation that for \(\epsilon = 0\), the final sum evaluates to 1, while for \(\mathbb{N}_0^n \ni \epsilon \neq 0\),

$$\sum_{\gamma + \delta = \epsilon} \frac{\epsilon!}{\gamma! \delta!} (-1)^{||\delta||} = \prod_{\epsilon_j \neq 0} (1 - 1)^{\epsilon_j} = 0. \quad (4.60)$$

This finishes the proof. \(\square\)

As a simple consequence, we can now prove the pseudolocality of ps.d.o.s:

**Proposition 4.17.** Let \(A \in \Psi^m(\mathbb{R}^n)\). Then

$$\text{sing supp } Au \subseteq \text{sing supp } u, \quad u \in \mathcal{S}'(\mathbb{R}^n). \quad (4.61)$$

To prove this, we record:

\(^1\)Since these are asymptotic sums, it suffices to consider only those terms which have symbolic order bigger than some fixed but arbitrary number; in particular, there are no convergence or rearrangement issues.
Lemma 4.18. A residual operator $A \in \Psi^{-\infty}(\mathbb{R}^n)$ is continuous as a map

$$A: \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n).$$

More precisely, for any $u \in \mathcal{S}'(\mathbb{R}^n)$ we have $Au \in \langle x \rangle^N C^\infty_0(\mathbb{R}^n)$ for some $N$ (depending on $u$).

Proof. Let $K$ denote the Schwartz kernel of $A$; recall that it satisfies the estimates (4.37). For $u \in \mathcal{S}'(\mathbb{R}^n)$, we then have, for some $N \in \mathbb{N}$,

$$|(Au)(x)| = |\langle K(x, \cdot), u \rangle| \leq C\|K(x, \cdot)\|_N = C \sup_{y \in \mathbb{R}^n} \|y^\alpha D_y^\beta K(x, y)\|,$$

$$\leq C \sup_{y \in \mathbb{R}^n} |\langle y \rangle^N D_y^\beta K(x, y)| \leq C \sup_{y \in \mathbb{R}^n} \langle y \rangle^N \langle x - y \rangle^{-N} \|\langle x - y \rangle^N D_y^\beta K(x, y)\|.$$

Using Lemma 4.3, we see that $\langle y \rangle^N \langle x - y \rangle^{-N} \leq C_N \langle x \rangle^N$, hence

$$|(Au)(x)| \leq C \langle x \rangle^N. \quad (4.64)$$

Derivatives in $x$ are estimated analogously, so $Au \in C^\infty(\mathbb{R}^n)$, and in fact

$$|\partial_x^\alpha (Au)(x)| \leq C_\alpha \langle x \rangle^N. \quad (4.65)$$

Note here that the number $N$ above only depends on $u$, not on $K$ itself. \hfill \Box

Proof of Proposition 4.17. Suppose $x \notin \text{sing supp } u$. There exist cutoffs $\chi, \tilde{\chi} \in C_c^\infty(\mathbb{R}^n)$ such that

$$\chi(x) \neq 0, \quad \tilde{\chi} \equiv 1 \text{ on } \text{supp } \chi, \quad \tilde{\chi} u \in C^\infty(\mathbb{R}^n). \quad (4.66)$$

Then

$$\chi Au = \chi A(\tilde{\chi} u) + \chi A(1 - \tilde{\chi})u. \quad (4.67)$$

Since $A$ acts on $\mathcal{S}(\mathbb{R}^n)$, we have $\chi A(\tilde{\chi} u) \in \mathcal{S}(\mathbb{R}^n)$. For the second term, note that $\chi$ and $1 - \tilde{\chi}$ have disjoint supports; hence we have

$$\sigma_L(\chi A \circ (1 - \tilde{\chi}))(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \chi(x) \partial_\xi^\alpha \sigma_L(A)(x, \xi) \cdot D_\xi^\alpha (1 - \tilde{\chi}(x)) = 0,$$

which implies

$$\chi A(1 - \tilde{\chi}) \in \Psi^{-\infty}(\mathbb{R}^n). \quad (4.69)$$

By Lemma 4.18, we conclude that $\chi A(1 - \tilde{\chi}) u \in C^\infty(\mathbb{R}^n)$, finishing the proof. \hfill \Box

