## Contents

1. **Introduction** ................................................................. 2

2. **Schwartz functions and tempered distributions** .................... 4  
   2.1. Fourier transform and its inverse  .................................. 6  
   2.2. Sobolev spaces and the Schwartz representation theorem ........ 7  
   2.3. The Schwartz kernel theorem ........................................ 8  
   2.4. Differential operators ................................................ 9  
   2.5. Exercises ............................................................... 10

3. **Symbols** ........................................................................... 11  
   3.1. Ellipticity ................................................................. 13  
   3.2. Classical symbols ....................................................... 14  
   3.3. Asymptotic summation .................................................. 15  
   3.4. Exercises ................................................................. 17

4. **Pseudodifferential operators** ............................................ 17  
   4.1. Left/right reduction, adjoints ........................................ 21  
   4.2. Topology on spaces of pseudodifferential operators ............ 24  
   4.3. Composition .............................................................. 24  
   4.4. Principal symbols ...................................................... 26  
   4.5. Classical operators ..................................................... 28  
   4.6. Elliptic parametrix ...................................................... 28  
   4.7. Boundedness on Sobolev spaces ..................................... 29  
   4.8. Exercises ................................................................. 32

5. **Pseudodifferential operators on manifolds** ............................ 35  
   5.1. Local coordinate invariance ........................................... 35  
   5.2. Manifolds, vector bundles, densities ................................ 37  
   5.3. Differential operators on manifolds ................................ 41  
   5.4. Definition of $\Psi^m(M)$ ................................................ 42  
   5.5. Principal symbol ......................................................... 47  
   5.6. Operators acting on sections of vector bundles ................. 48  
   5.7. Special classes of operators .......................................... 51

*Date*: March 21, 2019.  
*I am grateful to everybody who sends me corrections and comments via email!*
1. Introduction

Microlocal analysis is a paradigm for the study of distributions and their singularities. Interesting distributions mostly arise in two ways:

(1) as solutions of partial differential equations (PDE), and
(2) as integral kernels of operators used to localize, transform, or otherwise ‘test’ a partial differential operator.

In these notes, we explicitly mostly focus on the first kind, and prove very general results about solutions of linear PDE. The second kind will be present throughout, starting in §4, though mostly not explicitly so.

Following a quick reminder on Schwartz functions and tempered distributions in §2, the notes can be roughly divided into two parts. The first part (§§3–4) introduces pseudodifferential operators (ps.d.o.s) on $\mathbb{R}^n$ and their basic properties. Consider for example the Laplacian

\[ \Delta = \sum_{j=1}^{n} D_{x_j}^2, \quad D_{x_j} := \frac{1}{i} \partial_{x_j}, \]  

which is a differential operator of order 2:

\[ \Delta \in \text{Diff}^2(\mathbb{R}^n). \]
Consider the operator \( L \in \text{Diff}^2(\mathbb{R}^n) \) defined by
\[
L := \Delta + 1.
\] (1.3)
Then \( L : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \) is invertible (see Exercise 2.1); what kind of object is its inverse \( L^{-1} \)? Morally, it should be an operator of order \(-2\), since composing it with \( L \) gives the identity operator, which has order 0. And indeed, \( L^{-1} \) is a pseudodifferential operator of order \(-2\),
\[
L^{-1} \in \Psi^{-2}(\mathbb{R}^n). 
\] (1.4)
By means of the Fourier transform and its inverse (see §2.1), we can write
\[
(L^{-1}u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi}(1 + |\xi|^2)^{-1}u(y) \, dy \, d\xi 
\] (1.5)
More generally, we shall define spaces of operators
\[
\Psi^m(\mathbb{R}^n), \quad m \in \mathbb{R},
\] (1.6)
acting on Schwartz functions (and much larger function spaces too, such as tempered distributions), with \( \text{Diff}^m(\mathbb{R}^n) \subset \Psi^m(\mathbb{R}^n) \) for \( m = 0, 1, 2, \ldots \), and forming a graded algebra:
\[
\Psi^m(\mathbb{R}^n) \circ \Psi^{m'}(\mathbb{R}^n) \subset \Psi^{m+m'}(\mathbb{R}^n).
\] (1.7)
Roughly speaking, a typical element \( A \in \Psi^m(\mathbb{R}^n) \) is defined similarly to (1.5), but with \((1 + |\xi|^2)^{-1}\) replaced by a more general symbol \( a(x, \xi) \) with \( |a(x, \xi)| \lesssim (1 + |\xi|^2)^{m/2} \); see §3 for the definition of symbols. In §4, we will define \( \Psi^m(\mathbb{R}^n) \) precisely, prove (1.7), as well as the boundedness of ps.d.o.s on a variety of useful function spaces. We will also discuss generalizations of (1.4) for elliptic (pseudo)differential operators. (Ellipticity is a notion concerning only the principal symbol of \( A \); the latter is, roughly speaking, the leading order part of \( a \), i.e. a modulo symbols of order \( m - 1 \), and ellipticity is the requirement that the principal symbol be invertible.) In particular, we shall prove that on closed manifolds (compact without boundary) \( M \), every elliptic operator \( L \in \Psi^m(M) \) is Fredholm as a map on \( \mathcal{C}^\infty(M) \), or as a map \( L : H^s(M) \to H^{s-m}(M) \) (\( s \in \mathbb{R} \)); thus, we can solve the equation \( Lu = f \) provided \( f \) satisfies a finite number of linear constraints, and then \( u \) is unique modulo elements of the finite-dimensional space \( \ker L \).

While there are many more interesting things one can say about linear elliptic operators (index theory, Weyl law, degenerate or non-compact problems, etc.), we will switch gears in the second part (§§6–8) of the notes and study non-elliptic phenomena. We begin in §6 by defining the wave front set of a distribution \( u \in \mathcal{S}'(\mathbb{R}^n) \), which is a subset
\[
\text{WF}(u) \subset T^*\mathbb{R}^n \setminus o = \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}),
\] (1.8)
conic in the second factor. (Here, \( o \) is the zero section of the cotangent bundle \( T^*\mathbb{R}^n \).) Its projection onto \( \mathbb{R}^n \) coincides with the singular support, \( \text{sing supp} \, u \); roughly speaking, \( \text{WF}(u) \) measures where and in what (co)directions \( u \) is singular. As a basic example, see Exercise 6.3, the wave front set of the characteristic function \( 1_\Omega \) of a smooth domain \( \Omega \subset \mathbb{R}^n \) is given by the conormal bundle of the boundary (minus the zero section)
\[
\text{WF}(1_\Omega) = N^*\partial\Omega \setminus o.
\] (1.9)
Elliptic regularity can then be microlocalized: if \( L \in \Psi^m(\mathbb{R}^n) \) has principal symbol \( \ell \), and if \( u \in \mathcal{S}'(\mathbb{R}^n) \) is such that \( Lu \) is smooth, then \( \text{WF}(u) \) is contained in the characteristic set
Char($L$) of $L$: roughly speaking, the set of those $(x, \xi)$ where $\ell$ is not elliptic. For example, the wave operator
\[
\Box = -D_t^2 + \sum_{j=1}^n D_{x_j}^2
\] (1.10)
on $\mathbb{R}^{1+n}_{t,x}$ has (principal) symbol
\[
\ell = -\sigma^2 + |\xi|^2, \quad |\xi|^2 = \sum_{j=1}^n \xi_j^2,
\]
where we write $(\sigma, \xi)$ for the momentum variables (dual under the Fourier transform) to $(t, x)$. Thus,
\[
\text{Char} (\Box) = \{(t, x, \sigma, \xi) \in T^*\mathbb{R}^{1+n} : \sigma^2 = |\xi|^2 \}.
\] (1.11)

As a very concrete example, note that
\[
u = H(t - x_1) \implies \Box \nu = 0
\] (1.12)
and indeed WF($\nu$) $\subset$ Char($\Box$) by (1.9).

The theorem on the propagation of singularities, proved in §8, gives a complete description of the structure of WF($\nu$) for $\nu$ a distributional solution of an equation $Lu = f \in C^\infty$; it states that WF($\nu$) $\subset$ Char($L$) is invariant under the flow along the Hamilton vector field of the principal symbol of $L$. In the case of $\Box$, this flow, for time $s \in \mathbb{R}$, maps $(t, x, \sigma, \xi)$ to $(t - 2s\sigma, x + 2s\xi, \sigma, \xi)$; use this to verify the theorem for (1.12)!

We shall prove this using the method of positive commutators, which showcases the utility of ps.d.o.s as tools, rather than as interesting operators in their own right as in (1.4), and exploits the link between symplectic geometry and ps.d.o.s (a form of the ‘classical–quantum correspondence’). More importantly, this is a very flexible method, which allows one to control solutions of PDE also in more degenerate situations—which arise frequently in applications. We give one example concerning radial points in §9.

These notes draw material from Richard Melrose’s lecture notes [Mel07], available under www-math.mit.edu/~rbm/iml90.pdf, the textbook Microlocal Analysis for Differential Operators: an Introduction by Grigis and Sjöstrand [GS94], lecture notes by Jared Wunsch [Wun13], as well as from notes for lectures by András Vasy at Stanford University and Ingo Witt at the University of Göttingen.

2. Schwartz functions and tempered distributions

Let $k \in \mathbb{N} = \{1, 2, 3, \ldots \}$. For an open set $\Omega \subset \mathbb{R}^n$, we denote by $C^k(\Omega)$ the space of $k$ times continuously differentiable functions (with no growth restrictions), and $C^\infty(\Omega) = \bigcap_{k \in \mathbb{N}} C^k(\Omega)$. By $C^k_b(\Omega) \subset C^k(\Omega)$ we denote the space of functions which are bounded, together with their derivatives up to order $k$. We denote by $C^k_c(\Omega)$ the space of compactly supported elements of $C^k(\Omega)$. Unless otherwise noted, all functions will be complex-valued.

We use standard multiindex notation: for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we set
\[
x^\alpha := \prod_{j=1}^n x_j^{\alpha_j}, \quad \partial_x^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \quad D_x^\alpha := D_{x_1}^{\alpha_1} \cdots D_{x_n}^{\alpha_n}, \quad D = \frac{1}{i} \partial.
\] (2.1)
When the context is clear, we shall often simply write $D^\alpha := D_x^\alpha$, and $D_j := D_{x_j}$. We also put
\[
|\alpha| := \sum_{j=1}^n \alpha_j, \quad \alpha! := \prod_{j=1}^n \alpha_j!.
\] (2.2)
We will moreover use the *Japanese bracket*, defined for $x \in \mathbb{R}^n$ by

$$\langle x \rangle = (1 + |x|^2)^{1/2}.$$

**Definition 2.1.** The space $\mathcal{S}(\mathbb{R}^n)$ of *Schwartz functions* consists of all $\phi \in C^\infty(\mathbb{R}^n)$ such that for all $k \in \mathbb{N}_0$,

$$\|\phi\|_k := \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \phi(x)| < \infty.$$

**Example 2.2.** We have $\exp(-|x|^2) \in \mathcal{S}(\mathbb{R}^n)$. Moreover, we have a (continuous) inclusion $C^\infty_c(\mathbb{R}^n) \hookrightarrow \mathcal{S}(\mathbb{R}^n)$ with dense support; indeed, when $K \subset U \subset \mathbb{R}^n$ with $K$ compact and $U$ open and bounded, there exists $\phi \in C^\infty_c(\mathbb{R}^n)$ with $\phi \equiv 1$ on $K$ and $\phi \equiv 0$ on $\mathbb{R}^n \setminus U$.

Equipped with the countably many seminorms $\| \cdot \|_k$, $\mathcal{S}(\mathbb{R}^n)$ is a Fréchet space. Directly from the definition, we have continuous maps

$$x_j : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \quad (\phi \mapsto x_j \phi),
$$

$$D_j : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \quad (\phi \mapsto D_j \phi).$$

Given $a \in C^\infty_b(\mathbb{R}^n)$, pointwise multiplication by $a$ is also continuous. Furthermore, integration is a continuous map

$$\int : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}.$$  

(2.4)

Indeed, this follows from

$$\left| \int_{\mathbb{R}^n} \phi(x) \, dx \right| = \left| \int_{\mathbb{R}^n} \langle x \rangle^{-n-1} \langle x \rangle^{n+1} \phi(x) \, dx \right| \leq C_n \|\phi\|_{n+1}.$$  

(2.5)

Other useful operations are the pointwise product

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (\phi, \psi) \mapsto \phi \psi \in \mathcal{S}(\mathbb{R}^n), \quad (\phi \psi)(x) = \phi(x) \psi(x),$$

(2.6)

and the exterior product

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (\phi, \psi) \mapsto \phi \otimes \psi \in \mathcal{S}(\mathbb{R}^{2n}), \quad (\phi \otimes \psi)(x, y) = \phi(x) \psi(y).$$

(2.7)

**Definition 2.3.** The space $\mathcal{S}'(\mathbb{R}^n)$ of *tempered distributions* is the space of all continuous linear functionals $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$, equipped with the weak topology: the seminorms are $|u|_{\phi} := |\langle u, \phi \rangle|$ for $\phi \in \mathcal{S}(\mathbb{R}^n)$. We shall usually write $\langle u, \phi \rangle := u(\phi)$.

**Example 2.4.** The $\delta$-distribution is defined by $\langle \delta, \phi \rangle := \phi(0)$. We have $\delta \in \mathcal{S}'(\mathbb{R}^n)$ since $|\langle \delta, \phi \rangle| \leq \|\phi\|_0$.

Combining (2.4) and (2.6), we can define a continuous map

$$\mathcal{S}(\mathbb{R}^n) \ni \phi \mapsto T_\phi \in \mathcal{S}'(\mathbb{R}^n), \quad T_\phi(\psi) = \int_{\mathbb{R}^n} \phi(x) \psi(x) \, dx.$$  

(2.8)

**Proposition 2.5.** The map $\phi \mapsto T_\phi$ is injective, and has dense range in the weak topology.

**Proof.** Regarding injectivity: $T_\phi(\phi) = \int_{\mathbb{R}^n} |\phi(x)|^2 \, dx = 0$ implies $\phi = 0$. To prove the density, it suffices to show that, given $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\phi_1, \ldots, \phi_N \in \mathcal{S}(\mathbb{R}^n)$ as well as any $\epsilon > 0$, there exists $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $|\langle u, \phi_j \rangle - \langle \phi, \phi_j \rangle| < \epsilon$ for all $j = 1, \ldots, N$. Assuming, as one may, that the $\phi_j$ are orthonormal with respect to the $L^2(\mathbb{R}^n)$ inner product, this holds (with ‘$< \epsilon$’ replaced by ‘$= 0$’) for $\phi = \sum_{j=1}^N \langle u, \phi_j \rangle \phi_j$. \qed
This allows us to extend the maps (2.3) by continuity and duality to $\mathcal{S}'(\mathbb{R}^n)$: indeed, for $u, \phi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\langle x_j u, \phi \rangle = \langle u, x_j \phi \rangle, \quad \langle D_j u, \phi \rangle = \langle u, -D_j \phi \rangle,$$

and the right hand sides now make sense also for $u \in \mathcal{S}'(\mathbb{R}^n)$. Similarly, by duality and starting from (2.7), pointwise multiplication by a Schwartz function extends to a continuous map on $\mathcal{S}'(\mathbb{R}^n)$; more generally, this is true for multiplication by a function in $\mathcal{C}^\infty_0(\mathbb{R}^n)$.

Other notions, which will be significantly refined later, are:

**Definition 2.6.** Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then the support, supp $u$, is the complement of the set of $x \in \mathbb{R}^n$ such that there exists $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$, $\chi(x) \neq 1$, such that $\chi u = 0$.

The singular support, sing supp $u$, is the complement of the set of $x \in \mathbb{R}^n$ such that there exists $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$, $\chi(x) \neq 1$, such that $\chi u$ is smooth, i.e. $\chi u = T_\phi$, $\phi \in \mathcal{S}(\mathbb{R}^n)$.

2.1. **Fourier transform and its inverse.** We define the Fourier transform of $\phi \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}\phi(\xi) = \hat{\phi}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(x) \, dx, \quad \xi \in \mathbb{R}^n,$$  \hspace{2cm} (2.9)

and the inverse Fourier transform of $\psi \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}^{-1}\psi(\xi) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi(\xi) \, d\xi, \quad x \in \mathbb{R}^n.$$  \hspace{2cm} (2.10)

As in (2.5), one finds $\|\mathcal{F}\phi\|_0 \leq C_N\|\phi\|_{n+1}$ and $\|\mathcal{F}^{-1}\phi\|_0 \leq C_N\|\phi\|_{n+1}$. Moreover, we have

$$\mathcal{F}(D_j \phi) = \xi_j \mathcal{F}\phi, \quad \mathcal{F}(x_j \phi) = -D_j \mathcal{F}\phi,$$  \hspace{2cm} (2.11)

using integration by parts for the first and third statement; reading these from right to left shows that

$$\|\mathcal{F}\phi\|_k \leq C_N\|\phi\|_{k+n+1} \quad \forall \ k \in \mathbb{N}_0,$$  \hspace{2cm} (2.12)

hence the (inverse) Fourier transform preserves the Schwartz space:

$$\mathcal{F}, \mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$  \hspace{2cm} (2.13)

Note then that for $u, \psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\langle \mathcal{F}u, \psi \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) \, dx \psi(\xi) \, d\xi = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) \psi(\xi) \, dx \, d\xi = \langle u, \mathcal{F}\psi \rangle.$$  \hspace{2cm} (2.14)

This allows us to extend $\mathcal{F}, \mathcal{F}^{-1}$ to maps on tempered distributions,

$$\mathcal{F}, \mathcal{F}^{-1} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n),$$  \hspace{2cm} (2.15)

and the formulas (2.11) remain valid for $\phi \in \mathcal{S}'(\mathbb{R}^n)$.

**Example 2.7.** The Fourier transform of $\delta$ is calculated by $\langle \mathcal{F}\delta, \psi \rangle = \langle \delta, \mathcal{F}\psi \rangle = \hat{\psi}(0) = \int_{\mathbb{R}^n} \psi(x) \, dx$, so $\mathcal{F}\delta = 1$.

We recall the proof that $\mathcal{F}$ and $\mathcal{F}^{-1}$ are indeed inverses to each other.

**Theorem 2.8.** We have $\mathcal{F} \circ \mathcal{F}^{-1} = \mathcal{F}^{-1} \circ \mathcal{F} = I$ on $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$. 
Proof. By (2.11), we have $F^{-1}FD_{x_j} = F^{-1}F = D_{x_j}F^{-1}F$ and $F^{-1}Fx_j = F^{-1}(-D_{x_j})F = x_jF^{-1}F$, i.e. the map $A := F^{-1}F: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ commutes with differentiation along and multiplication by coordinates. Given $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $x_0 \in \mathbb{R}^n$, we can write

$$\phi(x) = \phi(x_0) + \sum_{j=1}^{n} \phi_j(x)(x_j - (x_0)_j), \quad \phi_j(x) = \int_0^1 (\partial_j \phi)(x_0 + t(x - x_0)) \, dt. \quad (2.16)$$

The fact that $\phi_j$ is in general not Schwartz is remedied by fixing a cutoff $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, identically 1 near $x_0$, and writing $\phi(x) = \chi(x)\phi(x) + (1 - \chi(x))\phi(x)$, so

$$\tilde{\phi}_j(x) = \chi(x)\phi_j(x) + \frac{(1 - \chi(x))\phi(x)}{|x - x_0|^2}(x_j - (x_0)_j). \quad (2.17)$$

Since $A$ annihilates every term in the sum, we have $(A\phi)(x_0) = \phi(x_0)(A\chi)(x_0)$; note that the constant $(A\chi)(x_0)$ here does not depend on $\phi$, and not on the cutoff $\chi$ either (since the left hand side does not involve $\chi$ at all).