Returning to the observation (4.69), note that if $A = \text{Op}(a)$ has Schwartz kernel $K \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$, then the Schwartz kernel of $\chi A(1 - \tilde{\chi})$ is $\chi(x) (1 - \tilde{\chi}(y)) K(x, y)$. Thus, (4.69) can equivalently be stated as:

Proposition 4.19. The Schwartz kernel $K$ of a pseudodifferential operator is smooth away from the diagonal $\Delta = \{(x, x) : x \in \mathbb{R}^n\}$. That is, sing supp $K \subset \Delta$. 

4.4. **Principal symbols.** Similarly to Proposition 2.19, the ‘leading order part’ of the left or right symbol of an operator $A \in \Psi^m(\mathbb{R}^n)$ has particularly simple properties.

**Definition 4.20.** Let $m \in \mathbb{R}$. The *principal symbol* $\sigma_m(A)$ of a ps.d.o. $A \in \Psi^m(\mathbb{R}^n)$ is the equivalence class

$$\sigma_m(A) := [\sigma_L(A)] \in S^m(\mathbb{R}^n; \mathbb{R}^n)/S^{m-1}(\mathbb{R}^n; \mathbb{R}^n).$$

(4.70)

We shall often omit from the notation the passage to the equivalence class.

Directly from the definition, this gives a short exact sequence for every $m \in \mathbb{R}$:

$$0 \to \Psi^{m-1}(\mathbb{R}^n; \mathbb{R}^n) \to \Psi^m(\mathbb{R}^n; \mathbb{R}^n) \xrightarrow{\sigma_m} S^m(\mathbb{R}^n; \mathbb{R}^n)/S^{m-1}(\mathbb{R}^n; \mathbb{R}^n) \to 0.$$  

(4.71)

The surjectivity of $\sigma_m$ is clear: given a representative $a \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ of an equivalence class of symbols, we have $\sigma_m(Op_L(a)) = [a]$.

**Proposition 4.21.** The principal symbol map has the following properties:

1. $\sigma_m(Op_R(a)) = [a]$, i.e. using the right symbol in (4.70) gives the same principal symbol map.

2. For $A \in \Psi^m(\mathbb{R}^n)$, we have $\sigma_m(A^*) = \overline{\sigma_m(A)}$.

3. For $A \in \Psi^m(\mathbb{R}^n)$, $B \in \Psi^{m'}(\mathbb{R}^n)$, we have $\sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B)$.

(The behavior under changes of variables will be discussed in §4.8.) Notice that the principal symbol map translates operator composition (a highly non-commutative operation) to the multiplication of (equivalence classes of) functions (a commutative operation), though of course at what seems to be an enormous loss of information compared to the full expansion (4.56) (which itself gives up information on the residual part of $A \circ B$). However, in most situations, the principal symbol, and sometimes a ‘subprincipal’ part of the full symbol, dominate the behavior of the operator, while lower order parts are irrelevant; cf. the discussion of ellipticity for symbols in §3.1.

One crucial calculation is the following. For $A \in \Psi^m(\mathbb{R}^n)$, $B \in \Psi^{m'}(\mathbb{R}^n)$, note that $\sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B) = \sigma_{m+m'}(B \circ A)$, so

$$\sigma_{m+m'}([A, B]) = 0, \quad [A, B] = A \circ B - B \circ A.$$  

(4.72)

In view of (4.71), we thus have $[A, B] \in \Psi^{m+m'-1}(\mathbb{R}^n)$, and it is natural to inquire about its principal symbol as an operator of order $m + m' - 1$. It turns out that it can be computed solely in terms of the principal symbols of $A$ and $B$:

**Proposition 4.22.** For $A \in \Psi^m(\mathbb{R}^n)$, $B \in \Psi^{m'}(\mathbb{R}^n)$, we have

$$\sigma_{m+m'-1}([A, B]) = \{\sigma_m(A), \sigma_{m'}(B)\},$$  

(4.73)

where the Poisson bracket of $a, b \in C^\infty(\mathbb{R}_x^m \times \mathbb{R}_x^n)$ is defined as

$$\{a, b\} := \sum_{j=1}^n (\partial_{\xi_j} a)(\partial_{x_j} b) - (\partial_{x_j} a)(\partial_{\xi_j} b).$$  

(4.74)

This will be the key connection between ‘quantum mechanics’ (quantizations of symbols, noncommutative algebra of operators) and ‘classical mechanics’ (symbols themselves, commutative algebra of functions), which will play a central role in §6.
Proof of Proposition 4.22. We leave it to the reader to verify that (4.73) is well-defined, i.e. that the image of the right hand side in the quotient space $S^{m+m'-1}/S^{m+m'-2}$ does not depend on the choice of representatives of the principal symbols of $A$ and $B$.

The proof is an immediate application of (4.56). Let $a = \sigma_L(A)$, $b = \sigma_L(B)$. Working modulo $S^{m+m'-2}(\mathbb{R}^n; \mathbb{R}^n)$, we have

$$\sigma_L(A \circ B) \equiv ab + \frac{1}{i} \sum_{j=1}^n (\partial_{x_j} a)(\partial_{x_j} b), \quad \sigma_L(B \circ A) \equiv ab + \frac{1}{i} \sum_{j=1}^n (\partial_{x_j} b)(\partial_{x_j} a),$$

(4.75)

and (4.73) follows. \(\square\)

4.5. Classical operators. Following Definition 3.11, we have a subclass of classical operators:

Definition 4.23. For $m \in \mathbb{C}$, we define the space of classical pseudodifferential operators of order $m$ by

$$\Psi^m_{cl}(\mathbb{R}^n) := \text{Op}_L(S^m_{cl}(\mathbb{R}^n; \mathbb{R}^n)) \subset \Psi^m(\mathbb{R}^n),$$

(4.76)

equipped with the structure of a Fréchet space which makes $\text{Op}_L$ into an isomorphism. We put $\Psi^{-\infty}_{cl}(\mathbb{R}^n) := \Psi^{-\infty}(\mathbb{R}^n)$.

The symbol expansions in Theorem 4.16 and Corollary 4.13 imply that compositions and adjoints of classical operators are still classical:

Proposition 4.24. Composition of ps.d.o.s restricts to a continuous bilinear map

$$\Psi^m_{cl}(\mathbb{R}^n) \times \Psi^{m'}_{cl}(\mathbb{R}^n) \ni (A, B) \mapsto A \circ B \in \Psi^{m+m'}_{cl}(\mathbb{R}^n).$$

(4.77)

Similarly, the map

$$\Psi^m_{cl}(\mathbb{R}^n) \ni A \mapsto A^* \in \Psi^m_{cl}(\mathbb{R}^n)$$

(4.78)

is a continuous conjugate-linear map.

For a classical operator $A = \text{Op}_L(a)$, with $a \in S^m_{cl}(\mathbb{R}^n)$, we can identify the principal symbol $\sigma_{\text{Re}m}(A)$ with the homogenous leading order part of $a$, as discussed after Lemma 3.12. The corresponding short exact sequence is

$$0 \to \Psi^{m-1}_{cl}(\mathbb{R}^n) \to \Psi^m_{cl}(\mathbb{R}^n) \to S^m_{\text{hom}}(\mathbb{R}^n; \mathbb{R}^n \setminus \{0\}) \to 0.$$  

(4.79)

4.6. Elliptic parametrix. Recall Definition 3.8 and the discussion around (3.19). Then:

Definition 4.25. We call an operator $A \in \Psi^m(\mathbb{R}^n)$ (uniformly) elliptic if its principal symbol $\sigma_m(A)$ is elliptic.