The same cutoff $\chi$ can be used to evaluate $A\phi$ at points $x$ close to $x_0$; but

$$D_{x_j}(A\chi)(x) = A(D_{x_j}\chi)(x) = 0 \quad (2.18)$$

for $x \in \chi^{-1}(1)$. We conclude that $A = cI$ for some constant $c \in \mathbb{C}$. One can find $c$ by explicitly evaluating

$$\mathcal{F}(e^{-|\xi|^2})(\xi) = \pi^{n/2}e^{-|\xi|^2/4}, \quad \mathcal{F}^{-1}(e^{-|\xi|^2/4})(x) = \pi^{n/2}e^{-|x|^2}, \quad (2.19)$$

so $c = 1$ indeed. The proof that $\mathcal{F}\mathcal{F}^{-1} = I$ is completely analogous. \hfill \Box

We also recall that $\mathcal{F}$ is an isomorphism on $L^2(\mathbb{R}^n)$; this follows from the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ and the following fact:

**Proposition 2.9.** For $\phi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\|\mathcal{F}\phi\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2}\|\phi\|_{L^2(\mathbb{R}^n)}. \quad (2.20)$$

**Proof.** Analogously to (2.14), we have

$$\int (\mathcal{F}\phi)(\xi)\overline{\psi}(\xi) \, d\xi = (2\pi)^n \int \phi(x)\mathcal{F}^{-1}\overline{\psi}(x) \, dx, \quad \phi, \psi \in \mathcal{S}(\mathbb{R}^n). \quad (2.21)$$

Plugging in $\psi = \mathcal{F}\phi$ proves the proposition. \hfill \Box

### 2.2. Sobolev spaces and the Schwartz representation theorem

Using the Fourier transform, we can define operators which differentiate a ‘fractional number of times’:

**Definition 2.10.** For $s \in \mathbb{R}$ (or $s \in \mathbb{C}$), we let

$$(D)^s = (1 + |D|^2)^{s/2} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n), \quad \langle D \rangle^s = \mathcal{F}^{-1}(\xi)^s \mathcal{F}. \quad (2.22)$$

This agrees for $s \in 2\mathbb{N}_0$ with the usual definition, and for $s = -2$ with the operator (1.4). What is implicitly used here is that multiplication by $(1 + |\xi|^2)^s$ is continuous on $\mathcal{S}(\mathbb{R}^n)$.
Definition 2.11. For $s \in \mathbb{R}$, the Sobolev space of order $s$ is defined by

$$H^s(\mathbb{R}^n) := \{ u \in \mathcal{S}'(\mathbb{R}^n) : \langle D \rangle^s u \in L^2(\mathbb{R}^n) \}. \quad (2.23)$$

With the norm

$$\| u \|_{H^s} := \| \langle D \rangle^s u \|_{L^2} = (2\pi)^{-n/2} \| \langle \xi \rangle^s \mathcal{F} u \|_{L^2}, \quad (2.24)$$

it is a Hilbert space.

Example 2.12. The $\delta$-distribution at $0 \in \mathbb{R}^n$ satisfies $\delta \in H^s(\mathbb{R}^n)$ for all $s < -n/2$.

Since multiplication by $\langle x \rangle^r$ is continuous on $\mathcal{S}'(\mathbb{R}^n)$ for any $r \in \mathbb{R}$, we can more generally define weighted Sobolev spaces,

$$\langle x \rangle^r H^s(\mathbb{R}^n) := \{ u \in \mathcal{S}'(\mathbb{R}^n) : \langle x \rangle^{-r} u \in H^s(\mathbb{R}^n) \}. \quad (2.25)$$

The second part of the following is (a version of) the Schwartz representation theorem:

Theorem 2.13. We have

$$\mathcal{S}(\mathbb{R}^n) = \bigcap_{s,r \in \mathbb{R}} \langle x \rangle^r H^s(\mathbb{R}^n), \quad \mathcal{S}'(\mathbb{R}^n) = \bigcup_{s,r \in \mathbb{R}} \langle x \rangle^{-r} H^s(\mathbb{R}^n). \quad (2.26)$$

Proof. See Exercises 2.2 and 2.3. \hfill \square

It easily implies (using Sobolev embedding, Exercise 2.2) that every tempered distribution is a sum of terms of the form $x^\alpha D^\beta a$, $a \in C_0^0(\mathbb{R}^n)$.

2.3. The Schwartz kernel theorem. The Schwartz kernel theorem is a philosophically useful fact, establishing a 1–1 correspondence between the ‘most general’ operators in the present context—mapping Schwartz functions to tempered distributions—and distributional integral kernels, also called Schwartz kernels. To state this, we note that any distribution $K \in \mathcal{S}'(\mathbb{R}^{n+m})$ induces a bounded linear operator $\mathcal{S}(\mathbb{R}^m) \to \mathcal{S}'(\mathbb{R}^n)$ by integration along the $\mathbb{R}^m$ factor, to wit

$$(O_K \phi)(\psi) := \langle K, \psi \otimes \phi \rangle = \int \left( \int_{\mathbb{R}^m} K(x,y) \phi(y) \, dy \right) \psi(x) \, dx, \quad \phi \in \mathcal{S}(\mathbb{R}^m), \ \psi \in \mathcal{S}(\mathbb{R}^n). \quad (2.27)$$

Formally, one usually writes

$$(O_K \phi)(x) = \int_{\mathbb{R}^m} K(x,y) \phi(y) \, dy. \quad (2.28)$$

Theorem 2.14. The map $K \mapsto O_K$ is a bijection between $\mathcal{S}'(\mathbb{R}^{n+m})$ and the space of continuous linear operators $\mathcal{S}(\mathbb{R}^m) \to \mathcal{S}'(\mathbb{R}^n)$.

Proof. See Exercises 2.4 and 2.5. \hfill \square

Example 2.15. The Schwartz kernel of the identity operator $I$ on $\mathcal{S}(\mathbb{R}^n)$ is given by

$$K(x,y) = \delta(x-y), \quad x, y \in \mathbb{R}^n. \quad (2.29)$$
2.4. Differential operators. Given \( a_\alpha \in C^\infty_0(\mathbb{R}^n) \) for \( \alpha \in \mathbb{N}^n_0 \), \( |\alpha| \leq m \), we can define the \( m \)-th order differential operator

\[
A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha.
\]

(2.30)

Since multiplication by \( a_\alpha \) is continuous on \( \mathcal{S}(\mathbb{R}^n) \), \( A \) defines a continuous linear operator on \( \mathcal{S}(\mathbb{R}^n) \). By duality, \( A \) extends (uniquely) to a continuous linear operator on \( \mathcal{S}'(\mathbb{R}^n) \).

**Definition 2.16.** By \( \text{Diff}^m(\mathbb{R}^n) \), we denote the space of all operators \( A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \) of the form (2.30).

Given \( A \) as in (2.30), let us define the full symbol of \( A \) to be

\[
\sigma(A)(x, \xi) := \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha.
\]

(2.31)

Then, in view of (2.11), we can write

\[
(Au)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x, \xi) \hat{u}(\xi) \, d\xi = (2\pi)^{-n} \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} a(x, \xi) u(y) \, dy \right) \, d\xi,
\]

which we read as an iterated integral. On the other hand, the Schwartz kernel \( K \) of \( A \) is easily verified to be

\[
K(x, y) = \sum_{|\alpha| \leq m} a_\alpha(x) (D^\alpha \delta)(x - y),
\]

(2.33)

so (formally) we have

\[
K(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} a(x, \xi) \, d\xi,
\]

(2.34)

which is indeed (rigorously) correct if one reads this as the Fourier transform of \( a \) in \( \xi \).

**Proposition 2.17.** Let \( A \in \text{Diff}^m(\mathbb{R}^n) \). Then \( A \) is local, that is,

\[
\text{supp } Au \subset \text{supp } u, \quad u \in \mathcal{S}'(\mathbb{R}^n),
\]

(2.35)

and \( A \) is pseudolocal, that is,

\[
\text{sing supp } Au \subset \text{sing supp } u, \quad u \in \mathcal{S}'(\mathbb{R}^n).
\]

(2.36)

The proof is straightforward. From the perspective of the Schwartz kernel \( K \) of \( A \), (2.35) is really due to the fact that \( K(x, y) \) is supported on the diagonal \( x = y \), while (2.36) is really due to the fact that \( K(x, y) \) is smooth away from \( x = y \). (That is, adding to \( K \) an element of \( \mathcal{S}(\mathbb{R}^{2n}) \) preserves (2.36), but destroys (2.35).) Since as microlocal analysts we are interested in singularities, it is the property (2.36) which we care about most; and this will persist when \( A \) is a pseudodifferential operator. On the other hand, the only continuous linear operators \( A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \) satisfying condition (2.35) are differential operators, see Exercise 2.6.

We mention three features of differential operators concerning their principal symbol.

**Definition 2.18.** Given \( m \in \mathbb{N}_0 \) and a differential operator \( A = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha \), its principal symbol is defined as

\[
\sigma_m(A)(x, \xi) := \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha.
\]

(2.37)
i.e. keeping only the top order terms.

**Proposition 2.19.** Let \( A \in \text{Diff}^m(\mathbb{R}^n) \).

1. Define the adjoint \( A^* \) of \( A \) by \( \int_{\mathbb{R}^n} (A^*u)(x)\overline{v(x)} \, dx = \int_{\mathbb{R}^n} u(x)(\overline{Av}(x)) \, dx \), \( u, v \in C^\infty_c(\mathbb{R}^n) \). Then \( A^* \in \text{Diff}^m(\mathbb{R}^n) \), and the principal symbol is
   \[
   \sigma_m(A^*)(x, \xi) = \overline{\sigma_m(A)(x, \xi)}. \tag{2.38}
   \]

2. Let \( B \in \text{Diff}^{m'}(\mathbb{R}^n) \). Then \( A \circ B \in \text{Diff}^{m+m'}(\mathbb{R}^n) \), and
   \[
   \sigma_{m+m'}(A \circ B)(x, \xi) = \sigma_m(A)(x, \xi)\sigma_{m'}(B)(x, \xi). \tag{2.39}
   \]

3. Let \( \kappa : \mathbb{R}^n \to \mathbb{R}^n \) be a diffeomorphism which is the identity outside of a compact set. Define \( A_\kappa : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n) \) by \( (A_\kappa u)(y) := (A(u \circ \kappa^{-1}))(\kappa(y)) \). Then \( A_\kappa \in \text{Diff}^m(\mathbb{R}^n) \), and the principal symbols are related via
   \[
   \sigma_m(A_\kappa)(y, \eta) = \sigma_m(A)(\kappa(y), (\kappa'(y)^T)^{-1}\eta). \tag{2.40}
   \]

**Proof.** Exercise 2.8. \( \square \)

Thus, the principal symbol is well-defined as a function on \( T^*\mathbb{R}^n \), and is a homomorphism from the (non-commutative) algebra \( \text{Diff}(\mathbb{R}^n) = \bigcup_{m \in \mathbb{N}_0} \text{Diff}^m(\mathbb{R}^n) \) into the commutative algebra of functions \( a(x, \xi) \) which are homogeneous polynomials in \( \xi \) with coefficients in \( C^\infty_b(\mathbb{R}^n) \).

### 2.5. Exercises.

**Exercise 2.1.**
1. Show that \( \Delta + 1 : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \) is an isomorphism.
2. Find a non-trivial solution \( u \in C^\infty(\mathbb{R}^n) \) of \( (\Delta + 1)u = 0 \). Why does this not contradict the first part?

**Exercise 2.2.** (Sobolev embedding.) Let \( s > n/2 \).

1. Prove that there exists a constant \( C_s < \infty \) such that for \( \phi \in \mathcal{S}(\mathbb{R}^n) \), the estimate
   \[
   \|\phi\|_{L^\infty(\mathbb{R}^n)} \leq C_s\|\phi\|_{H^s(\mathbb{R}^n)}. \tag{2.41}
   \]
   holds. (Hint. Pass to the Fourier transform.) Deduce that \( H^s(\mathbb{R}^n) \subset C^0_b(\mathbb{R}^n) \).
2. Show more generally that \( H^s(\mathbb{R}^n) \subset C^k_b(\mathbb{R}^n) \) for \( s > n/2 + k \).
3. Prove the first equality in Theorem 2.13.

**Exercise 2.3.** (Schwartz representation theorem.) Prove the second equality in Theorem 2.13 as follows.

1. Given \( u \in \mathcal{S}'(\mathbb{R}^n) \), there exist \( C, k \) such that \( |u(\phi)| \leq C\|\phi\|_k \).
2. Let \( R_q = (x)^{-q}(D)^{-q} \). Then \( R_q \) is an isomorphism on \( \mathcal{S}(\mathbb{R}^n) \) and \( \mathcal{S}'(\mathbb{R}^n) \). Moreover, for sufficiently large \( q \), we have \( \|R_q\phi\|_k \leq C\|\phi\|_{L^2(\mathbb{R}^n)} \) (for some other constant \( C \)). (Hint. Use the previous exercise. It may be convenient to take \( s \) there and \( q \) here to be even integers.)
3. Denoting \( R_q^\dagger = (D)^{-q}(x)^{-q} \), deduce that \( R_q^\dagger u \in L^2(\mathbb{R}^n) \), and conclude that \( u \in \langle x\rangle^q H^{-q}(\mathbb{R}^n) \).
Exercise 2.4. (Schwartz kernel theorem I.) Prove the injectivity claim of Theorem 2.14. (Hint. Let \( K \in \mathcal{S}'(\mathbb{R}^{n+m}) \) be given with \( O_K = 0 \). Given \( \phi \in \mathcal{S}(\mathbb{R}^{n+m}) \), you need to show that \( \langle K, \phi \rangle = 0 \). You know that this is true when \( \phi \) is a finite linear combination of exterior products \( \psi_1 \otimes \psi_2, \psi_1 \in \mathcal{S}(\mathbb{R}^n), \psi_2 \in \mathcal{S}(\mathbb{R}^m) \). Try to use the Fourier transform, or Fourier series, to approximate \( \phi \) by such linear combinations. It may help to first reduce to the case that \( \text{supp} \, K \) is compact.)

Exercise 2.5. (Schwartz kernel theorem II.) Let \( A: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^m) \) be continuous. Prove the surjectivity claim of Theorem 2.14 as follows.

(1) The continuity of \( A \) is equivalent to the statement that for all \( \psi \in \mathcal{S}(\mathbb{R}^n) \) there exists \( N > 1 \) such that \( |\langle A\phi, \psi \rangle| \leq N\|\phi\|_N \) for all \( \phi \in \mathcal{S}(\mathbb{R}^m) \).
(2) There exist \( N, M \in \mathbb{R} \) such that \( A \) extends by continuity to a bounded operator
\[
A: \langle x \rangle^{-M}H^M(\mathbb{R}^m) \to \langle x \rangle^N H^{-N}(\mathbb{R}^n).
\]
(3) The operator
\[
A' := \langle D \rangle^{-N-n/2-1} \langle x \rangle^{-N} A \langle D \rangle^{-M-m/2-1} \langle x \rangle^{-M}
\]
is bounded from \( H^{-m/2-1}(\mathbb{R}^m) \) to \( C^0_b(\mathbb{R}^n) \).
(4) Evaluate \( A'\delta_y \) for \( y \in \mathbb{R}^m \) and deduce that \( A' \) has a Schwartz kernel \( K' \in C^0_b(\mathbb{R}^{n+m}) \).
(5) By relating the Schwartz kernels of \( A' \) and \( A \), prove that \( A = O_K \) for some \( K \in \mathcal{S}'(\mathbb{R}^{n+m}) \).

Exercise 2.6. Let \( A: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \) be a continuous linear operator, and suppose for all \( u \in \mathcal{S}(\mathbb{R}^n) \), we have \( \text{supp} \, Au \subset \text{supp} \, u \). Prove that \( A \) is a differential operator. (Hint. Show that the Schwartz kernel \( K \) of \( A \) has support in the diagonal \( \{x = y\} \). Then show/recall that distributions with support on a submanifold \( S \) are locally finite linear combinations of (differentiated) \( \delta \)-distributions (with coefficients in \( C^\infty(S) \)) at \( S \). Cf. (2.33). To prove that \( A \) is a differential operator of finite order, exploit that \( K \) is a tempered distribution.)

Exercise 2.7. Show that the principal symbol \( \sigma_m(A) \) of \( A \in \text{Diff}^m(\mathbb{R}^n) \) captures the ‘high frequency behavior’ of \( A \) in the following sense: for \( x_0, \xi_0 \in \mathbb{R}^n \), we have
\[
\sigma_m(A)(x_0, \xi_0) = \lim_{\lambda \to \infty} \lambda^{-m} e^{-i\lambda \xi_0 \cdot x} A e^{i\lambda \xi_0 \cdot x}(x_0),
\]
where \( e^{i\xi_0 \cdot x} \) is the function \( x \mapsto e^{i\xi_0 \cdot x} \).


3. Symbols

As a first step towards the definition of pseudodifferential operators, we generalize the class of symbols \( a(x, \xi) \) from polynomials in \( \xi \) to more general functions:

**Definition 3.1.** Let \( m \in \mathbb{R}, n, N \in \mathbb{N} \). Then the space of \( (\text{uniform}) \) symbols of order \( m \)
\[
S^m(\mathbb{R}^n; \mathbb{R}^N) \subset C^\infty(\mathbb{R}^n \times \mathbb{R}^N)
\]
consists of all functions \( a = a(x, \xi) \) which for all \( \alpha \in \mathbb{N}_0^n, \beta \in \mathbb{N}_0^N \) satisfy the estimate
\[
|\partial_\xi^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha \beta} \langle \xi \rangle^{m-|\beta|}.
\]
for some constants \( C_{\alpha \beta} \). We also write
\[
S^m(\mathbb{R}^N) := S^m(\mathbb{R}^0; \mathbb{R}^N).
\]
for symbols only depending on the symbolic variable $\xi$.

The gain of decay upon differentiation in the $\xi$-variables is often called *symbolic behavior* (in $\xi$).

**Remark 3.2.** Sometimes these symbol classes are denoted $S^m_\infty(\mathbb{R}^n; \mathbb{R}^N)$, the subscript ‘$\infty$’ indicating the uniform boundedness in $C^\infty$ of the ‘coefficients’, i.e. the $x$-variables. There exist many generalizations and variants of the class $S^m(\mathbb{R}^n; \mathbb{R}^N)$, such as: symbols of type $\rho, \delta$; symbols which in addition have symbolic behavior in $x$ (these are symbols of scattering (pseudo)differential operators); or symbols with joint symbolic behavior in $(x, \xi)$ (symbols of isotropic operators). See [Mel07, §4] and [Hör71, §1.1].

Equipped with the norms given by the best constants in (3.2), or more concisely,

$$
\|a\|_{m,k} := \sup_{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^N} \max_{|\alpha|+|\beta| \leq k} \langle \xi \rangle^{-m+|\beta|}\langle \partial_x^\alpha \partial_\xi^\beta a(x,\xi) \rangle,
$$

the space $S^m(\mathbb{R}^n; \mathbb{R}^N)$ is a Fréchet space. Directly from the definition, we note that differentiations

$$
D^\alpha_x : S^m(\mathbb{R}^n; \mathbb{R}^N) \rightarrow S^m(\mathbb{R}^n; \mathbb{R}^N),
$$

$$
D^\beta_\xi : S^m(\mathbb{R}^n; \mathbb{R}^N) \rightarrow S^{m-|\beta|}(\mathbb{R}^n; \mathbb{R}^N)
$$

(3.5)

are continuous.

**Example 3.3.** Full symbols of differential operators of order $m$ on $\mathbb{R}^n$, see (2.31), lie in $S^m(\mathbb{R}^n; \mathbb{R}^N)$. A special case of this is: given $a \in C^\infty_0(\mathbb{R}^n)$, the function $(x, \xi) \mapsto a(x)$ lies in $S^0(\mathbb{R}^n; \mathbb{R}^N)$ (for any $N$).

**Example 3.4.** Let $m \in \mathbb{R}$. Then $\langle \xi \rangle^m \in S^m(\mathbb{R}^n; \mathbb{R}^n)$. (See Exercise 3.1.)

**Proposition 3.5.** **Pointwise multiplication of symbols is a continuous bilinear map**

$$
S^m(\mathbb{R}^n; \mathbb{R}^N) \times S^{m'}(\mathbb{R}^n; \mathbb{R}^N) \rightarrow S^{m+m'}(\mathbb{R}^n; \mathbb{R}^N).
$$

(3.6)

**Proof.** This follows from the Leibniz rule: for $a \in S^m(\mathbb{R}^n; \mathbb{R}^N)$, $b \in S^{m'}(\mathbb{R}^n; \mathbb{R}^N)$, and $\alpha \in \mathbb{N}_0^n$, $\beta \in \mathbb{N}_0^N$, we have

$$
|\partial_\xi^\alpha \partial_x^\beta (a \cdot b)| = \sum_{\alpha' + \alpha'' = \alpha, \beta' + \beta'' = \beta} C_{\alpha' \beta' \alpha'' \beta''} \langle \xi \rangle^{m+m' - |\beta'|-|\beta''|} \leq C_{\alpha \beta} \langle \xi \rangle^{m+m' - |\beta|}.
$$

We note the trivial continuous inclusion

$$
m \leq m' \implies S^m(\mathbb{R}^n; \mathbb{R}^N) \subseteq S^{m'}(\mathbb{R}^n; \mathbb{R}^N),
$$

(3.7)

hence the $S^m(\mathbb{R}^n; \mathbb{R}^N)$ give a filtration of the space of all symbols $\bigcup_{m \in \mathbb{R}} S^m(\mathbb{R}^n; \mathbb{R}^N)$. In the other direction, we define the space of residual symbols by

$$
S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N) := \bigcap_{m \in \mathbb{R}} S^m(\mathbb{R}^n; \mathbb{R}^N).
$$

(3.8)

Equipped with the norms $\| \cdot \|_{m,k}$, $m, k \in \mathbb{N}$, this is again a Fréchet space.
Example 3.6. We have \( \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^N) \subset S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N) \). Moreover, given a cutoff \( \chi \in \mathcal{C}_c^\infty(\mathbb{R}^N) \), the pullback of \((x, \xi) \mapsto \chi(\xi)\) to \(\mathbb{R}^n \times \mathbb{R}^N\) defines a residual symbol.

While the inclusion (3.7) never has dense range for \( m < m' \), there is a satisfying replacement:

**Proposition 3.7.** Let \( m < m' \). Then \( S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N) \) is a dense subspace of \( S^m(\mathbb{R}^n; \mathbb{R}^N) \) in the topology of \( S^{m'}(\mathbb{R}^n; \mathbb{R}^N) \). More precisely, for any \( a \in S^m(\mathbb{R}^n; \mathbb{R}^N) \) there exists a sequence \( a_j \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N) \) which is uniformly bounded in \( S^m(\mathbb{R}^n; \mathbb{R}^N) \) and converges to \( a \) in the topology of \( S^{m'}(\mathbb{R}^n; \mathbb{R}^N) \).

**Proof.** Fix a cutoff function \( \chi \in \mathcal{C}_c^\infty(\mathbb{R}^N) \subset S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N) \) (see Example 3.6) which is identically 1 in \( |\xi| \leq 1 \) and identically 0 when \( |\xi| \geq 2 \). By Proposition 3.5, we have

\[
a_j(x, \xi) := a(x, \xi) \chi(\xi/j) \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N). \tag{3.9}
\]

To prove the proposition, it suffices to show, in view of Proposition 3.5, that

\[
\chi_j(\xi) := \chi(\xi/j) \tag{3.10}
\]

is bounded in \( S^0(\mathbb{R}^N) \) and converges to 1 in the topology of \( S^\epsilon(\mathbb{R}^N) \) for all \( \epsilon > 0 \). Regarding the former, we have \( |\chi_j(\xi)| \leq \|\chi\|_{0,0} \) for all \( j \), while for \( |\beta| \geq 1 \) we have \( \partial_\xi^\beta \chi_j(\xi) \equiv 0 \) for \( |\xi| \leq 1 \), and

\[
|\xi|^{|\beta|} \partial_\xi^\beta \chi_j(\xi) = \chi_\beta(\xi/j), \quad \chi_\beta(\xi) = [\xi]|^{|\beta|} (\partial_\xi^\beta \chi)(\xi) \in \mathcal{C}_c^\infty(\mathbb{R}^N). \tag{3.11}
\]

Regarding the latter, we note that \( \text{supp}(\chi_j - 1) \subset \{ j \leq |\xi| \leq 2j \} \), hence

\[
|\chi(\xi/j) - 1| \leq j^{-\epsilon} |\xi|^\epsilon. \tag{3.12}
\]

For derivatives, we note that the support observation and (3.11) give

\[
|\xi|^{|\beta| - \epsilon} |\partial_\xi^\beta \chi_j(\xi)| \leq j^{-\epsilon} |\chi_\beta(\xi/j)|. \tag{3.13}
\]

Thus, \( \|\chi_j\|_{\epsilon,k} \leq C_{k\epsilon} j^{-\epsilon} \rightarrow 0, j \rightarrow \infty \), as desired. \( \square \)

### 3.1. Ellipticity

We now generalize the key property of the symbol of the operator \( L = \Delta + 1 \) in (1.3).

**Definition 3.8.** Let \( m \in \mathbb{R} \). A symbol \( a \in S^m(\mathbb{R}^n; \mathbb{R}^N) \) is (uniformly) elliptic if there exists a symbol \( b \in S^{-m}(\mathbb{R}^n; \mathbb{R}^N) \) such that \( ab - 1 \in S^{-1}(\mathbb{R}^n; \mathbb{R}^N) \).

**Proposition 3.9.** Let \( m \in \mathbb{R} \), and \( a \in S^m(\mathbb{R}^n; \mathbb{R}^N) \). Then the following are equivalent:

1. \( a \) is elliptic.
2. There exist constants \( C, c > 0 \) such that
   \[
   |\xi| \geq C \implies |a(x, \xi)| \geq c|\xi|^m. \tag{3.14}
   \]
3. There exist constants \( C, c > 0 \) such that
   \[
   |a(x, \xi)| \geq c|\xi|^m - C|\xi|^{m-1}, \quad |\xi| \geq 1. \tag{3.15}
   \]
If $a$ is elliptic, then in the notation of Definition 3.8, we have
\[ 1 - C(\xi)^{-1} \leq |a(x,\xi)||b(x,\xi)| \leq C|a(x,\xi)|\langle\xi\rangle^{-m}, \tag{3.16} \]
for some constant $C > 0$, that is,
\[ |a(x,\xi)| \geq c(\xi)^m - \langle\xi\rangle^{m-1}. \tag{3.17} \]
This proves (3.15). This in turn implies (3.14) since for all $c > 0$, there exists $C > 0$ such that $|\xi|^{m-1} \leq c|\xi|^m$ for $|\xi| \geq C$ (indeed, this holds for $C = c^{-1}$).

Conversely, if (3.14) holds, choose a cutoff $\chi \in C^\infty(\mathbb{R}^n)$, $\chi(\xi) = 0$ for $|\xi| \leq 2C$, $\chi(\xi) = 1$ for $|\xi| \geq 3C$, then (see Exercise 3.2)
\[ b(x,\xi) := \chi(\xi)/a(x,\xi) \in S^{-m}(\mathbb{R}^n;\mathbb{R}^N), \tag{3.18} \]
and $a(x,\xi)b(x,\xi) = \chi(\xi) \in S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N)$. \hfill \square

Note that if $a \in S^m(\mathbb{R}^n;\mathbb{R}^N)$ is elliptic, then so is $a + a'$ for any $a' \in S^{m-1}(\mathbb{R}^n;\mathbb{R}^N)$. Thus, ellipticity is only a condition on the equivalence class
\[ [a] \in S^m(\mathbb{R}^n;\mathbb{R}^N)/S^{m-1}(\mathbb{R}^n;\mathbb{R}^N). \tag{3.19} \]
For full symbols of differential operators, we can identify $[a]$ with the leading order, homogeneous of degree $m$, part of $a$. Compare with Definition 2.18 and Proposition 2.19.

### 3.2. Classical symbols.

An important subclass of symbols mimics those of differential operators: they are sums of homogeneous (in $\xi$) functions. More precisely, we call a function $a(x,\xi)$, defined for $\xi \neq 0$, (positively) homogeneous of order $m \in \mathbb{C}$ iff
\[ a(x,\lambda\xi) = \lambda^m a(x,\xi), \quad \lambda > 0. \tag{3.20} \]

**Definition 3.10.** Let $m \in \mathbb{C}$. Then $S^m_{\text{hom}}(\mathbb{R}^n;\mathbb{R}^N \setminus \{0\})$ is the space of all functions $a(x,\xi) \in C^\infty(\mathbb{R}^n \times (\mathbb{R}^N \setminus \{0\}))$, positively homogeneous of order $m$ in $\xi$, such that for all $\alpha, \beta \in \mathbb{N}_0^n$
\[ |\partial_\alpha^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta} |\xi|^{\text{Re} m - |\beta|}, \quad \xi \neq 0. \tag{3.21} \]

**Definition 3.11.** Let $m \in \mathbb{C}$, and fix a cutoff $\chi \in C^\infty(\mathbb{R}^N)$ which is identically 1 near 0. A symbol $a \in S^{\text{Re} m}(\mathbb{R}^n;\mathbb{R}^N)$ is called a classical symbol of order $m$ if there exist functions $a_{m-j} \in S^{m-j}_{\text{hom}}(\mathbb{R}^n \times (\mathbb{R}^N \setminus \{0\}))$ such that for all $J \in \mathbb{N}$, we have
\[ a - \sum_{j=0}^{J-1} (1 - \chi)a_{m-j} \in S^{\text{Re} m-J}(\mathbb{R}^n;\mathbb{R}^N). \tag{3.22} \]

The space of classical symbols of order $m$ is denoted $S^m_{\text{cl}}(\mathbb{R}^n;\mathbb{R}^N)$. Finally, we put
\[ S^{-\infty}_{\text{cl}}(\mathbb{R}^n;\mathbb{R}^N) := S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N). \tag{3.23} \]

Equipped with the seminorms of $a_{m-j}$ and the remainders $a - \sum_{j=0}^{J-1} (1 - \chi)a_{m-j}$ in the respective spaces, $S^m_{\text{cl}}(\mathbb{R}^n;\mathbb{R}^N)$ is a Fréchet space. Proposition 3.7 fails dramatically for classical symbols; indeed (Exercise 3.4),
\[ S^{-\infty}(\mathbb{R}^n;\mathbb{R}^N) \subset S^m_{\text{cl}}(\mathbb{R}^n;\mathbb{R}^N) \] is closed for any $m \in \mathbb{C}. \tag{3.24}

We have the following straightforward lemma (Exercise 3.5):

**Lemma 3.12.** The homogeneous terms $a_{m-j}$ in (3.22) are uniquely determined by $a$. 

For $a \in S^m_{\text{cl}}(\mathbb{R}^n; \mathbb{R}^N)$ as in Definition 3.11, we can thus identify the equivalence class $[a] \in S_{\text{Rem}}^m(\mathbb{R}^n; \mathbb{R}^N)/S_{\text{Rem}}^{m-1}(\mathbb{R}^n; \mathbb{R}^N)$ with the leading order homogeneous part $a_m$, or even more simply with the function $\mathbb{R}^n \times S^{N-1} \ni (x, \xi) \mapsto a_m(x, \xi)$, where $S^{N-1} = \{ \xi \in \mathbb{R}^N : |\xi| = 1 \}$ is the unit sphere. Cf. (2.37).

3.3. Asymptotic summation. There is a (general) ‘converse’ to (3.22) which is very useful when performing iterative constructions which yield lower order corrections:

**Proposition 3.13.** Let $a_j \in S^{m_j}(\mathbb{R}^n; \mathbb{R}^N)$, $j \geq 0$, and suppose $\limsup_{j \to \infty} m_j = -\infty$. Let $\bar{m}_j := \sup_{j \geq j'} m_{j'}$, and $m = \bar{m}_0$. Then there exists a symbol $a \in S^m(\mathbb{R}^n; \mathbb{R}^N)$ such that for all $J \in \mathbb{N}$

$$a - \sum_{j=0}^{J-1} a_j \in S^{m_J}(\mathbb{R}^n; \mathbb{R}^N).$$

(3.25)

Moreover, $a$ is unique modulo $S^{-\infty}(\mathbb{R}^n; \mathbb{R}^N)$.

We call $a$ ‘the’ asymptotic sum of the $a_j$, and write

$$a \sim \sum_{j=0}^{\infty} a_j.$$  

(3.26)

**Proof of Proposition 3.13.** This is similar to Borel’s theorem about the existence of a smooth function with prescribed Taylor series at 0. Uniqueness is clear, since any two asymptotic sums $a, a'$ satisfy $a - a' \in S^{m_J}(\mathbb{R}^n; \mathbb{R}^N)$, with $\bar{m}_J \to -\infty$, hence $a - a'$ is residual indeed.

For existence, we may partially sum finitely many of the $a_j$ and thereby reduce to the case that $a_j \in S^{m-j}(\mathbb{R}^n; \mathbb{R}^N)$, $j \geq 0$, and $\bar{m}_j = m - j$. Fix a cutoff $\chi \in C^\infty(\mathbb{R}^n)$, identically 0 in $|\xi| \leq 1$ and equal to 1 for $|\xi| \geq 2$. With $\epsilon_j > 0$, $\epsilon_j \to 0$, to be determined, we wish to set

$$a(x, \xi) := \sum_{j=0}^{\infty} \chi(\epsilon_j \xi) a_j(x, \xi).$$

(3.27)

This sum is locally finite, hence $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^N)$. Choosing $\epsilon_j$ more precisely, we can arrange that

$$\| \chi(\epsilon_j \cdot) a_j \|_{m-j'-j'} \leq 2^{-j'}, \quad j > j' \geq 0.$$  

(3.28)

Indeed, for fixed $j, j'$, we can choose $\epsilon_j > 0$ such that this holds since $(1 - \chi(\epsilon_j \cdot)) a_j \to 0$ in $S^{m-j}(\mathbb{R}^n; \mathbb{R}^N)$ as $\epsilon_j \to 0$, as in the proof of Proposition 3.7; but for any fixed $j$, (3.28) gives a finite number of conditions on $\epsilon_j$, one for each $0 \leq j' < j$.

But then $\chi(\epsilon_j \xi) a_j(x, \xi) + \sum_{j=j'+1}^{\infty} \chi(\epsilon_j \xi) a_j(x, \xi)$ converges in $S^{m-j'}(\mathbb{R}^n; \mathbb{R}^N)$. Thus, (3.27) converges in $S^m(\mathbb{R}^n; \mathbb{R}^N)$, and we have

$$a(x, \xi) - \sum_{j=0}^{J-1} a_j(x, \xi) = \sum_{j=0}^{J-1} (1 - \chi(\epsilon_j \xi)) a_j(x, \xi) + \sum_{j=J}^{\infty} \chi(\epsilon_j \xi) a_j(x, \xi) \in S^{m-J}(\mathbb{R}^n; \mathbb{R}^N).$$

(3.29)

as desired.  

The space $S^m_{\text{cl}}(\mathbb{R}^n; \mathbb{R}^N)$ can then be characterized as the space of symbols in $S^m_{\text{Rem}}(\mathbb{R}^n; \mathbb{R}^N)$ which are asymptotic sums of symbols which in $|\xi| \geq 1$ are positively homogeneous of degree $m - j$, $j \in \mathbb{N}_0$. 

For completeness and later use, we refine the previous result to ensure the continuous dependence of $a$ on the sequence $(a_j)$.

**Proposition 3.14.** Let $\ell S^\infty(\mathbb{R}^n; \mathbb{R}^N) = \prod_{j=0}^\infty S^{m-j}(\mathbb{R}^n; \mathbb{R}^N)$ be the space of all sequences $(a_0, a_1, \ldots)$ of symbols $a_j \in S^{m-j}(\mathbb{R}^n; \mathbb{R}^N)$. Equip $\ell S^\infty$ with the topology generated by the seminorms $\|(a_j)\|_J := \max_{1 \leq k \leq J} \|a_k\|_{m-k,J}$. Then there exists a continuous (nonlinear) map

$$\sum_A : \ell S^\infty(\mathbb{R}^n; \mathbb{R}^N)$$

with the property that $\sum_A ((a_j)_{j \in \mathbb{N}_0}) \sim \sum_{j=0}^\infty a_j$.

**Remark 3.15.** The topology on $\ell S^\infty(\mathbb{R}^n; \mathbb{R}^N)$ is akin to e.g. the standard topology on $C^\infty(\mathbb{R}^n)$ which is given by seminorms $\|\cdot\|_{C^k(B(0,\kappa))}$. To verify convergence of a sequence of symbols in this topology, one merely needs to check that for any fixed $J \in \mathbb{N}$, the first $J$ terms of the sequence converge in the respective symbol spaces.

**Proof of Proposition 3.14.** Fix $\chi \in C^\infty(\mathbb{R}^N)$, $\chi(\xi) = 0$ for $|\xi| \leq 1$ and $\chi(\xi) = 1$ for $|\xi| \geq 2$. As in the previous proof, we shall set, for $a = (a_j)_{j \in \mathbb{N}_0} \in \ell S^\infty(\mathbb{R}^n; \mathbb{R}^N)$,

$$\left(\sum_A a\right)(x, \xi) := \sum_{j=0}^\infty \chi(\epsilon_j(a))a_j(x, \xi),$$

(3.31)

where $\epsilon_j(a)$, as in (3.28), is chosen so that for all $j \in \mathbb{N}$

$$\max_{0 \leq j' \leq j-1} \|\chi(\epsilon_j(a))a_j(x, \xi)\|_{m-j', j'} \leq 2^{-j},$$

(3.32)

and we set $\epsilon_0(a) = 1$. We now need to make a concrete choice of $\epsilon_j(a)$: to this effect, we note that for $|\alpha| + |\beta| \leq j' \leq j - 1$,

$$\langle \xi \rangle^{-m+j'} \partial_\alpha^{(1)} \partial_\beta \left(\chi(\epsilon_j(a))a_j(x, \xi)\right) \leq C_j \langle \xi \rangle^{-m+j'} \langle \xi \rangle^{m-j} \|a_j\|_{m-j', j'} 1_{|\xi| \geq \epsilon_j(a)-1} \leq C_j \|a_j\|_{m-j,j},$$

(3.33)

where $C_j$ only depends on $\chi$ and $j$ of course. Therefore (3.32) holds provided we take

$$\epsilon_j(a) := 2^{-j} \left(1 + C_j \|a_j\|_{m-j,j}\right)^{-1}.$$  

(3.34)

With this choice, $\sum_A a$ is well-defined, and $\sum_A a \sim \sum_{j=0}^\infty a_j$.