As a first, and important, application of the symbol calculus we have developed above, we construct parametrices (approximate inverses—a term which, almost by nature, has no precise definition, but rather depends on the context) of uniformly elliptic operators.

Theorem 4.26. Let $A \in \Psi^m(\mathbb{R}^n)$ be uniformly elliptic. Then there exists an operator $B \in \Psi^{-m}(\mathbb{R}^n)$ which is unique modulo $\Psi^{-\infty}(\mathbb{R}^n)$, such that

$$AB - I, \quad BA - I \in \Psi^{-\infty}(\mathbb{R}^n).$$

(4.80)

We call an operator $B$ satisfying (4.80) a parametrix of $A$. 

Proof of Theorem 4.26. Let $b \in S^{-m}(\mathbb{R}^n; \mathbb{R}^n)$ be such that $\sigma_m(A)b - 1 \in S^{-1}(\mathbb{R}^n; \mathbb{R}^n)$. Put $B_0 = \text{Op}(b) \in \Psi^{-m}(\mathbb{R}^n)$, then

$$A \circ B_0 = I - R, \quad R \in \Psi^{-1}(\mathbb{R}^n). \quad (4.81)$$

Indeed, this follows from $\sigma_0(AB_0 - I) = 0$. We approximately invert $I - R$ using a Neumann series: we choose

$$B_0' \sim \sum_{j=1}^{\infty} R^j \in \Psi^{-1}(\mathbb{R}^n), \quad (4.82)$$

i.e. the left symbol of $B_0'$ is an asymptotic sum of the left symbols of $R^j = R \circ \cdots \circ R$ ($j$ times). Since $(I - R)(I + \sum_{j=1}^{N} R^j) = I - R^{N+1}$ for all $N$, we have

$$(I - R)(I + R') = I + E, \quad E \in \Psi^{-\infty}(\mathbb{R}^n). \quad (4.83)$$

Therefore, if we put

$$B := B_0(I + R') \in \Psi^{-m}(\mathbb{R}^n), \quad (4.84)$$

then $AB = I + E$, as desired.

An analogous argument produces $B' \in \Psi^{-m}(\mathbb{R}^n)$ with $B'A = I + E'$, $E' \in \Psi^{-\infty}(\mathbb{R}^n)$. But then abstract ‘group theory’ gives

$$B = IB = (B'A - E')B = B'AB - E'B = B'(I + E) - E'B = B' + (B'E - E'B). \quad (4.85)$$

Therefore $B - B' \in \Psi^{-\infty}(\mathbb{R}^n)$. In particular, any two parametrices differ by an element of $\Psi^{-\infty}(\mathbb{R}^n)$. \hfill \Box

As a simple application, we prove:

**Proposition 4.27.** Let $A \in \Psi^{m}(\mathbb{R}^n)$ be uniformly elliptic, and suppose 

$$u \in \mathcal{S}'(\mathbb{R}^n), \quad Au = f \in C^\infty(\mathbb{R}^n). \quad (4.86)$$

Then $u \in C^\infty(\mathbb{R}^n)$. More precisely, we have

$$\text{sing supp } u = \text{sing supp } Au. \quad (4.87)$$

**Proof.** We prove (4.87). Let $B \in \Psi^{-m}(\mathbb{R}^n)$ be a parametrix of $A$, with $BA = I + R$, $R \in \Psi^{-\infty}(\mathbb{R}^n)$. Then by Proposition 4.17, we have

$$\text{sing supp } u = \text{sing supp } (BAu + Ru) = \text{sing supp } BAu \subset \text{sing supp } Au \subset \text{sing supp } u. \quad (4.88)$$

Therefore, equality must hold at each step. \hfill \Box

4.7. **Boundedness on Sobolev spaces.** TBC boundedness on $L^2$: square root trick 

TBC (weighted) Sobolev spaces. 