We now check continuity. Define

$$\chi_j(q, \xi) := \chi\left(2^{-j} (1 + C_j q)^{-1} \xi\right).$$

(3.35)

Fix $a = (a_j)_{j \in \mathbb{N}_0} \in \ell S^\infty(\mathbb{R}^n; \mathbb{R}^N)$, and fix $k \in \mathbb{N}_0$, $\epsilon > 0$. We need to show that there exist $\delta > 0$ and $J \in \mathbb{N}$ such that

$$a' \in \ell S^\infty(\mathbb{R}^n; \mathbb{R}^N), \quad \|a - a'\|_J \leq \delta \quad \Rightarrow \quad \left\|\sum_A a - \sum_A a'\right\|_{m,k} < \epsilon,$$

(3.36)

which holds provided

$$\sum_{j=0}^\infty \left\|\chi_j(\|a_j\|_{m-j,j}, \xi)a_j - \chi_j(\|a'_j\|_{m-j,j}, \xi)a'_j\right\|_{m,k} < \epsilon.$$

(3.37)
The $j$-th summand can individually be estimated by

\[
\| \chi_j(\|a_j\|_{m-j,j}, \xi)(a_j - a'_j)\|_{m,k} + \| (\chi_j(\|a_j\|_{m-j,j}, \xi) - \chi_j(\|a'_j\|_{m-j,j}, \xi))a'_j\|_{m,k} \\
\leq C_j\|a_j - a'_j\|_{m,k} + \| a_j\|_{m-j,j} - \|a'_j\|_{m-j,j} \|a'_j\|_{m,k} \\
\leq (C_j + \|a_j\|_{m,k} + \|a - a'\|_{\max(j,k)})\|a - a'\|_{\max(j,k)},
\]

which tends to zero as $a' \to a$ in $\ell S^m(\mathbb{R}^n; \mathbb{R}^N)$.

The tail of the sum (3.37) on the other hand is estimated simply using (3.32)

\[
\| \chi_j(\|a_j\|_{m-j,j}, \xi) a_j\|_{m,k} + \| \chi_j(\|a'_j\|_{m-j,j}, \xi) a'_j\|_{m,k} \leq 2^{-j} + 2^{-j} = 2^{-j+1}
\]

provided $j > k$. Thus, we first choose $J_0 \in \mathbb{N}$, $J_0 > k$, such that $\sum_{j=J_0}^{\infty} 2^{-j+1} < \epsilon/2$,
and then $\delta > 0$, $J \in \mathbb{N}$ such that $a' \in \ell S^m(\mathbb{R}^n; \mathbb{R}^N)$, $\|a - a'\|_j < \delta$ implies that the $j$-th summand in (3.37) is bounded by $\epsilon/(2J_0)$ for $j = 0, \ldots, J_0 - 1$. This achieves (3.36) and thus finishes the proof.

\[\square\]

3.4. Exercises.

Exercise 3.1. (1) Let $m \in \mathbb{R}$. Show that $(\xi)^m \in S^m(\mathbb{R}^N)$. By expanding into Taylor series in $1/|\xi|$, show that indeed $(\xi)^m \in S^m_{\cl}(\mathbb{R}^N)$.

(2) More generally, let $\mu \in \mathbb{C}$. Show that $(\xi)^\mu \in S^m_{\cl}(\mathbb{R}^N)$.

Exercise 3.2. (1) Show that if $a \in S^m(\mathbb{R}^n; \mathbb{R}^N)$ satisfies (3.14), and $\chi \in S^0(\mathbb{R}^N)$ vanishes for $|\xi| \leq 2C$, then $\chi/a \in S^{-m}(\mathbb{R}^n; \mathbb{R}^N)$.

(2) If in addition $a$ and $\chi$ are classical symbols, show that $\chi/a$ is classical as well.

Exercise 3.3. (1) Let $f \in C^\infty(\mathbb{R})$. Show that if $a \in S^0(\mathbb{R}^n; \mathbb{R}^N)$, then also $f \circ a \in S^0(\mathbb{R}^n; \mathbb{R}^N)$.

(2) Show that if $a \in S^0(\mathbb{R}^n; \mathbb{R}^N)$ is elliptic and positive, then there exists $b \in S^0(\mathbb{R}^n; \mathbb{R}^N)$ such that $a - b^2 \in S^{-1}(\mathbb{R}^n; \mathbb{R}^N)$.

Exercise 3.4. Prove (3.24).

Exercise 3.5. Prove Lemma 3.12. (Hint. Use induction on $j$; the case $j = 0$ is the main content.)

4. Pseudodifferential operators

For developing the theory of ps.d.o.s, it is useful to consider slightly more general symbols, in the class

\[
(x - y)^w S^m(\mathbb{R}^n_x \times \mathbb{R}^n_y; \mathbb{R}^\xi) = \{ (x - y)^w \hat{a}: \hat{a} \in S^m(\mathbb{R}^n_x \times \mathbb{R}^n_y; \mathbb{R}^\xi) \},
\]

where $w \in \mathbb{R}$. Our immediate goal will be to make sense of the following definition.

Definition 4.1. Let $m, w \in \mathbb{R}$, and $a \in (x - y)^w S^m(\mathbb{R}^n_x \times \mathbb{R}^n_y; \mathbb{R}^\xi)$. Then we define its quantization $\text{Op}(a)$ by

\[
(\text{Op}(a)u)(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(x, y, \xi) u(y) dy d\xi, \quad u \in \mathcal{S}(\mathbb{R}^n).
\]
Previously, see (2.32), we only considered the special case of the left quantization of a left symbol \( a \in S^m(\mathbb{R}_x^n;\mathbb{R}_s^n)\), independent of \( y \):

\[
(\text{Op}_L(a)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} a(x,\xi) u(y) \, dy \, d\xi; \tag{4.3}
\]

this immediately makes sense as an iterated integral for \( u \in \mathcal{S}(\mathbb{R}^n) \), and should be thought of as ‘differentiate first, then multiply by coefficients’. Dually, we can consider the right quantization of a right symbol \( a \in S^m(\mathbb{R}_y^n;\mathbb{R}_s^n)\),

\[
(\text{Op}_R(a)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} a(y,\xi) u(y) \, dy \, d\xi, \tag{4.4}
\]

which does not immediately make sense (similarly to (4.2)); this should be thought of as ‘multiply by coefficients, then differentiate’. (Take \( a(z,\xi) = \xi^m a_\alpha(z) \) with \( a_\alpha \in C^\infty_b(\mathbb{R}^n) \) and evaluate \( \text{Op}_L(a)u \) and \( \text{Op}_R(a)u \!)

The quantization map (4.2) should be read as ‘multiply \( y \), then differentiate \( \xi \), then multiply \( x \)’. (Try this with \( a(x,y,\xi) = a_1(x)\xi^m a_2(y) \).) We shall see below that every operator \( \text{Op}(a) \) can be written as \( \text{Op}(a) = \text{Op}_L(a_L) = \text{Op}_R(a_R) \) for suitable left and right symbols \( a_L \) and \( a_R \) of the same order as \( a \), see §4.1. (You have done most of the work for proving this for differential operators, i.e. in the case that \( a \) is a polynomial in \( \xi \), in Exercise 2.8.)

**Lemma 4.2.** Let \( w \in \mathbb{R}, \ m < -n, \) and let \( a = \langle x - y \rangle^w \hat{a}, \ \hat{a} \in S^m(\mathbb{R}^n \times \mathbb{R}^n;\mathbb{R}^n) \). Then the integral (4.2) is absolutely convergent and defines a continuous operator

\[
\text{Op}(a): \mathcal{S}(\mathbb{R}^n) \rightarrow \langle x \rangle^w C_0^0(\mathbb{R}^n). \tag{4.5}
\]

More precisely, for \( N > n + |w| \), there exists a constant \( C < \infty \) such that

\[
\| \text{Op}(a)u \|_{\langle x \rangle^w C_0^0(\mathbb{R}^n)} \leq C \| \hat{a} \|_{m,0} \| u \|_N, \quad u \in \mathcal{S}(\mathbb{R}^n). \tag{4.6}
\]

For the proof, we need a simple lemma:

**Lemma 4.3.** Let \( w \in \mathbb{R} \). Then \( \langle x + y \rangle^w \leq 2^{|w|/2} \langle x \rangle^w \langle y \rangle^{|w|} \).

**Proof.** By the triangle and Cauchy–Schwarz inequalities, we have

\[
1 + |x + y|^2 \leq 1 + 2|x|^2 + 2|y|^2 \leq 2(1 + |x|^2)(1 + |y|^2). \tag{4.7}
\]

If \( w > 0 \), then taking this to the power \( w/2 \) proves the lemma. For \( w = 0 \), the lemma is the equality \( 1 = 1 \). For \( w < 0 \), hence \(-w > 0\), we obtain, analogously to (4.7),

\[
\langle x \rangle^{-w} \leq 2^{-w/2} \langle x + y \rangle^{-w} \langle y \rangle^{-w}, \tag{4.8}
\]

which upon multiplication by \( \langle x \rangle^w \langle x + y \rangle^w \) gives the desired result. \( \square \)

**Proof of Lemma 4.2.** Since \( u \) is Schwartz, we have \( |u(y)| \leq C_N \| u \|_N \langle y \rangle^{-N} \) for all \( N \in \mathbb{N}_0 \). Therefore, the integrand in (4.2) satisfies

\[
|e^{i(x-y)\cdot \xi} a(x,y,\xi) u(y)| \leq C (x-y)^w \| \hat{a} \|_{m,0} \xi^m \cdot \| u \|_N \langle y \rangle^{-N} \leq C \langle x \rangle^w \cdot \langle \xi \rangle^m \langle y \rangle^{w-N} \cdot \| \hat{a} \|_{m,0} \| u \|_N. \tag{4.9}
\]

This is integrable in \( (y,\xi) \) provided \( m < -n \) and \( |w| - N < -n \), proving the lemma. \( \square \)
Proposition 4.4. Let $w \in \mathbb{R}$ and $a = \langle x - y \rangle^w \tilde{a}$, $\tilde{a} \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. Then the quantization $\text{Op}(a) : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ is continuous. In fact, for all $k \in \mathbb{N}_0$, $m \in \mathbb{R}$, there exists $N \in \mathbb{N}$ and a constant $C$ such that

$$\| \text{Op}(a)u \|_k \leq C \| \tilde{a} \|_{m,N} \| u \|_N.$$ (4.10)

Lemma 4.5. Differentiations $D_x^a$ and $D_y^a$ are continuous maps $\langle x - y \rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \rightarrow \langle x - y \rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. More precisely,

$$\| \langle x - y \rangle^{-w} D_x^a a \|_{m,k} \leq C \| \langle x - y \rangle^{-w} a \|_{m,k+|\alpha|},$$ (4.11)

likewise for $D_y^a a$.

Proof. It suffices to prove the claim for $D_x$. For $a(x, y, \xi) = \langle x - y \rangle^w \tilde{a}(x, y, \xi)$, $\tilde{a} \in S^m(\mathbb{R}^n \times \mathbb{R}^n)$, we have

$$\partial_x a = \langle x - y \rangle^w (\partial_x \tilde{a}) + w \langle x - y \rangle^{-2} (x_1 - y_1) \tilde{a}. \hspace{1cm} (4.12)$$

The first summand lies in $\langle x - y \rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n)$, and the second summand even lies in the smaller space $\langle x - y \rangle^w - 1 S^m(\mathbb{R}^n \times \mathbb{R}^n)$.

Proof of Proposition 4.4. The key is that for $\xi \neq 0$, the phase $(x - y) \cdot \xi$ has no critical points in $y$. We exploit this by writing

$$(1 - \xi \cdot D_y) e^{i(x-y) \cdot \xi} = (\xi)^2 e^{i(x-y) \cdot \xi}, \hspace{1cm} (4.13)$$

so upon integrating by parts in $y$, one gains decay in $\xi$. Concretely, for $N \in \mathbb{N}$, we have

$$\text{Op}(a)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} ((1 - \xi \cdot D_y)^N e^{i(x-y) \cdot \xi} (\xi)^{-2N} a(x, y, \xi) u(y) dy d\xi$$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} (1 + \xi \cdot D_y)^N (\xi)^{-2N} a(x, y, \xi) u(y)) dy d\xi.$$ (4.14)

By the Leibniz rule, we have

$$(1 + \xi \cdot D_y)^N (\xi)^{-2N} a(x, y, \xi) u(y)) = \sum_{|\gamma| \leq N} a_{\gamma}(x, y, \xi) \cdot D_y^\gamma u,$$ (4.15)

where

$$a_{\gamma}(x, y, \xi) = \sum_{|\delta|, |\epsilon| \leq N} c_{\gamma \delta \epsilon} (\xi)^{-2N} \xi^\epsilon D_y^\gamma a(x, y, \xi) \hspace{1cm} (4.16)$$

for some combinatorial constants $c_{\gamma \delta \epsilon}$. By Lemma 4.5, $\tilde{a}_\gamma := \langle x - y \rangle^{-w} a_{\gamma} \in S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$, and for any $m \in \mathbb{R}$,

$$\| \tilde{a}_\gamma \|_{m-N,0} \leq C \| \tilde{a} \|_{m,N}.$$ (4.17)

Thus, if $N > m + n$, Lemma 4.2 applies, giving

$$\| \text{Op}(a_{\gamma}) D^\gamma u \|_{(x)^w C^0(\mathbb{R}^n)} \leq C \| \tilde{a}_\gamma \|_{m-N,0} \| D^\gamma u \|_M, \hspace{1cm} M > n + |w|,$$ (4.18)

and therefore

$$\| \text{Op}(a)u \|_{(x)^w C^0(\mathbb{R}^n)} \leq C \| \tilde{a} \|_{m,N} \| u \|_M, \hspace{1cm} M > n + N + |w|. \hspace{1cm} (4.19)$$
To get higher regularity and decay, let now $\alpha, \beta \in \mathbb{N}_0^n$, then
\[
x^{\alpha}D_x^{\beta} \text{Op}(a)u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} ((D_\xi + y)^\alpha e^{i(x-y)\cdot \xi})(\xi + D_x)^\beta a(x, y, \xi) u(y) \, dy \, d\xi
\]
\[
= (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi}(y - D_\xi)^\alpha (\xi + D_x)^\beta a(x, y, \xi) u(y) \, dy \, d\xi.
\] 
(4.20)

This can be expanded using the Leibniz rule; note that powers of $y$ are acceptable since $u$ is Schwartz. We thus obtain
\[
\|x^{\alpha}D_x^{\beta} \text{Op}(a)u\|_{(x)wC^0(\mathbb{R}^n)} \leq C\|\tilde{a}\|_{m,N}\|u\|_N
\] 
for $N$ sufficiently large (depending on $m, n, \alpha, \beta$). Thus, $\text{Op}(a)u \in \mathcal{S}(\mathbb{R}^n)$, finishing the proof. □

This shows that the map
\[
\langle x - y \rangle^w S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (a, u) \mapsto \text{Op}(a)u \in \mathcal{S}(\mathbb{R}^n)
\] 
(4.22) is a continuous bilinear map when putting the topology of $\langle x - y \rangle^w S^{m'}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ on the first factor (for any $m' \in \mathbb{R}$). By Proposition 3.7, it thus extends by continuity to a continuous bilinear map
\[
\langle x - y \rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \ni (a, u) \mapsto \text{Op}(a)u \in \mathcal{S}(\mathbb{R}^n).
\] 
(4.23)

Identifying $\text{Op}(a)$ with its Schwartz kernel, we thus get a continuous map
\[
\text{Op}: \langle x - y \rangle^w S^{m}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n),
\] 
(4.24)

which is given (interpreted as a limit along a sequence of residual symbols) by
\[
\text{Op}(a)(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi}a(x, y, \xi) \, d\xi.
\] 
(4.25)

(This is of course much weaker than (4.23).)

Remark 4.6. Let $\chi \in C^\infty_c(\mathbb{R}^n)$ be identically 1 near 0. Given $a \in \langle x - y \rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, (the proof of) Proposition 3.7 implies that
\[
\text{Op}(a)u(x) = \lim_{j \to \infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot \xi} \chi(\xi/j)a(x, y, \xi) u(y) \, dy \, d\xi,
\] 
(4.26)

with convergence in $\mathcal{S}(\mathbb{R}^n)$.

Definition 4.7. Let $m \in \mathbb{R}$. The space of pseudodifferential operators of order $m$,
\[
\Psi^m(\mathbb{R}^n),
\] 
(4.27)
is the space of all operators of the form $\text{Op}(a): \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, where $a \in \langle x - y \rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ and $w \in \mathbb{R}$. (As we show in the next section, one can take $w = 0$. See Exercise 4.1 for the case of differential operators.) We set
\[
\Psi^{-\infty}(\mathbb{R}^n) := \bigcap_{m \in \mathbb{R}^n} \Psi^m(\mathbb{R}^n).
\] 
(4.28)
Note that a priori it is not clear that $\Psi^{-\infty}(\mathbb{R}^n)$ is equal to the space of quantizations of residual symbols (it is certainly contained in the latter); we show this in Proposition 4.10 below.

By duality, we can define the action of $A = \text{Op}(a) \in \Psi^m(\mathbb{R}^n)$ on tempered distributions: for $u, v \in \mathcal{S}(\mathbb{R}^n)$ and $a \in \langle x - y \rangle^w S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, we have

$$\langle \text{Op}(a)u, v \rangle = \iint_{\mathbb{R}^{3n}} e^{i(x-y) \cdot \xi} a(x, y, \xi) u(y) v(y) \, dy \, d\xi \, dx$$

where we put

$$a^\dagger(x, y, \xi) = a(y, x, -\xi).$$

Since $a \mapsto a^\dagger$ is an isomorphism on $\langle x - y \rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, the equality

$$\text{Op}(a)^\dagger = \text{Op}(a^\dagger), \quad \text{that is,} \quad \langle \text{Op}(a)u, v \rangle = \langle u, \text{Op}(a^\dagger)v \rangle$$

continuous to hold for $a \in \langle x - y \rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. By the density $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, we can thus uniquely extend, by continuity, $\text{Op}(a)$ to an operator on $\mathcal{S}'(\mathbb{R}^n)$ via (4.31).

4.1. Left/right reduction, adjoints. In this section, we shall prove:

**Theorem 4.8.** Let $a \in \langle x - y \rangle^w S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$. Then there exists a unique left symbol $a_L \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$\text{Op}(a) = \text{Op}(a_L) = \text{Op}_L(a_L),$$

and a unique right symbol $a_R \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ such that

$$\text{Op}(a) = \text{Op}(a_R) = \text{Op}_R(a_R).$$

The symbols $a_L, a_R$ depend continuously on $a$. Modulo residual symbols, they are given by asymptotic sums

$$a_L(x, \xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} (\partial_\xi^\alpha D_y^\alpha a(x, y, \xi))|_{y=x},$$

$$a_R(y, \xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{(-1)^{|\alpha|}}{\alpha!} (\partial_\xi^\alpha D_x^\alpha a(x, y, \xi))|_{x=y}. \quad (4.34, 4.35)$$

(The summands are ordered by increasing $|\alpha|$.)

**Definition 4.9.** In the notation of Theorem 4.8, we call $a_L$, resp. $a_R$ the left, resp. right reduction of the full symbol $a$. Writing $A = \text{Op}(a)$, we write

$$a_L =: \sigma_L(A), \quad a_R =: \sigma_R(A). \quad (4.36)$$

We first consider the case ‘$m = -\infty$’ of Theorem 4.8 and give a description of kernels of residual operators:
Proposition 4.10. An operator $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ is a residual operator, $A \in \Psi^{-\infty}(\mathbb{R}^n)$, if and only if its Schwartz kernel $K(x, y)$ is smooth and satisfies
\[
|\partial_x^\alpha \partial_y^\beta K(x, y)| \leq C_{\alpha\beta N} \langle x - y \rangle^{-N} \quad \forall \alpha, \beta, N. \tag{4.37}
\]
Moreover, any such $A$ can be written as $A = \text{Op}_L(a_L) = \text{Op}_R(a_R)$ for unique symbols $a_L, a_R \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$.

Proof. Since $A \in \Psi^N(\mathbb{R}^n)$ for all $N \in \mathbb{R}$, we can write $A = \text{Op}(a_N)$ with $a_N \in \langle x - y \rangle^{w_N} S^{-N}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ for some $w_N \in \mathbb{R}$. Taking $N$ large, the Schwartz kernel $K$ of $A$ is then given by
\[
K(x, y) = (\mathcal{F}_3^{-1} a_N)(x, y, x - y), \tag{4.38}
\]
with $\mathcal{F}_3$ denoting the Fourier transformation in the third argument ($\xi$). For any fixed $k \in \mathbb{N}_0$, the symbol $a_N(x, y, \xi)$ has Schwartz seminorms $\|a_N(x, y, \cdot)\|_k$ bounded by $\langle x - y \rangle^{w_N}$ for $N \geq N(k)$; estimates for the (inverse) Fourier transform as in (2.12) thus imply that
\[
|\partial_x^\alpha \partial_y^\beta K(x, y)| \leq C_{\alpha\beta N} \langle x - y \rangle^{w_N} \langle x - y \rangle^{-k} \tag{4.39}
\]
for all $\alpha, \beta, k$, giving (4.37).

For the converse, note that if $K$ satisfies (4.37), we can define
\[
a_L(x, \xi) = \int_{\mathbb{R}^n} e^{-iz \cdot \xi} K(x, x - z) \, dz. \tag{4.40}
\]
Then $\text{Op}(a_L) = K$ by the Fourier inversion formula, and the estimates (4.37) imply $a_L \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$. Similarly, we have $K = \text{Op}(a_R)$ for
\[
a_R(y, \xi) = \int_{\mathbb{R}^n} e^{-iz \cdot \xi} K(y + z, y) \, dz. \tag{4.41}
\]

Remark 4.11. Define seminorms on the space of all $K \in \mathcal{C}^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^n)$ satisfying the estimates (4.37) to be the optimal constants there, i.e. $|K|_{\alpha\beta N} = \sup_{x,y \in \mathbb{R}^n} \langle x - y \rangle^N |\partial_x^\alpha \partial_y^\beta K(x, y)|$. Then the proof of Proposition 4.10 shows that the maps $K \mapsto a_{L/R} \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$ and $S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n) \ni a \mapsto K = \text{Op}_{L/R}(a)$ are continuous.

To handle the case of general orders $m \in \mathbb{R}$, we note that integration by parts in $\xi$ implies the equality of Schwartz kernels
\[
\text{Op}((y - x)^\alpha a)(x, y) = (2\pi)^{-n} \int ((-D_\xi)^\alpha e^{i(x - y) \cdot \xi}) a(x, y, \xi) \, d\xi
\]
\[
= (2\pi)^{-n} \int e^{i(x - y) \cdot \xi} D_\xi^\alpha a(x, y, \xi) \, d\xi
\]
\[
= \text{Op}(D_\xi^\alpha a)(x, y), \tag{4.42}
\]
first for $a \in \langle x - y \rangle^m S^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$, and then for symbols of order $m$ by density and continuity. The additional off-diagonal growth of $(y - x)^\alpha a$ is the reason for working with the more general symbol class (4.1).
Proof of Theorem 4.8. Let $N \in \mathbb{N}$, then Taylor’s formula states
\[
a(x, y, \xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} (y-x)^{\alpha} (\partial_y^\alpha a(x, y, \xi))|_{y=x} + r_N(x, y, \xi),
\]
and then
\[
r_N(x, y, \xi) = \sum_{|\alpha|=N} \frac{N}{\alpha!} (y-x)^{\alpha} \int_0^1 (1-t)^{N-1}(\partial_y^\alpha a)(x, x+t(y-x), \xi) \, dt.
\]
Using the identity (4.42), we have
\[
\text{Op}\left(a - \sum_{|\alpha|<N} \frac{1}{\alpha!} (D_{\xi}^\alpha \partial_y^\alpha a)|_{y=x}\right) = \text{Op}(\tilde{r}_N) \in \Psi^{m-N}(\mathbb{R}^n),
\]
where
\[
\tilde{r}_N(x, y, \xi) = \sum_{|\alpha|=N} \frac{N}{\alpha!} \int_0^1 (1-t)^{N-1}(D_{\xi}^\alpha \partial_y^\alpha a)(x, x+t(y-x), \xi) \, dt.
\]
In view of the symbolic estimates for $a$, the remainder here satisfies the estimate
\[
|\partial_y^\alpha \partial_x^\beta \partial_{\xi}^\gamma \tilde{r}_N(x, y, \xi)| \leq C_{\beta, \gamma, \delta}N (x-y)^{\delta} (\xi)^{m-N-|\delta|},
\]
and hence
\[
\tilde{r}_N \in (x-y)^u S^{m-N}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n). \tag{4.47}
\]
for all $N$. Note that for $|\alpha| = k$, we have $D_{\xi}^\alpha \partial_y^{\alpha} a|_{y=x} \in S^{m-k}(\mathbb{R}^n; \mathbb{R}^n)$. Thus, we can let $b \in S^m(\mathbb{R}^n; \mathbb{R}^n)$ be an asymptotic sum
\[
b \sim \sum_{\alpha} \frac{1}{\alpha!} (D_{\xi}^\alpha \partial_y^\alpha a)|_{y=x},
\]
and then
\[
R := \text{Op}(a - b) \in \bigcup_{N \in \mathbb{N}} \Psi^{m-N}(\mathbb{R}^n) = \Psi^{-\infty}(\mathbb{R}^n). \tag{4.49}
\]
By Proposition 4.10, we then have $R = \text{Op}_L(r)$ for some $r \in S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$. Therefore,
\[
A = \text{Op}_L(a_L), \quad a_L := b + r. \tag{4.50}
\]
The continuous dependence of $a_L$ on $a$ follows by using the explicit asymptotic summation procedure of Proposition 3.14 to define $b$, which thus depends continuously on $a$, and then noting that the optimal constants for the Schwartz kernel $K$ of $R$ in (4.37), and thus the $S^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$ seminorms of $r$ (see Remark 4.11), depend continuously on $a, b$.

Reduction to a right symbol is proved analogously. Instead of going through the argument, one can instead use duality as in (4.29), the idea being that the adjoint of a left quantization is a right quantization (and vice versa). Namely, using (4.30), we write the adjoint of $\text{Op}(a)$ as $\text{Op}(a)^\dagger = \text{Op}(a^\dagger) = \text{Op}(a_L')$ for $a_L' \in S^m(\mathbb{R}^n; \mathbb{R}^n)$, and then
\[
\text{Op}(a) = \text{Op}(a_L')^\dagger = \text{Op}(a_L')^\dagger = \text{Op}(a_L')^\dagger = \text{Op}_R(a_R), \tag{4.51}
\]
where $a_R(y, \xi) = a_L'(y, -\xi)$. The formula for right reductions gives
\[
a_L'(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} ((-\partial_\xi)^\alpha D_\xi^\alpha a)(y, x, -\xi)|_{y=x},
\]
yielding the asymptotic description (4.35) of $a_R$. 

MICROLOCAL ANALYSIS 23
It remains to prove the uniqueness of $a_L, a_R$. For this, note that a left symbol $a_L$ can be viewed as an element $a_L \in C^\infty(\mathbb{R}_+^n; \mathcal{S}'(\mathbb{R}_+^n))$, and the Schwartz kernel of $\text{Op}(a_L)$ is

$$\text{Op}(a_L)(x, x - z) = (\mathcal{F}_2^{-1}a_L)(x, z). \tag{4.53}$$

Since $\mathcal{F}_2$ is an isomorphism of $C^\infty(\mathbb{R}_+^n; \mathcal{S}'(\mathbb{R}_+^n))$, $\text{Op}(a_L) = 0$ implies $a_L = 0$. The proof for $a_R$ is similar.

**Corollary 4.12.** Let $m \in \mathbb{R}$ or $m = -\infty$. Then $\Psi^m(\mathbb{R}^n) = \text{Op}_{L/R}(S^m(\mathbb{R}^n; \mathbb{R}^n))$.

A slight variant of (4.29) gives the first part of the following corollary; the second part is an immediate application of Theorem 4.8.

**Corollary 4.13.** Let $A \in \Psi^m(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} (A^*u)(x) \overline{v(x)} \, dx = \int_{\mathbb{R}^n} u(x)(Av)(x) \, dx, \quad u, v \in \mathcal{S}(\mathbb{R}^n). \tag{4.54}$$

defines an operator $A^* \in \Psi^m(\mathbb{R}^n)$. If $A = \text{Op}(a)$, then $A^* = \text{Op}(a^*)$, $a^*(x, y, \xi) = \bar{a}(y, x, \xi)$. If $A = \text{Op}_L(a_L)$, then $A^* = \text{Op}_L(a_L^*)$ with

$$a_L^*(x, \xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_\xi^\alpha \pi_L(x, \xi) \tag{4.55}$$

4.2. **Topology on spaces of pseudodifferential operators.** Let $m \in \mathbb{R}$ or $m = -\infty$. Since $\text{Op}_L: S^m(\mathbb{R}^n; \mathbb{R}^n) \rightarrow \Psi^m(\mathbb{R}^n)$ is an isomorphism of vector spaces, it is natural to transport the Fréchet space structure of $S^m(\mathbb{R}^n; \mathbb{R}^n)$ to $\Psi^m(\mathbb{R}^n)$ via $\text{Op}_L$. For instance:

**Lemma 4.14.** Let $\chi \in C^\infty_c(\mathbb{R}_+^n)$ be identically 1 near 0, and put $J_\epsilon = \text{Op}(\chi(\epsilon \cdot))$, $\epsilon > 0$. Then $J_\epsilon \in \Psi^{-\infty}(\mathbb{R}^n)$ is uniformly bounded in $\Psi^0(\mathbb{R}^n)$ and converges to the identity operator $I = \text{Op}(1)$ in the topology of $\Psi^0(\mathbb{R}^n)$ for any $\eta > 0$.

**Proof.** This is equivalent to the main part of (the proof of) Proposition 3.7. \qed

It is reassuring to note that one can equally well define the topology on $\Psi^m(\mathbb{R}^n)$ using the right quantization. This is a consequence of the following result.

**Proposition 4.15.** Let $m \in \mathbb{R}$ or $m = -\infty$. Then the isomorphism of vector spaces $\text{Op}_R: S^m(\mathbb{R}^n; \mathbb{R}^n) \rightarrow \Psi^m(\mathbb{R}^n)$ is an isomorphism of Fréchet spaces.

**Proof.** Right reduction $\sigma_R$ is the inverse of $\text{Op}_R$. By definition of the Fréchet space structure of $\Psi^m(\mathbb{R}^n)$, the proposition is thus equivalent to the continuity of $\sigma_R \circ \text{Op}_L$, which is part of Theorem 4.8. \qed

4.3. **Composition.** Proving that composition of ps.d.o.s produces another ps.d.o. is now straightforward:

**Theorem 4.16.** Let $A \in \Psi^m(\mathbb{R}^n)$, $B \in \Psi^{m'}(\mathbb{R}^n)$. Then $A \circ B: \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ is a pseudodifferential operator,

$$A \circ B \in \Psi^{m+m'}(\mathbb{R}^n), \tag{4.56}$$

and its left symbol is given as an asymptotic sum

$$\sigma_L(A \circ B) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_L(A) \cdot D_x^\alpha \sigma_L(B). \tag{4.57}$$
The bilinear map \((A, B) \mapsto A \circ B\) is continuous.

Note that the symbolic expansion (4.57) is local in \((x, \xi)\): the symbols of \(A\) and \(B\) do not ‘interact’ at all, modulo residual terms, at distinct points in phase space \(\mathbb{R}^n_x \times \mathbb{R}^n_\xi\).

**Proof of Theorem 4.16.** Write \(A = \text{Op}_L(a)\) and \(B = \text{Op}_R(b_R)\). Assume first that \(A, B \in \Psi^{-\infty}(\mathbb{R}^n)\), then for \(u, v \in \mathscr{S}(\mathbb{R}^n)\), we have

\[
Av(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} a(x, \xi) \hat{v}(\xi) \, d\xi,
\]

\[
\hat{B}u(\xi) = \int_{\mathbb{R}^n} e^{-iy \cdot \xi} b_R(y, \xi) u(y) \, dy.
\]

Thus,

\[
ABv(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} a(x, \xi) b_R(y, \xi) u(y) \, dy \, d\xi,
\]

giving \(A \circ B = \text{Op}(c), c(x, y, \xi) = a(x, \xi) b_R(y, \xi)\). (This is one of the reasons for considering such general symbols!) By density and continuity, this continues to hold for \(A, B\) as in the statement of the theorem.

To get the asymptotic expansion (4.57), let us write \(a = \sigma_L(A), b = \sigma_L(B)\), then\(^1\)

\[
\sigma_C(A \circ B)(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial^\alpha_x \left( a(x, \xi) D^\alpha_y b_R(y, \xi) \right)_{y=x}
\]

\[
\sim \sum_{\beta, \gamma} \frac{1}{\beta! \gamma!} \partial^\beta_x a(x, \xi) \cdot \partial^\gamma_x D^\beta+\gamma \left( \sum_{\delta} \frac{(-1)^{\delta}}{\delta!} (\partial^\beta_x D^\delta_y b)(x, \xi) \right)
\]

\[
\sim \sum_{\beta} \frac{1}{\beta!} \partial^\beta_x a(x, \xi) \cdot D^\beta_x \left( \sum_{\epsilon} \frac{1}{\epsilon!} \partial^\epsilon_x D^\epsilon_y b(x, \xi) \sum_{\gamma+\delta=\epsilon} \frac{\epsilon!}{\gamma! \delta!} (-1)^{\delta} \right)
\]

and the observation that for \(\epsilon = 0\), the final sum evaluates to 1, while for \(\mathbb{N}_0^n \ni \epsilon \neq 0\),

\[
\sum_{\gamma+\delta=\epsilon} \frac{\epsilon!}{\gamma! \delta!} (-1)^{\delta} = \prod_{\epsilon_j \neq 0} (1 - 1)^{\epsilon_j} = 0.
\]

This finishes the proof. \(\square\)

As a simple consequence, we can now prove the pseudolocality of ps.d.o.s:

**Proposition 4.17.** Let \(A \in \Psi^m(\mathbb{R}^n)\). Then

\[
\text{sing supp } Au \subset \text{sing supp } u, \quad u \in \mathscr{S}'(\mathbb{R}^n).
\]

To prove this, we record:

**Lemma 4.18.** A residual operator \(A \in \Psi^{-\infty}(\mathbb{R}^n)\) is continuous as a map

\[
A : \mathscr{S}'(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n) \cap \mathcal{C}^\infty(\mathbb{R}^n).
\]

More precisely, for any \(u \in \mathscr{S}'(\mathbb{R}^n)\) we have \(Au \in \langle x \rangle^N \mathcal{C}^\infty(\mathbb{R}^n)\) for some \(N\) (depending on \(u\)).

---

\(^1\)Since these are asymptotic sums, it suffices to consider only those terms which have symbolic order bigger than some fixed but arbitrary number; in particular, there are no convergence or rearrangement issues.
Proof. Let $K$ denote the Schwartz kernel of $A$; recall that it satisfies the estimates (4.37). For $u \in \mathscr{S}'(\mathbb{R}^n)$, we then have, for some $N \in \mathbb{N}$,

\[ |(Au)(x)| = |\langle K(x, \cdot), u \rangle| \leq C\|K(x, \cdot)\|_N = C \sup_{y \in \mathbb{R}^n} |y^\alpha D_y^\beta K(x, y)| \]

\[ \leq C \sup_{y \in \mathbb{R}^n} |\langle y \rangle^N D_y^\beta K(x, y)| = C \sup_{y \in \mathbb{R}^n} \langle y \rangle^N (x - y)^{-N} |\langle x - y \rangle^N D_y^\beta K(x, y)|. \]

(4.64)

Using Lemma 4.3, we see that $\langle y \rangle^N (x - y)^{-N} \leq C_N(x)^N$, hence

\[ |(Au)(x)| \leq C(x)^N. \]

(4.65)

Derivatives in $x$ are estimated analogously, so $Au \in \mathcal{C}^\infty(\mathbb{R}^n)$, and in fact

\[ |\partial_x^\alpha (Au)(x)| \leq C_\alpha(x)^N. \]

(4.66)

Note here that the number $N$ above only depends on $u$, not on $K$ itself. \(\square\)

Proof of Proposition 4.17. Suppose $x \notin \text{sing supp } u$. There exist cutoffs $\chi, \tilde{\chi} \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that

\[ \chi(x) \neq 0, \quad \tilde{\chi} \equiv 1 \text{ on supp } \chi, \quad \tilde{\chi} u \in \mathcal{C}_c^\infty(\mathbb{R}^n). \]

(4.67)

Then

\[ \chi Au = \chi A(\tilde{\chi} u) + \chi A(1 - \tilde{\chi}) u. \]

(4.68)

Since $A$ acts on $\mathcal{S}(\mathbb{R}^n)$, we have $\chi A(\tilde{\chi} u) \in \mathcal{S}(\mathbb{R}^n)$. For the second term, note that $\chi$ and $1 - \tilde{\chi}$ have disjoint supports; hence we have

\[ \sigma_L(\chi A \circ (1 - \tilde{\chi}))(x, \xi) \sim \sum_\alpha \frac{1}{\alpha!} \chi(x) \partial_\xi^\alpha \sigma_L(A)(x, \xi) \cdot D_\xi^\alpha (1 - \tilde{\chi}(x)) = 0, \]

(4.69)

which implies

\[ \chi A(1 - \tilde{\chi}) \in \Psi^{-\infty}(\mathbb{R}^n). \]

(4.70)

By Lemma 4.18, we conclude that $\chi A(1 - \tilde{\chi}) u \in \mathcal{C}^\infty(\mathbb{R}^n)$, finishing the proof. \(\square\)

Returning to the observation (4.70), note that if $A = \text{Op}(a)$ has Schwartz kernel $K \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$, then the Schwartz kernel of $\chi A(1 - \tilde{\chi})$ is $\chi(x)(1 - \tilde{\chi}(y))K(x, y)$. Thus, (4.70) can equivalently be stated as:

Proposition 4.19. The Schwartz kernel $K$ of a pseudodifferential operator is smooth away from the diagonal $\Delta = \{(x, x) : x \in \mathbb{R}^n\}$. That is, $\text{sing supp } K \subset \Delta$.

4.4. Principal symbols. Similarly to Proposition 2.19, the ‘leading order part’ of the left or right symbol of an operator $A \in \Psi^m(\mathbb{R}^n)$ has particularly simple properties.

Definition 4.20. Let $m \in \mathbb{R}$. The principal symbol $\sigma_m(A)$ of a ps.d.o. $A \in \Psi^m(\mathbb{R}^n)$ is the equivalence class

\[ \sigma_m(A) := \{ \sigma_L(A) \} \in S^m(\mathbb{R}^n; \mathbb{R}^n)/S^{m-1}(\mathbb{R}^n; \mathbb{R}^n). \]

(4.71)

We shall often omit from the notation the passage to the equivalence class.
Directly from the definition, this gives a short exact sequence for every \( m \in \mathbb{R} \):
\[
0 \to \Psi^{m-1}\mathbb{R}^n ; \mathbb{R}^n \to \Psi^m\mathbb{R}^n ; \mathbb{R}^n \xrightarrow{\sigma^m} S^m\mathbb{R}^n ; \mathbb{R}^n / S^{m-1}\mathbb{R}^n ; \mathbb{R}^n \to 0. \tag{4.72}
\]

The surjectivity of \( \sigma_m \) is clear: given a representative \( a \in S^m\mathbb{R}^n ; \mathbb{R}^n \) of an equivalence class of symbols, we have \( \sigma_m(\text{Op}_L(a)) = [a] \).