TBC upgrade elliptic regularity 

TBC point out that $\Psi^{-\infty}(\mathbb{R}^n) \colon \mathcal{S}'(\mathbb{R}^n) \to \bigcup_{r \in \mathbb{R}} (x)^r H^\infty(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$ by Theorem 2.13

4.8. **Coordinate invariance, pseudodifferential operators on manifolds, vector bundles.** TBC invariant integration on manifolds 

TBC operators on vector bundles
4.9. Elliptic operators on closed manifolds, Fredholm theory. TBC generalized inverse is psdo

4.10. Exercises.

Exercise 4.1. Let \( m \in \mathbb{N}_0 \), and let \( a \in S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \) be a polynomial in the symbolic variable \( \xi \).

1. Show, starting from the definition as a limit of quantizations of residual symbols, that \( \text{Op}(a) \in \text{Diff}^m(\mathbb{R}^n) \).
2. Prove that \( \text{Op}((x-y)^w a) \in \text{Diff}^m(\mathbb{R}^n) \) (which in particular entails the boundedness of the coefficients). (Hint. Compute its Schwartz kernel.)

Exercise 4.2. Let \( A \in \Psi^m(\mathbb{R}^n) \), and denote by \( K \) its Schwartz kernel.

1. Give another, direct, proof that \( K \in C^\infty((\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta) \), where \( \Delta = \{(x, x) : x \in \mathbb{R}^n\} \) is the diagonal. (Hint. For \( \phi, \psi \in C^\infty(\mathbb{R}^n) \) with \( \text{supp} \phi \cap \text{supp} \psi = \emptyset \), rewrite the pairing \( \langle A \phi, \psi \rangle \) for \( A \in \Psi^{-\infty}(\mathbb{R}^n) \) using integrations by parts as in the proof of Proposition 4.4. Then use a density argument.)
2. Prove that for every \( \epsilon > 0 \), \( N \in \mathbb{R} \) there exists a constant \( C \) such that
   \[
   |K(x, y)| \leq C|x - y|^{-N}, \quad |x - y| \geq \epsilon. \tag{4.89}
   \]

Exercise 4.3. Suppose \( K(x, z) \in C^\infty(\mathbb{R} \times (\mathbb{R} \setminus \{0\})) \) satisfies \( K(x, \lambda z) = \lambda^{-1}K(x, z) \), \( \lambda > 0 \), and \( K(x, -z) = -K(x, z) \). Let \( \chi \in C^\infty_c(\mathbb{R}^n) \) be identically 1 near 0. Show that the operator
   \[
   Au(x) = \lim_{\epsilon \to 0} \int_{|x-y| \geq \epsilon} \chi(x-y)K(x, x-y)u(y) \, dy, \quad u \in C^\infty_c(\mathbb{R}^n), \tag{4.90}
   \]
is well-defined and defines an element \( A \in \Psi^0_{cl}(\mathbb{R}) \). Compute its principal symbol.

Exercise 4.4. Prove Gårding’s inequality. Let \( A \in \Psi^{2m}(\mathbb{R}^n) \), and suppose \( \text{Re} \sigma_{2m}(A) \geq c(\xi)^{2m} \) for some \( c \in \mathbb{R} \). Then for every \( \epsilon > 0 \) and \( N \in \mathbb{R} \), there exists a constant \( C \) such that
   \[
   \text{Re} \langle Au, u \rangle_{L^2(\mathbb{R}^n)} \geq (c - \epsilon)\|u\|^2_{H^m(\mathbb{R}^n)} - C\|u\|^2_{H^{-N}(\mathbb{R}^n)}, \quad u \in \mathcal{S}(\mathbb{R}^n). \tag{4.91}
   \]
   (Hint. Use the ‘square root trick’.)