**Proposition 4.21.** The principal symbol map has the following properties:

1. \( \sigma_m(\text{Op}_R(a)) = [a] \), i.e. using the right symbol in (4.71) gives the same principal symbol map.
2. For \( A \in \Psi^m\mathbb{R}^n \), we have \( \sigma_m(A^*) = \overline{\sigma_m(A)} \).
3. For \( A \in \Psi^m\mathbb{R}^n \), \( B \in \Psi^{m'}\mathbb{R}^n \), we have \( \sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B) \).

(The behavior under changes of variables will be discussed in §5.1.) Notice that the principal symbol map translates operator composition (a highly non-commutative operation) to the multiplication of (equivalence classes of) functions (a commutative operation), though of course at what seems to be an enormous loss of information compared to the full expansion (4.57) (which itself gives up information on the residual part of \( A \circ B \)). However, in most situations, the principal symbol, and sometimes a ‘subprincipal’ part of the full symbol, dominate the behavior of the operator, while lower order parts are irrelevant; cf. the discussion of ellipticity for symbols in §3.1.

One crucial calculation is the following. For \( A \in \Psi^m\mathbb{R}^n \), \( B \in \Psi^{m'}\mathbb{R}^n \), note that \( \sigma_{m+m'}(A \circ B) = \sigma_m(A)\sigma_{m'}(B) = \sigma_{m+m'}(B \circ A) \), so
\[
\sigma_{m+m'}([A, B]) = 0, \quad [A, B] = A \circ B - B \circ A. \tag{4.73}
\]

In view of (4.72), we thus have \([A, B] \in \Psi^{m+m'-1}\mathbb{R}^n \), and it is natural to inquire about its principal symbol as an operator of order \( m + m' - 1 \). It turns out that it can be computed solely in terms of the principal symbols of \( A \) and \( B \):

**Proposition 4.22.** For \( A \in \Psi^m\mathbb{R}^n \), \( B \in \Psi^{m'}\mathbb{R}^n \), we have
\[
\sigma_{m+m'-1}(i[A, B]) = \{\sigma_m(A), \sigma_{m'}(B)\}, \tag{4.74}
\]
where the Poisson bracket of \( a, b \in C^\infty\mathbb{R}^n \times \mathbb{R}^n \) is defined as
\[
\{a, b\} := \sum_{j=1}^n (\partial_{\xi_j} a)(\partial_{x_j} b) - (\partial_{x_j} a)(\partial_{\xi_j} b). \tag{4.75}
\]

This will be the key connection between ‘quantum mechanics’ (quantizations of symbols, noncommutative algebra of operators) and ‘classical mechanics’ (symbols themselves, commutative algebra of functions), which will play a central role in §8.

**Proof of Proposition 4.22.** We leave it to the reader to verify that (4.74) is well-defined, i.e. that the image of the right hand side in the quotient space \( S^{m+m'-1}/S^{m+m'-2} \) does not depend on the choice of representatives of the principal symbols of \( A \) and \( B \).

The proof is an immediate application of (4.57). Let \( a = \sigma_L(A) \), \( b = \sigma_L(B) \). Working modulo \( S^{m+m'-2}\mathbb{R}^n \), we have
\[
\sigma_L(A \circ B) \equiv ab + \frac{1}{i} \sum_{j=1}^n (\partial_{\xi_j} a)(\partial_{x_j} b), \quad \sigma_L(B \circ A) \equiv ab + \frac{1}{i} \sum_{j=1}^n (\partial_{\xi_j} b)(\partial_{x_j} a) \tag{4.76}
\]
and (4.74) follows.

4.5. Classical operators. Following Definition 3.11, we have a subclass of classical operators:

**Definition 4.23.** For \( m \in \mathbb{C} \), we define the space of classical pseudodifferential operators of order \( m \) by

\[
\Psi^m_{cl}(\mathbb{R}^n) := \text{Op}_L(S^m_{cl}(\mathbb{R}^n; \mathbb{R}^n)) \subset \Psi^{Re m}(\mathbb{R}^n),
\]

equipped with the structure of a Fréchet space which makes \( \text{Op}_L \) into an isomorphism. We put \( \Psi^{-\infty}_{cl}(\mathbb{R}^n) := \Psi^{-\infty}(\mathbb{R}^n) \).

The symbol expansions in Theorem 4.16 and Corollary 4.13 imply that compositions and adjoints of classical operators are still classical:

**Proposition 4.24.** Composition of ps.d.o.s restricts to a continuous bilinear map

\[
\Psi^m_{cl}(\mathbb{R}^n) \times \Psi^m'_{cl}(\mathbb{R}^n) \ni (A, B) \mapsto A \circ B \in \Psi^{m+m'}_{cl}(\mathbb{R}^n).
\]

Similarly, the map

\[
\Psi^m_{cl}(\mathbb{R}^n) \ni A \mapsto A^* \in \Psi^m_{cl}(\mathbb{R}^n)
\]

is a continuous conjugate-linear map.

For a classical operator \( A = \text{Op}_L(a) \), with \( a \in S^m_{cl}(\mathbb{R}^n; \mathbb{R}^n) \), we can identify the principal symbol \( \sigma_{Re m}(A) \) with the homogenous leading order part of \( a \), as discussed after Lemma 3.12. The corresponding short exact sequence is

\[
0 \to \Psi^{m-1}_{cl}(\mathbb{R}^n) \to \Psi^m_{cl}(\mathbb{R}^n) \to S^m_{hom}(\mathbb{R}^n; \mathbb{R}^n \setminus \{0\}) \to 0.
\]

4.6. Elliptic parametrix. Recall Definition 3.8 and the discussion around (3.19). Then:

**Definition 4.25.** We call an operator \( A \in \Psi^m(\mathbb{R}^n) \) (uniformly) elliptic if its principal symbol \( \sigma_m(A) \) is elliptic.

As a first, and important, application of the symbol calculus we have developed above, we construct parametrices (approximate inverses—a term which, almost by nature, has no precise definition, but rather depends on the context) of uniformly elliptic operators.

**Theorem 4.26.** Let \( A \in \Psi^m(\mathbb{R}^n) \) be uniformly elliptic. Then there exists an operator \( B \in \Psi^{-m}(\mathbb{R}^n) \) which is unique modulo \( \Psi^{-\infty}(\mathbb{R}^n) \), such that

\[
AB - I, \ BA - I \in \Psi^{-\infty}(\mathbb{R}^n).
\]

We call an operator \( B \) satisfying (4.81) a parametrix of \( A \).

**Proof of Theorem 4.26.** Let \( b \in S^{-m}(\mathbb{R}^n; \mathbb{R}^n) \) be such that \( \sigma_m(A)b - 1 \in S^{-1}(\mathbb{R}^n; \mathbb{R}^n) \). Put \( B_0 = \text{Op}(b) \in \Psi^{-m}(\mathbb{R}^n) \), then

\[
A \circ B_0 = I - R, \quad R \in \Psi^{-1}(\mathbb{R}^n).
\]

Indeed, this follows from \( \sigma_0(AB_0 - I) = 0 \). We approximately invert \( I - R \) using a Neumann series: we choose

\[
R' \sim \sum_{j=1}^{\infty} R^j \in \Psi^{-1}(\mathbb{R}^n),
\]
Therefore, if we put $R$, i.e. the left symbol of $\Psi^m$ via $(\mathcal{C} = I \times)$. Since $(\mathcal{C} = I \times)$, then Proposition 4.27.

Let $B = \mathcal{C}$. Therefore, $\mathcal{C}$ is $\mathcal{C}$-linear, and $\mathcal{C}_\mathcal{L}$ spaces related to smooth developing microlocal analysis in the $\mathcal{C}_\mathcal{L}$ category, hence do not directly recover this stronger conclusion.

As a simple application, we prove:

**Proposition 4.27.** Let $A \in \Psi^m(\mathbb{R}^n)$ be uniformly elliptic, and suppose

$$u \in \mathcal{S}'(\mathbb{R}^n), \quad Au = f \in \mathcal{C}^\infty(\mathbb{R}^n).$$

Then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$. More precisely, we have

$$\text{sing supp } u = \text{sing supp } Au.$$  

**(Proof.** We prove (4.88). Let $B \in \Psi^{-m}(\mathbb{R}^n)$ be a parametrix of $A$, with $BA = I + R$, $R \in \Psi^{-\infty}(\mathbb{R}^n)$. Then by Proposition 4.17, we have

$$\text{sing supp } u = \text{sing supp } (BAu + Ru) = \text{sing supp } BAu \subset \text{sing supp } Au \subset \text{sing supp } u.$$  

Therefore, equality must hold at each step. 

**(Example 4.28.** Examples to which Proposition 4.27 applies are the Laplacian $\Delta \in \Psi^2(\mathbb{R}^n)$ and the Cauchy–Riemann operator $\partial = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2}) \in \Psi^1(\mathbb{R}^2)$, which is identified with $\mathbb{C}$ via $(x_1, x_2) \mapsto x_1 + ix_2$. For the latter, we deduce that if $\partial u = 0$ for $u \in \mathcal{S}'(\mathbb{R}^n)$, then $u \in \mathcal{C}^\infty(\mathbb{R}^n)$. In complex analysis we learn that in fact $u$ is analytic; here we are only developing microlocal analysis in the smooth category, hence do not directly recover this stronger conclusion.

4.7. **Boundedness on Sobolev spaces.** In applications, one typically uses function spaces other than $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$, such as Hölder or $L^p$ spaces. Here, we focus on function spaces related to $L^2$, in parts because they are the most natural for the study of non-elliptic operators in Chapter 8.

As usual, we first consider residual operators:

**Proposition 4.29.** Let $A \in \Psi^{-\infty}(\mathbb{R}^n)$. Then $A$ extends by continuity from $\mathcal{S}(\mathbb{R}^n)$ to a bounded linear operator $A: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$.  

This will follow from the estimates (4.37) and Schur’s lemma:

---

2We use here that $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ is a dense subspace.
Lemma 4.30. Let \((X, \mu)\) and \((Y, \nu)\) be measure spaces. Suppose \(K(x, y)\) is measurable on \(X \times Y\) and
\[
\int_X |K(x, y)| \, d\mu(x) \leq C_1, \quad \int_Y |K(x, y)| \, d\nu(y) \leq C_2
\]
for almost all \(y \in Y\) and \(x \in X\), respectively. Let
\[
Tu(x) = \int_Y K(x, y)u(y) \, d\nu(y).
\]
Then \(T: L^2(Y) \to L^2(X)\) is bounded. Quantitatively,
\[
\|Tu\|_{L^2(X)} \leq (C_1 C_2)^{1/2}\|u\|_{L^2(Y)}.
\]

Proof. Let \(u \in L^2(Y)\) and \(v \in L^2(X)\), then by Cauchy–Schwarz
\[
\left| \int_X \int_Y K(x, y)u(y)\overline{v(x)} \, d\nu(y) \, d\mu(x) \right| \\
\leq \left( \int_{X \times Y} |K(x, y)||u(y)||v(x)|^2 \, d\mu(x) \, d\nu(y) \right)^{1/2} \\
\leq C_1^{1/2}\|u\|_{L^2(Y)} \cdot C_2^{1/2}\|v\|_{L^2(X)}.
\]

Proof of Proposition 4.29. The Schwartz kernel \(K\) of \(A\) satisfies \(|K(x, y)| \leq C\langle x-y \rangle^{-n-1}\), hence
\[
\int_{\mathbb{R}^n} |K(x, y)| \, dx \leq C \int_{\mathbb{R}^n} \langle z \rangle^{-n-1} \, dz < \infty,
\]
and likewise \(\int_{\mathbb{R}^n} |K(x, y)| \, dy < \infty\). The claim then follows from Lemma 4.30.

Using ‘Hörmander’s square root trick’, we can now prove:

Theorem 4.31. Let \(A \in \Psi^0(\mathbb{R}^n)\). Then \(A: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\) is bounded.

Proof. By Corollary 4.13 and Theorem 4.16, we have \(A^*A \in \Psi^0(\mathbb{R}^n)\). With \(a = \sigma_0(A)\) (that is, \(a\) is any representative of \(\sigma_0(A)\)), we have \(\sigma_0(A^*A) = |a|^2\), which is real, non-negative, and bounded. Thus, for \(C > \sup_{x, t \in \mathbb{R}^n} |a|^2\), the symbol \(C - |a|^2 \in S^0(\mathbb{R}^n; \mathbb{R}^n)\) is elliptic and positive. By Exercise 3.3, it has an approximate square root \(0 < b_0 \in S^0(\mathbb{R}^n; \mathbb{R}^n)\), so \(C - |a|^2 - b_0^2 \in S^{-1}(\mathbb{R}^n; \mathbb{R}^n)\). Let \(B_0 = \text{Op}(b_0)\), then
\[
C - A^*A = B_0^*B_0 + R_1, \quad R_1 \in \Psi^{-1}(\mathbb{R}^n).
\]

Assume inductively that we have found \(B_j \in \Psi^{-j}(\mathbb{R}^n)\), \(j = 0, \ldots, k - 1\), such that
\[
R_k := C - A^*A - (B_0 + \cdots + B_{k-1})^*(B_0 + \cdots + B_{k-1}) \in \Psi^{-k}(\mathbb{R}^n).
\]
This holds for \(k = 1\). We try to improve the error term by finding the next correction \(B_k = \text{Op}(b_k) \in \Psi^{-k}(\mathbb{R}^n)\); we compute
\[
R_{k+1} = C - A^*A - (B_0 + \cdots + B_k)^*(B_0 + \cdots + B_k) \\
= R_k - (B_k^*(B_0 + \cdots + B_{k-1}) + (B_0 + \cdots + B_{k-1})^*B_k) \in \Psi^{-k}(\mathbb{R}^n).
\]
Thus, the requirement \(R_{k+1} \in \Psi^{-k-1}(\mathbb{R}^n)\) is equivalent to a principal symbol condition,
\[
\overline{b_kb_0} + \overline{b_0b_k} = \sigma_{-k}(R_k) \quad (\text{in } S^{-k}(\mathbb{R}^n; \mathbb{R}^n)/S^{-k-1}(\mathbb{R}^n; \mathbb{R}^n)).
\]
Since $R_k = R_k^*$, the principal symbol $\sigma_{-k}(R_k)$ is real; hence we can take $b_k = \frac{1}{2} \sigma_{-k}(R_k)/b_0 \in S^{-k}(\mathbb{R}^n; \mathbb{R}^n)$.

Finally, we let $B \in \Psi^0(\mathbb{R}^n)$ be the asymptotic sum

$$B \sim \sum_{k=0}^{\infty} B_k.$$  

(4.98)

We have then arranged

$$R := C - A^*A - B^*B \in \Psi^{-\infty}(\mathbb{R}^n).$$  

(4.99)

(Thus, we have constructed a square root, modulo residual operators, of $C - A^*A$.)

Given $u \in \mathcal{S}(\mathbb{R}^n)$, we then have

$$\|Au\|_{L^2(\mathbb{R}^n)}^2 = \langle A^*Au, u \rangle_{L^2(\mathbb{R}^n)}$$

$$= C\|u\|_{L^2(\mathbb{R}^n)}^2 - \|Bu\|_{L^2(\mathbb{R}^n)}^2 - \langle Ru, u \rangle$$

$$\leq C\|u\|_{L^2(\mathbb{R}^n)}^2 + \|Ru\|_{L^2(\mathbb{R}^n)}\|u\|_{L^2(\mathbb{R}^n)}$$

(4.100)

by Proposition 4.29. Thus, $A$ extends by continuity to a bounded operator on $L^2(\mathbb{R}^n)$. □

Boundedness of ps.d.o.s on Sobolev spaces is a straightforward consequence:

**Corollary 4.32.** Let $s, m \in \mathbb{R}$, and $A \in \Psi^m(\mathbb{R}^n)$. Then $A: H^s(\mathbb{R}^n) \to H^{s-m}(\mathbb{R}^n)$ is bounded.

**Proof.** Recall the operators $\langle D \rangle^\sigma = \mathcal{F}^{-1}\langle \xi \rangle^\sigma \mathcal{F}$ for $\sigma \in \mathbb{R}$ from Definition 2.10; note that $\langle D \rangle^\sigma \in \Psi^\sigma(\mathbb{R}^n)$. Moreover, $\langle D \rangle^{-s}: L^2(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ and $\langle D \rangle^{s-m}: H^{s-m}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are isometric isomorphisms. Now

$$\langle D \rangle^{s-m}A\langle D \rangle^{-s} \in \Psi^0(\mathbb{R}^n)$$

(4.101)

is bounded on $L^2(\mathbb{R}^n)$ by Theorem 4.31, which is equivalent to the statement of the corollary. □

In fact, this can be generalized to weighted Sobolev spaces, see (2.25):

**Theorem 4.33.** Let $s, m, r \in \mathbb{R}$, and $A \in \Psi^m(\mathbb{R}^n)$. Then $A: \langle x \rangle^r H^s(\mathbb{R}^n) \to \langle x \rangle^r H^{s-m}(\mathbb{R}^n)$ is bounded.

**Proof.** Since $\langle x \rangle^r \langle D \rangle^{-s}: L^2(\mathbb{R}^n) \to \langle x \rangle^s H^s(\mathbb{R}^n)$ and $\langle D \rangle^{s-m} \langle x \rangle^{-r}: \langle x \rangle^r H^s(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ are isomorphisms, we need to show that

$$A' := \langle D \rangle^{s-m} \langle x \rangle^{-r} \circ A \circ \langle x \rangle^r \langle D \rangle^{-s} \in \Psi^0(\mathbb{R}^n).$$

(4.102)

If $a = \sigma_1(A)$, then the full symbol $a^\sharp(x, y, \xi)$ of $A^\sharp := \langle x \rangle^{-r} \circ A \circ \langle x \rangle^r$ is given by $a^\sharp(x, y, \xi) = \langle x \rangle^{-r} \langle y \rangle^r a(x, \xi)$. By Lemma 4.3, we have

$$|a^\sharp(x, y, \xi)| \leq 2^{|r|/2}|x - y|^{|r|}|a(x, \xi)| \leq C|x - y|^{|r|}|\xi|^{m-s},$$

(4.103)
which is the first step towards showing that $a^s \in (x - y)^{|r|} S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$; it remains to consider derivatives. The essence of this is contained in
\[
|\partial_y^j a^s(x, y, \xi)| = \left| -r(x)^r \langle y \rangle^{-r-2} y_j a(x, \xi) \langle \xi \rangle^{-s} \right| \\
\leq C \langle x - y \rangle^{r |r|} \frac{|y_j|}{|y|^2} (\langle \xi \rangle^m)
\]
(4.104)
\[
\leq C \langle x - y \rangle^{r |r|} (\langle \xi \rangle)^m.
\]
We conclude that $A^s \in \Psi^m(\mathbb{R}^n)$, hence $A' \in \Psi^0(\mathbb{R}^n)$, finishing the proof.
\]

In view of the Schwartz representation theorem, Theorem 2.13, we thus obtain another proof of Lemma 4.18. Indeed, a residual operator maps $\mathscr{S}'(\mathbb{R}^n) = \bigcup_{r,s} (x)^r H^s(\mathbb{R}^n)$ into $\bigcup_r (x)^r H^\infty(\mathbb{R}^n) = \bigcup_r (x)^r C_0^\infty(\mathbb{R}^n)$ (using Sobolev embedding, Exercise 2.2).

We can sharpen and upgrade the elliptic regularity result, Proposition 4.27:

**Corollary 4.34.** Let $A \in \Psi^m(\mathbb{R}^n)$ be uniformly elliptic, and suppose $u \in \langle x \rangle^r H^{-N}(\mathbb{R}^n)$ for some $r, N \in \mathbb{R}$. If $Au = f = \langle x \rangle^r H^{s-m}(\mathbb{R}^n)$, then $u \in \langle x \rangle^r H^s(\mathbb{R}^n)$.

**Proof.** With $B \in \Psi^{-m}(\mathbb{R}^n)$ denoting a parametrix of $A$, so $I = BA + R, R \in \Psi^{-\infty}(\mathbb{R}^n)$, we have
\[
u = BAu + Ru = Bf + Ru,
\]
(4.105) with $Bf \in \langle x \rangle^r H^s(\mathbb{R}^n)$ and $Ru \in \langle x \rangle^r \bigcup_{r'} H^r(\mathbb{R}^n)$. \qed

**Remark 4.35.** It is important that the assumption on $u$ already has the weight factor $\langle x \rangle^r$. Indeed, the conclusion would be false in general if we merely assumed $u \in \langle x \rangle^{r'} H^{-N}(\mathbb{R}^n)$ for some $r' < r$. (Convince yourself of this. For example, take $A = \Delta$, the Laplacian on $\mathbb{R}^n$, and $u = 1$.)

4.8. **Exercises.**

**Exercise 4.1.** Let $m \in \mathbb{N}_0$, and let $a \in S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)$ be a polynomial in the symbolic variable $\xi$.

1. Show, starting from the definition as a limit of quantizations of residual symbols, that $\text{Op}(a) \in \text{Diff}^m(\mathbb{R}^n)$.

2. Prove that $\text{Op}(\langle x - y \rangle^u a) \in \text{Diff}^m(\mathbb{R}^n)$ (which in particular entails the boundedness of the coefficients). (Hint. Compute its Schwartz kernel.)

**Exercise 4.2.** Let $A \in \Psi^m(\mathbb{R}^n)$, and denote by $K$ its Schwartz kernel.

1. Give another, direct, proof that $K \in C^\infty((\mathbb{R}^n \times \mathbb{R}^n) \setminus \Delta)$, where $\Delta = \{(x, x) : x \in \mathbb{R}^n\}$ is the diagonal. (Hint. For $\phi, \psi \in C^\infty(\mathbb{R}^n)$ with $\text{supp} \phi \cap \text{supp} \psi = \emptyset$, rewrite the pairing $\langle A\phi, \psi \rangle$ for $A \in \Psi^{-\infty}(\mathbb{R}^n)$ using integrations by parts as in the proof of Proposition 4.4. Then use a density argument.)

2. Prove that for every $\epsilon > 0, N \in \mathbb{R}$ there exists a constant $C$ such that
\[
|K(x, y)| \leq C|x - y|^{-N}, \quad |x - y| \geq \epsilon.
\]
(4.106)

**Exercise 4.3.** Suppose $K(x, z) \in C^\infty(\mathbb{R} \times (\mathbb{R} \setminus \{0\})$) satisfies $K(x, \lambda z) = \lambda^{-1} K(x, z), \lambda > 0$, and $K(x, -z) = -K(x, z)$. Let $\chi \in C^\infty_0(\mathbb{R}^n)$ be identically 1 near 0. Show that the operator
\[
Au(x) = \lim_{\epsilon \to 0} \int_{|x - y| \geq \epsilon} \chi(x - y) K(x, x - y) u(y) \, dy, \quad u \in C_c^\infty(\mathbb{R}^n),
\]
(4.107)
is well-defined and defines an element \( A \in \Psi_0^0(\mathbb{R}) \). Compute its principal symbol.

**Exercise 4.4.** Prove Gårding’s inequality. Let \( A \in \Psi^{2m}(\mathbb{R}^n) \), and suppose \( \text{Re} \sigma_{2m}(A) \geq c(\xi)^{2m} \) for some \( c \in \mathbb{R} \). Then for every \( \epsilon > 0 \) and \( N \in \mathbb{R} \), there exists a constant \( C \) such that

\[
\text{Re} \langle Au, u \rangle_{L^2(\mathbb{R}^n)} \geq (c - \epsilon)\|u\|_{H^m(\mathbb{R}^n)}^2 - C\|u\|_{H^{-N}(\mathbb{R}^n)}^2, \quad u \in \mathcal{S}(\mathbb{R}^n).
\]  

**(Hint.** Use the ‘square root trick’.) The sharp Gårding inequality states that (4.108) holds for \( \epsilon = 0 \), but then with \(-N = m - 1/2\); see [Hör03, Theorem 18.1.14]. (This can be further refined to the Fefferman–Phong inequality, which allows \(-N = m - 1\).)

**Exercise 4.5.** TBC stationary phase for quadratic phases

**Exercise 4.6.** TBC characterization of principal symbol by oscillatory testing

**Exercise 4.7.** TBC symbols and operators of class \( \rho, \delta \)

**Exercise 4.8.** TBC variable order symbols and operators

**Exercise 4.9.** TBC semiclassical operators

The following series of exercises introduces the basic properties of scattering pseudodifferential operators on \( \mathbb{R}^n \).

**Exercise 4.10.** (Scattering symbols.) For \( m, r_1, r_2 \in \mathbb{R} \), define the space of symbols

\[
S^{m,r_1,r_2}(\mathbb{R}^n_x \times \mathbb{R}^n_y; \mathbb{R}^n_\xi) \quad (4.109)
\]

to consist of all \( a \in \mathcal{C}_c^\infty(\mathbb{R}^{3n}) \) such that the seminorms

\[
\|a\|_{m,r_1,r_2,k} := \sup_{|\alpha_1| + |\alpha_2| + |\beta| \leq k} \langle x \rangle^{-r_1 + |\alpha_1|} \langle y \rangle^{-r_2 + |\alpha_2|} \langle \xi \rangle^{-m + |\beta|} |\partial_2^{\alpha_1} \partial_y^{\alpha_2} \partial_\xi^\beta a(x,y,\xi)| \quad (4.110)
\]

are finite for all \( k \in \mathbb{N}_0 \). \( ^3 \) Let

\[
S^{-\infty,-\infty,-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) := \bigcap_{m,r_1,r_2 \in \mathbb{R}} S^{m,r_1,r_2}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n). \quad (4.111)
\]

1. Prove that \( S^{-\infty,-\infty,-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \subset S^{m,r_1,r_2}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \) whenever \( m < m', r_1 < r_1', r_2 < r_2' \).

2. Prove the following variant of Proposition 3.13: given \( a_j \in S^{m-j,r_1-j,r_2-j}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \), there exists \( a \in S^{m,r_1,r_2}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \), unique modulo \( S^{-\infty,-\infty,-\infty}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \), such that \( a - \sum_{j=0}^{J-1} \rho_j a_j \in S^{m-J,r_1-J,r_2-J}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \) for all \( J \in \mathbb{N} \).

**Exercise 4.11.** (Scattering ps.d.o.s, I: boundedness.) Let \( m, r_1, r_2 \in \mathbb{R} \). Prove that

\[
\text{Op}(a): \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \quad a \in S^{m,r_1,r_2}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n). \quad (4.112)
\]

Prove this more generally for \( a \in \langle x - y \rangle^w S^{m,r_1,r_2}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n), \) \( w \in \mathbb{R} \).

**Exercise 4.12.** (Scattering ps.d.o.s, II: residual operators.) Let \( r_2 \in \mathbb{R} \). Show that an operator \( A \) can be written as \( A = \text{Op}(a_N) \), \( a_N \in S^{-N,-r_2,-N}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \), for all \( N \) if and only if its Schwartz kernel \( K = K(x,y) \) satisfies \( K \in \mathcal{S}(\mathbb{R}^n_x \times \mathbb{R}_y^n) \). Show that in this case, there exist unique \( a_L, a_R \in S^{-\infty,-\infty}(\mathbb{R}^n \times \mathbb{R}^n) \) such that \( A = \text{Op}_L(a_L) = \text{Op}_R(a_R) \).

\(^3\text{That is, such a are symbolic not only in } \xi, \text{ but also in } x \text{ and } y.\)
Exercise 4.13. (Scattering ps.d.o.s, III: reduction.) We write

\[ S^{m,r}(\mathbb{R}^n; \mathbb{R}^n) \]  

(4.113)

for the space of \( a = a(x, \xi) \in C^\infty(\mathbb{R}^{2n}) \) satisfying \( |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} |x|^{-|\alpha|} |\xi|^{-|\beta|} \) for all \( \alpha, \beta \in \mathbb{N}_0^n \).

Let \( A = \text{Op}(a) \), \( a \in S^{m,r_1,r_2}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n) \). Prove that there exists a unique left symbol \( a_L \in S^{m,r_1+1,r_2}(\mathbb{R}^n; \mathbb{R}^n) \) such that \( A = \text{Op}_L(a_L) \).

Exercise 4.14. (Scattering ps.d.o.s, IV: algebra.) Define

\[ \Psi_{sc}^{m,r}(\mathbb{R}^n) := \text{Op}(S^{m,r}(\mathbb{R}^n; \mathbb{R}^n)). \]  

(4.114)

(1) Prove that \( A \in \Psi_{sc}^{m,r}(\mathbb{R}^n) \) implies \( A^* \in \Psi_{sc}^{m,r}(\mathbb{R}^n) \).
(2) Suppose \( A \in \Psi_{sc}^{m,r}(\mathbb{R}^n) \), \( B \in \Psi_{sc}^{m',r'}(\mathbb{R}^n) \). Prove that

\[ A \circ B \in \Psi_{sc}^{m+m',r+r'}(\mathbb{R}^n). \]  

(4.115)

Exercise 4.15. (Scattering ps.d.o.s, V: principal symbol.) Define the principal symbol of \( A = \text{Op}_L(a_L) \in \Psi_{sc}^{m,r}(\mathbb{R}^n) \) by

\[ c_{sc}^{m,r}(A) := [a_L] \in S^{m,r}(\mathbb{R}^n; \mathbb{R}^n) / S^{m-1,r-1}(\mathbb{R}^n; \mathbb{R}^n). \]  

(4.116)

State and prove the analogue of Proposition 4.21 for scattering ps.d.o.s.

Exercise 4.16. (Scattering ps.d.o.s, VI: ellipticity.) Suppose \( A = \text{Op}_L(a_L) \in \Psi_{sc}^{m,r}(\mathbb{R}^n) \) is elliptic, that is, there exists \( b \in S^{-m,-r}(\mathbb{R}^n; \mathbb{R}^n) \) such that \( a_L b - 1 \in S^{-1,-1}(\mathbb{R}^n; \mathbb{R}^n) \).

(1) Prove that there exists \( B \in \Psi_{sc}^{-m,-r}(\mathbb{R}^n) \) such that \( BA - I \in \Psi_{sc}^{-\infty,-\infty}(\mathbb{R}^n; \mathbb{R}^n) = \bigcap_{m,r \in \mathbb{R}} \Psi_{sc}^{m,r}(\mathbb{R}^n) \).
(2) Suppose \( u \in \mathcal{S}'(\mathbb{R}^n) \), and \( Au = f \in \mathcal{S}(\mathbb{R}^n) \). Prove that \( u \in \mathcal{S}(\mathbb{R}^n) \). (Notice the difference to the statements of Proposition 4.27 or Corollary 4.34! For example, the Laplacian \( \Delta \in \Psi^2(\mathbb{R}^n) \) is uniformly elliptic, but \( \Delta u = 0 \) for \( u = 1 \), \( u = x_1 x_2 \), etc. However, \( \Delta \) is not elliptic as an element of \( \Psi_{sc}^2(\mathbb{R}^n) \). (Check!) What about \( \Delta + 1 \)?)

Exercise 4.17. (Scattering ps.d.o.s, VII: boundedness on Sobolev spaces.)

(1) Prove that elements of \( \Psi_{sc}^{0,0}(\mathbb{R}^n) \) are bounded as maps on \( L^2(\mathbb{R}^n) \).
(2) Show that \( \Lambda_{m,r} := \langle x \rangle^r (D)^m \in \Psi_{sc}^{m,r}(\mathbb{R}^n) \) and \( \Lambda_{m,r}' := \langle D \rangle^m \langle x \rangle^r \in \Psi_{sc}^{m,r}(\mathbb{R}^n) \).
(3) Let \( A \in \Psi_{sc}^{m,r}(\mathbb{R}^n) \). Show that for all \( \rho, \sigma \in \mathbb{R}, A \) is a bounded operator

\[ A : \langle x \rangle^{\rho} H\sigma(\mathbb{R}^n) \rightarrow \langle x \rangle^{\rho+r} H^{\sigma-m}(\mathbb{R}^n). \]  

(4.117)

Exercise 4.18. (Scattering ps.d.o.s, VIII: elliptic scattering ps.d.o.s are Fredholm.)

(1) Let \( m < m' \) and \( r > r' \). Show that the inclusion \( \langle x \rangle^{r'} H^{m'}(\mathbb{R}^n) \hookrightarrow \langle x \rangle^{r} H^{m}(\mathbb{R}^n) \) is compact.
(2) Let \( A \in \Psi_{sc}^{m,r}(\mathbb{R}^n) \) be elliptic (see Exercise 4.16). Show that for any \( \rho, \sigma \in \mathbb{R} \), the operator

\[ A : \langle x \rangle^{\rho} H\sigma(\mathbb{R}^n) \rightarrow \langle x \rangle^{\rho+r} H^{\sigma-m}(\mathbb{R}^n) \]  

is a Fredholm operator.
(3) Show that the index of \( A \) in \((4.118)\) is independent of \( \rho, \sigma \).

\[ \text{Thus, the principal symbol is more powerful in the scattering world: it not only captures the high frequency, i.e. large } \xi, \text{ behavior of an operator, but also the large } x \text{ behavior.} \]
5. PSEUDODIFFERENTIAL OPERATORS ON MANIFOLDS

We now show how the ps.d.o. algebra on $\mathbb{R}^n$ can be transferred to smooth manifolds by using local coordinate charts. The key ingredient for showing that this is a reasonable thing to do is the invariance of the class of $m$-th order ps.d.o.s under changes of coordinates on $\mathbb{R}^n$.

### 5.1 Local coordinate invariance

We now prove the analogue of the final part of Proposition 2.19 for ps.d.o.s.

**Definition 5.1.** Let $\Omega \subset \mathbb{R}^n$ be an open set. Then

$$\Psi^m_c(\Omega) := \{ A \in \Psi^m(\Omega) : \supp K_A \Subset \Omega \times \Omega \},$$

where $K_A \in \mathcal{S}'(\mathbb{R}^{2n})$ denotes the Schwartz kernel of $A$.

**Theorem 5.2.** Suppose $\Omega, \Omega' \subset \mathbb{R}^n$ are open, and $\kappa: \Omega \to \Omega'$ is a diffeomorphism. Given $A \in \Psi^m_c(\Omega')$, define $A_\kappa u := \kappa^* A(\kappa^{-1})^*(u|_\kappa)$. Then $A_\kappa \in \Psi^m_c(\Omega)$, and the map $\Psi^m_c(\Omega') \ni A \mapsto A_\kappa \in \Psi^m_c(\Omega)$ is bijective. Moreover,

$$\sigma_m(A_\kappa)(x, \xi) = \sigma_m(A)(\kappa(x), (\kappa'(x)^T)^{-1}\xi).$$

**Proof.** We have $A = \text{Op}_L(a)$ for some $a \in S^m(\mathbb{R}^n; \mathbb{R}^n)$. Choose $\psi \in C^\infty_c(\Omega')$ such that $\psi(x)\psi(y) = 1$ on $\supp K_A$; thus $K_A(x, y) = \psi(x)K_A(x, y)\psi(y)$, and therefore

$$K_A = \text{Op}(\psi(x)a(x, \xi)\psi(y)).$$

We localize near the diagonal: for $\epsilon > 0$ (to be determined), let $\chi_\epsilon(x, y) \in C^\infty(\mathbb{R}^{2n})$ be such that $\chi_\epsilon(x, y) = 1$ for $|x - y| < \epsilon$ and $\chi_\epsilon(x, y) = 0$ for $|x - y| > 2\epsilon$. Then

$$K_{A_\epsilon} := \text{Op}(a_\epsilon), \quad a_\epsilon(x, y, \xi) = \chi_\epsilon(x, y)\psi(x)\psi(y)a(x, \xi),$$

is the Schwartz kernel of an operator $A_\epsilon \in \Psi^m_c(\Omega')$, and

$$R_\epsilon := A - A_\epsilon$$

is a ps.d.o. with Schwartz kernel supported away from the diagonal, hence $R_\epsilon \in \Psi^{-\infty}(\mathbb{R}^n)$, and its Schwartz kernel satisfies $K_{R_\epsilon} \in C^\infty_c(\Omega' \times \Omega')$. We then have

$$(R_\epsilon)_\kappa u(x) = \int_{\Omega'} K_{R_\epsilon}(\kappa(x), y')u(\kappa^{-1}(y'))\,dy' = \int_{\Omega} K_{R_\epsilon}(\kappa(x), \kappa(y))|\det \kappa'(y)|u(y)\,dy.$$

Therefore, the Schwartz kernel of $(R_\epsilon)_\kappa$ is $K_{(R_\epsilon)_\kappa}(x, y) = K_{R_\epsilon}(\kappa(x), \kappa(y))|\det \kappa'(y)|$ for $x, y \in \Omega$, and 0 otherwise. Thus, $(R_\epsilon)_\kappa \in \Psi^{-\infty}_c(\Omega)$.

It remains to show that $(A_\epsilon)_\kappa \in \Psi^m_c(\Omega)$. To this end, note that

$$(A_\epsilon)_\kappa u(x) = (2\pi)^{-n} \iint e^{i(x-y)\xi'} a_\epsilon(\kappa(x), y', \xi')u(\kappa^{-1}(y'))\,dy'\,d\xi' = (2\pi)^{-n} \iint e^{i(x-y)\xi'} a_\epsilon(\kappa(x), \kappa(y), \xi')|\det \kappa'(y)|u(y)\,dy\,d\xi'.$$
thus the Schwartz kernel of \((A_\epsilon)_\kappa\) is

\[
K_{(A_\epsilon)_\kappa}(x, y) = (2\pi)^{-n} \int e^{i(\kappa(x) - \kappa(y))\xi'} b_\epsilon(x, y, \xi') \, d\xi',
\]

where \(\Phi(x, y)\) is invertible for \((x, y)\) in the set (5.12). Note now that

\[
\Phi(x, x) = \kappa'(x)
\]

is invertible for \(x \in \Omega\) since \(\kappa\) is a diffeomorphism. For

\[
(x, y) \in \text{supp} \, \chi_\epsilon(\kappa(x), \kappa(y))\psi(\kappa(x))\psi(\kappa(y)),
\]

we have \((x, y) \in \kappa^{-1}(\text{supp} \, \psi) \times \kappa^{-1}(\text{supp} \, \psi) \subseteq \Omega \times \Omega\) and \(|\kappa(x) - \kappa(y)| \leq 2\epsilon\). Therefore, we can choose \(\epsilon > 0\) such that \(\Phi(x, y)\) is invertible for \((x, y)\) in the set (5.12). In (5.8), we can then make the change of variables \(\xi' = (\Phi(x, y)^T)^{-1}\xi\), so

\[
(A_\epsilon)_\kappa = \text{Op}(c_\epsilon), \quad c_\epsilon(x, y, \xi) = b_\epsilon(x, y, (\Phi(x, y)^T)^{-1}\xi)|\text{det}(\Phi(x, y))|^{-1}
= a_\epsilon(\kappa(x), \kappa(y), (\Phi(x, y)^T)^{-1}\xi)|\text{det}(\Phi(x, y))|^{-1}|\text{det} \kappa'(y)|;
\]

it remains to check that \(c_\epsilon \in S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)\). We can drop the (smooth) Jacobian factors. We then compute

\[
\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma(a_\epsilon(\kappa(x), \kappa(y), (\Phi(x, y)^T)^{-1}\xi))
= \sum_{|\alpha| + |\beta| + |\gamma| \leq |\alpha'| + |\beta'| + |\gamma'| + |\beta''| + |\beta'''|} F^{\beta''\beta'''}_{\alpha''\alpha'''}(x, y)\xi^{\alpha'' + \beta''} \partial_x^{\alpha'} \partial_y^{\beta'} \partial_\xi^{\gamma + \alpha'' + \beta'''} a_\epsilon(\kappa(x), \kappa(y), (\Phi(x, y)^T)^{-1}\xi)\]

for some smooth functions \(F^{\beta''\beta'''}_{\alpha''\alpha'''}(x, y) \in C^\infty_\epsilon(\Omega \times \Omega)\). Clearly, this is bounded by a constant times \(|(\Phi(x, y)^T)^{-1}\xi|^{m-n}|\xi|\). Now, since \(\Phi(x, y)^T\) and its inverse are uniformly bounded on \(\text{supp} \, a_\epsilon\), there exist \(c, C > 0\) such that \(c|\xi| \leq |(\Phi(x, y)^T)^{-1}\xi| \leq C|\xi|\) on \(\text{supp} \, a_\epsilon\). Therefore, (5.14) is bounded by a constant times \(|\xi|^m|\xi|^{-n}\) on \(\text{supp} \, a_\epsilon\), proving \(c_\epsilon \in S^m(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n)\).