Exercise 4.5. (Sharp Gårding inequality.) For \( A \in \Psi^{2m}(\mathbb{R}^n) \) with \( \text{Re} \sigma_{2m}(A) \geq 0 \), there exists a constant \( C \) such that
   \[
   \text{Re} \langle Au, u \rangle \geq -C\|u\|^2_{H^{m-1/2}}, \quad u \in \mathcal{S}(\mathbb{R}^n). \tag{4.92}
   \]
   (Hint. Read [Hör03, Theorem 18.1.14].)

Exercise 4.6. Let \( M \) denote a smooth manifold, and denote by
   \[
   d: C^\infty(M; \Lambda^kT^*M) \to C^\infty(M; \Lambda^{k+1}T^*M) \tag{4.93}
   \]
the exterior derivative. Show that \( d \in \text{Diff}^1(M; \Lambda^kT^*M, \Lambda^{k+1}T^*M) \), and compute its principal symbol.

Exercise 4.7. Let \( \Gamma \subset \mathbb{C} \) be a smooth, simple closed curve. Let \( K \in C^\infty(\Gamma \times \Gamma) \). Prove that
   \[
   Au(t) := \lim_{\epsilon \to 0} \int_{|t-s| \geq \epsilon} \frac{K(t, s)}{t-s}u(s) \, ds, \quad u \in C^\infty(\Gamma) \tag{4.94}
   \]
is well-defined and defines an element \( A \in \Psi^0_{cl}(\Gamma) \). Compute its principal symbol.
Exercise 4.8. TBC $\sqrt{\Delta} \in \Psi^1(X)$ (Wunsch Proposition 3.5)

Exercise 4.9. TBC symbols and operators of class $\rho, \delta$

Exercise 4.10. TBC variable order symbols and operators

Exercise 4.11. TBC scattering symbols and scattering ps.d.o.s

The following two exercises give an indirect characterization of ps.d.o.s. For two operators $A, B$, we write

$$\text{ad}_A B := [A, B] = AB - BA.$$  \hspace{1cm} (4.95)

Exercise 4.12. Let $B \in \Psi^m(\mathbb{R}^n)$. Show that the linear operator $\text{ad}_{x_j} B$ on $\mathcal{S}(\mathbb{R}^n)$, defined using (4.95), is an element of $\Psi^{m-1}(\mathbb{R}^n)$.

Exercise 4.13. (Beals’ theorem for $\Psi^m(\mathbb{R}^n)$.) Let $A: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ be a continuous linear operator. Then $A \in \Psi^m(\mathbb{R}^n)$ if and only if for some $s \in \mathbb{R}$ and for all $N, M \in \mathbb{N}_0$ and all $i_1, \ldots, i_N$ and $j_1, \ldots, j_M \in \{1, \ldots, n\}$,

$$\text{ad}_{x_{i_1}} \cdots \text{ad}_{x_{i_N}} \text{ad}_{D_{j_1}} \cdots \text{ad}_{D_{j_M}} A \quad \text{is a bounded map} \quad H^s(\mathbb{R}^n) \to H^{s+N}(\mathbb{R}^n).$$  \hspace{1cm} (4.96)

(1) Prove the simple direction: $A \in \Psi^m(\mathbb{R}^n)$ implies (4.96). (Hint. Use the previous exercise.)

(2) TBC

5. Microlocalization

5.1. Wave front set. Wave front set of distributions ($C^\infty$ and $H^s$)

- Coordinate invariance of the wave front set, relative wave front set

5.2. Microlocal ellipticity. TBC Microlocal ellipticity; elliptic set, characteristic set of ps.d.o.s

5.3. Exercises.

Exercise 5.1. TBC WF($1_\Omega$)

Exercise 5.2. TBC polarization set?

TBC prove directly the invariance of the wave front set under coordinate changes

6. Propagation of singularities

6.1. Symplectic geometry.

6.2. Real principal type propagation.

6.3. Radial points.

6.4. Exercises. TBC

$|g'| \lesssim g^{1/2}$ uniformly on compacts for $C^2 \ni g \geq 0$. Optimality in general?
References


Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA

E-mail address: phintz@mit.edu