As for the principal symbol, we have \(\sigma_m((A_\epsilon)_\kappa) = \sigma_m(A_\epsilon) = \sigma_m(\text{Op}(c_\epsilon))\), which can be read off from the (first term of the) reduction formula (4.34): using (5.11), it is given by the equivalence class in \(S^m(\mathbb{R}^n; \mathbb{R}^n)/S^{m-1}(\mathbb{R}^n; \mathbb{R}^n)\) of

\[
c_\epsilon(x, x, \xi) = b_\epsilon(x, x, (\kappa'(x)^T)^{-1}\xi)|\text{det} \kappa'(x)|^{-1}
= a_\epsilon(\kappa(x), \kappa(x), (\kappa'(x)^T)^{-1}\xi)
= a(\kappa(x), (\kappa'(x)^T)^{-1}\xi).
\]

The proof is complete. \(\square\)
5.2. Manifolds, vector bundles, densities. We shall only work with smooth manifolds: they are locally diffeomorphic to the unit ball \( B(0,1) = \{ x \in \mathbb{R}^n : |x| < 1 \} \). We recall the ‘hands-on’ definition of smooth manifolds:

**Definition 5.3.** Let \( n \in \mathbb{N} \). A smooth manifold of dimension \( n \) is a second countable, paracompact Hausdorff space \( M \) such that

1. for each point \( p \in M \), there exist an open neighborhood \( U_p \ni p \) and a homeomorphism \( F_p : U_p \to B(0,1) \subset \mathbb{R}^n \);
2. for all \( p, q \in M \) such that \( U_p \cap U_q \neq \emptyset \), the transition map \( F_p \circ F_q^{-1} \big|_{U_p \cap U_q} : F_q(U_p \cap U_q) \to F_p(U_p \cap U_q) \) is smooth (as a map between open subsets of \( \mathbb{R}^n \)).

**Definition 5.4.** Let \( M \) be a smooth manifold. A atlas on \( M \) is a collection \( \{ (U_\alpha, F_\alpha) \} \) of pairs \( (U_\alpha, F_\alpha) \), with \( U_\alpha \neq \emptyset \) and \( M = \bigcup_\alpha U_\alpha \), such that \( F_\alpha : U_\alpha \to \mathbb{R}^n \) is a diffeomorphism onto an open subset \( F_\alpha(U_\alpha) \) of \( \mathbb{R}^n \). A maximal atlas, or smooth structure, is an atlas with the property that any other atlas is contained in it. An element of the\(^5\) maximal atlas is called a (local coordinate) chart.

The ‘hands-on’ definition of vector bundles is the following.

**Definition 5.5.** Let \( M \) be a smooth \( n \)-dimensional manifold. A real vector bundle of rank \( k \) over \( M \) is a triple \((\pi, E, M)\) with the following properties:

1. \( E \) is a smooth \((n+k)\)-dimensional manifold, and \( \pi : E \to M \) is smooth;
2. for each \( p \in M \) there exist an open neighborhood \( U_p \subset M \), \( p \in U_p \), and a diffeomorphism \( \tau_p : \pi^{-1}(U_p) \to U_p \times \mathbb{R}^k \) such that \( \pi(\tau_p^{-1}(q,v)) = q \) is the projection onto the first factor;
3. for all \( p, q \in M \) such that \( U_p \cap U_q \neq \emptyset \), the transition map

\[ \tau_{pq} := \tau_p \circ \tau_q^{-1} \big|_{(U_p \cap U_q) \times \mathbb{R}^k} : (U_p \cap U_q) \times \mathbb{R}^k \to (U_p \cap U_q) \times \mathbb{R}^k \]

(5.17)

takes the form \( \tau_{pq}(r,v) = (r, \Phi_{pq}(r)v) \), where \( \Phi_{pq} : U_p \cap U_q \to GL(k) \) is smooth.

The fiber of \( E \) over \( p \in M \) is denoted \( E_p := \pi^{-1}(p) \); it is a \( k \)-dimensional real vector space. The zero section of \( E \) is the submanifold of \( E \) given locally in \( \pi^{-1}(U_p) \) by \( \tau_p^{-1}(U_p \times \{0\}) \). A smooth section of \( E \) is a smooth map \( s : M \to E \) such that \( \pi \circ s = Id_M \). The space of smooth sections is denoted \( \mathcal{C}^\infty(M;E) \).

Another useful notion for later is the pullback of vector bundles:

**Definition 5.6.** Let \( M, N \) be smooth manifolds (not necessarily of the same dimension), and let \( f : M \to N \) be smooth. If \( \pi : E \to N \) is a vector bundle, then the pullback of \( E \) by \( f \) is the vector bundle \( \pi^* : f^* E \to N \)

(5.18)

given by \( f^* E = M_f \times_N E = \{(p,e) \in M \times E : f(p) = \pi(e)\} \), with projection map \( \pi^*(p,e) = p \), and with linear structure on \( (f^* E)_p = E_{f(p)} \) equal to that on \( E_{f(p)} \).

To specify a real rank \( k \) vector bundle uniquely (up to vector bundle isomorphisms), it suffices to have the following data and conditions:

\(^5\) A maximal atlas always exists and is unique.
For instance, given a linear map $A$, the resulting vector bundle is the cotangent bundle of Example 5.7, we thus take a bundle of $A$ on the base $M$ formula (5.22) appears in (5.2) (except in the latter we also use/change local coordinates $df$).

Indeed, one can then set

$$E := \left( \bigsqcup_{\alpha} U_{\alpha} \times \mathbb{R}^k \right) / \sim,$$

(5.19)

where we define the equivalence relation $\sim$ by

$$U_{\alpha} \times \mathbb{R}^k \ni (p,v) \sim (q,w) \in U_{\beta} \times \mathbb{R}^k \iff p = q, \; \tau_{\alpha\beta}(q,w) = (p,v).$$

(5.20)

(The cocycle condition guarantees that this is transitive, while reflexivity follows from the cocycle condition together with $\tau_{\alpha\alpha} = \text{Id}$.) The projection map $\pi: E \to M$ is simply given by $\pi([(p,v)]) = p$. As local trivializations, we can take

$$\tau_\alpha: \{(p,v)\}: p \in U_\alpha \mapsto (p,v) \in U_\alpha \times \mathbb{R}^k.$$ (5.21)

Example 5.7. Taking a cover of an $n$-dimensional manifold $M$ by coordinate charts $F_i: U_i \to \mathbb{R}^n$, we take $\tau_{ij}(p,v) = (p,(F_i \circ F_j^{-1})'(\Phi_j(p)v))$. The resulting vector bundle is the tangent bundle of $M$, denoted $TM$.

Functorial operations on vector spaces give corresponding operations on vector bundles. For instance, given a linear map $A: V \to W$ between to vector spaces, the adjoint is $A^T: W^* \to V^*$; if $A$ is invertible, then this gives a map $(A^T)^{-1}: V^* \to W^*$. In the notation of Example 5.7, we thus take

$$\tau_{ij}(p,v) = (p,((\kappa_{ij}|_{F_i(p)v})^T)^{-1}v), \quad \kappa_{ij} := F_i \circ F_j^{-1}.$$ (5.22)

The resulting vector bundle is the cotangent bundle, denoted $\pi: T^*M \to M$. Note that formula (5.22) appears in (5.2) (except in the latter we also use/change local coordinates on the base $M$ via $\kappa_{ij}$). We recall then that given a smooth function $f \in C^\infty(M)$, we can define its exterior derivative $df \in C^\infty(M; T^*M)$ as follows: if $F_i: U_i \to F_i(U_i) \subset \mathbb{R}^n$ is a coordinate chart, we define

$$\tau_i(df(p)) := (p, (\partial_{x_j}((F_i^{-1})^*f))_{j=1,\ldots,n}) \in U_i \times \mathbb{R}^n, \quad p \in U_i.$$ (5.23)

That this indeed gives a well-defined section of $T^*M$ follows from the change of variables formula.

A natural choice for $f$ in local coordinates near $p$ is $f = x_k$ (i.e. $f = F_i^*x_k$), in which case (5.23) defines the differential $dx_k$ with $\tau_i(dx_k) = (p, (0,\ldots,1,\ldots,0))$ (with the 1 in the $j$-th slot). Thus, a coordinate system $F: U \to F(U) \subset \mathbb{R}^n$ induces a canonical trivialization of $T^*_U M = \pi^{-1}(U)$:

$$\tau: T^*_UM \to F(U) \times \mathbb{R}^n, \quad \sum_{k=1}^n \xi_k(x)dx_k \mapsto (x,\xi) = ((x_1,\ldots,x_n),(\xi_1,\ldots,\xi_n)).$$ (5.24)

(Thus (5.23) is usually written as $df = \sum_{j=1}^n (\partial_{x_j}f)dx^j$, dropping the coordinate and trivialization maps from the notation.)
**Example 5.8.** Let $E \to M$ and $F \to M$ denote two vector bundles.

1. The fiberwise direct sum of vector spaces produces the vector bundle $E \oplus F \to M$, with fibers $(E \oplus F)_p = E_p \oplus F_p$.
2. Likewise, taking the fiberwise tensor product gives $E \otimes F \to M$, with fibers $(E \otimes F)_p = E_p \otimes F_p$.
3. The vector bundle $\text{Hom}(E, F) \to M$ has fibers $\text{Hom}(E_p, F_p)$. We have $\text{Hom}(E, F) = E \otimes F^*$.
4. Let $q \in \mathbb{N}$. The fiberwise $q$-th exterior power of $E$ produces the vector bundle $\Lambda^q E \to M$. In the special case $E = T^* M$, one often writes $\Lambda^q M := \Lambda^q T^* M$.

We discuss another important vector bundle, closely related to the top exterior power $\Lambda^n T^* M$ of the cotangent bundle of an $n$-dimensional manifold $M$:

**Definition 5.9.** Let $\alpha \in \mathbb{R}$. in the notation of Example 5.7, the $\alpha$-density bundle on $M$ is the vector bundle

$$\Omega^\alpha M \to M$$

with transition functions $\tau_{ij}(p, v) = (p, |\det \kappa_{ij}^\alpha|_{F_i(p)}^{-\alpha} p)$, $\kappa_{ij} = F_i \circ F_j^{-1}$. We also write

$$\Omega M := \Omega^1 M.$$  

**Remark 5.10.** $\Omega^\alpha M \to M$ arises functorially from the following operation on vector spaces, applied to $TM$: given a real $n$-dimensional vector space $V$, we define

$$\Omega^\alpha V := \{ \omega : \Lambda^\alpha V \to \mathbb{R} : \omega(\mu v) = |\mu|^\alpha \omega(v), \ v \in \Lambda^\alpha V, \ \mu \in \mathbb{R} \}.$$  

(5.27)

To see the relationship, note first that $\Lambda^q V$ is 1-dimensional. Then, given another $n$-dimensional vector space $W$ and a map $\kappa : V \to W$, let us fix bases $e_1, \ldots, e_n$ of $V$ and $f_1, \ldots, f_n$ of $W$. Consider, as a warm-up, the top exterior powers: $e_1 \wedge \cdots \wedge e_n$ and $f_1 \wedge \cdots \wedge f_n$ are bases of $\Lambda^n V$ and $\Lambda^n W$, and the map $\Lambda^n \kappa : \Lambda^n V \to \Lambda^n W$ is given by $e_1 \wedge \cdots \wedge e_n \mapsto \kappa(e_1) \wedge \cdots \wedge \kappa(e_n) = (\det \kappa) f_1 \wedge \cdots \wedge f_n$, where $\det \kappa$ is the determinant of the matrix of $\kappa$ in these bases: that is, in the stated basis, $\Lambda^n \kappa$ is simply multiplication by $\det \kappa$. Similarly, $\Omega^\alpha V$ and $\Omega^\alpha W$ are 1-dimensional, with basis elements $\omega_V : \mu e_1 \wedge \cdots \wedge e_n \mapsto |\mu|^\alpha$ and $\omega_W : \mu f_1 \wedge \cdots \wedge f_n \mapsto |\mu|^\alpha$. Now, the map

$$\Omega^\alpha \kappa : \Omega^\alpha V \to \Omega^\alpha W$$

(5.28)

is given by

$$\Omega^\alpha \kappa(\omega)(f_1 \wedge \cdots \wedge f_n) = \omega(\kappa^{-1}(f_1) \wedge \cdots \wedge \kappa^{-1}(f_n)),$$

hence $\Omega^\alpha \kappa(\omega_V) = |\det \kappa|^{-\alpha} \omega_W$.

(5.29)

The proof of the following simple lemmas is left to the reader as a simple exercise.

**Lemma 5.11.** Let $\alpha, \beta \in \mathbb{R}$. Then

1. $(\Omega^\alpha)^* M = \Omega^{-\alpha} M$,
2. $\Omega^\alpha M \otimes \Omega^\beta M = \Omega^{\alpha+\beta} M$,
3. $\Omega^0 M = M \times \mathbb{R}$.

If $x \in \mathbb{R}^n$ denotes local coordinates on a manifold $M$, then a typical $\alpha$-density is

$$|dx|^\alpha : \mu \partial_{x_1} \wedge \cdots \wedge \partial_{x_n} \mapsto |\mu|^\alpha.$$  

(5.30)

Similarly to differential forms, $\alpha$-densities can be pulled back by smooth maps:
Lemma 5.12. Let $f: M \to N$ be a smooth map between smooth manifolds. In local coordinates $x, y$ on $M, N$, and $u(y) = u_0(y)|dy|^{\alpha}$, define $(f^* u)(x) = u_0(f(x))|\det f'(x)|^{\alpha}|dx|^{\alpha}$. Then $f^*$ is a well-defined map

$$f^*: C^\infty(N; \Omega^\alpha N) \to C^\infty(M; \Omega^\alpha M).$$

(5.31)

For us, 1-densities are the most useful: sections of $\Omega M$ can be invariantly integrated. On $\mathbb{R}^n$, we write for $u \in C^\infty_c(\mathbb{R}^n; \Omega\mathbb{R}^n)$, $u = u_0(x)|dx|:

$$\int_{\mathbb{R}^n} u := \int_{\mathbb{R}^n} u_0(x) \, dx$$

(5.32)

Let $\{\phi_i\}$ be a partition of unity on $M$ subordinate to a cover by coordinate systems $F_i: U_i \to F_i(U_i) \subset \mathbb{R}^n$ with $\overline{U}_i$ compact. Define the map

$$\int_{M} : C^\infty_c(M; \Omega M) \to \mathbb{R}, \quad u \mapsto \sum_i \int_{\mathbb{R}^n} (F_i^{-1})^*(\phi_i u).$$

(5.33)

(Note here that $(F_i^{-1})^*(\phi_i u) \in C^\infty_c(\mathbb{R}^n; \Omega\mathbb{R}^n)$.)

Proposition 5.13. The map (5.33) is independent of the choice of local coordinates and the partition of unity.

Proof. First, suppose $u \in C^\infty_c(M; \Omega M)$ is supported in the intersection of two coordinate charts, with local coordinates $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ and transition function $x = \kappa(y)$, then

$$u(x) = u_0(x)|dx| = u_1(y)|dy|.$$

(5.34)

But at $x = \kappa(y)$, we have $(\Omega^1 \kappa)|dy| = |\det \kappa'(y)|^{-1}|dx|$, so $|dx| = |\det \kappa'(y)||dy|$. Therefore,

$$u_1(y) = u_0(\kappa(y))|\det \kappa'(y)|,$$

and thus

$$\int_{\mathbb{R}^n} u_1(y) \, dy = \int_{\mathbb{R}^n} u_0(\kappa(y))|\det \kappa'(y)| \, dy = \int_{\mathbb{R}^n} u_0(x) \, dx.$$

(5.35)

The proposition follows easily from this: if $\{\psi_j\}$ is another partition of unity subordinate to a cover by coordinate systems $G_j: V_j \to G_j(U_j) \subset \mathbb{R}^n$, then $\int_M u = \sum_j \int_M \psi_j u$, and

$$\int_M u = \sum_j \int_M \psi_j u$$

$$= \sum_{i,j} \int_{\mathbb{R}^n} (F_i^{-1})^*(\phi_i \psi_j u)$$

$$= \sum_{i,j} \int_{\mathbb{R}^n} (G_j \circ F_i^{-1})^*((G_j^{-1})^*(\psi_j \phi_i u))$$

$$= \sum_{i,j} \int_{\mathbb{R}^n} (G_j^{-1})^*(\psi_j \phi_i u)$$

$$= \sum_j \int_{\mathbb{R}^n} (G_j^{-1})^* \psi_j u.$$

In analogy with the case of $\mathbb{R}^n$, this leads us to define distributions on a manifold as follows:
Definition 5.14. The space $\mathcal{D}'(M)$ consists of all continuous linear maps $C^\infty_c(M; \Omega M) \to \mathbb{C}$. More generally, if $E \to M$ is a vector bundle, then $\mathcal{D}'(M; E)$ consists of all continuous linear functionals $C^\infty_c(M; E^* \otimes \Omega M) \to \mathbb{C}$. The space $\mathcal{E}'(M; E)$ consists of all continuous linear functionals $C^\infty(M; E^* \otimes \Omega M) \to \mathbb{C}$.

Thus, $C^\infty(M; E) \hookrightarrow \mathcal{D}'(M; E)$ via the pairing

$$C^\infty_c(M; E^* \otimes \Omega M) \times C^\infty(M; E) \ni (u, \phi) \mapsto \int_M \langle u(p), \phi(p) \rangle,$$

(5.36)

where $\langle \cdot, \cdot \rangle : E^* \times E \to \mathbb{R}$ is the dual pairing; note that $\langle u, \phi \rangle \in C^\infty_c(M; \Omega M)$ can indeed be invariantly integrated by Proposition 5.13.

The support and singular of a distribution are defined analogously to the local (\(\mathbb{R}^n\)) case, see Definition 2.6. The space $\mathcal{E}'(M; E) \subset \mathcal{D}'(M; E)$ is, as in the local theory (on $\mathbb{R}^n$), the space of distributions with compact support. (Without further structure, there is no natural analogue of the space of Schwartz functions or tempered distributions on a general smooth manifold.)

Example 5.15. Let $p \in M$, then $\delta_p \in \mathcal{E}'(M; \Omega M)$ is the distribution defined by mapping $\phi \in C^\infty_c(M)$ to $\phi(p)$.

The Schwartz kernel theorem in this context is the following.

Theorem 5.16. Let $M$ be a smooth $n$-dimensional manifold, and let $E, F \to M$ be two vector bundles. Then there is a one-to-one correspondence between continuous linear operators $A : C^\infty_c(X; E) \to \mathcal{D}'(X; F)$ and distributional Schwartz kernels $K \in \mathcal{D}'(M \times M; \pi_L^* F \otimes \pi_R^* (E^* \otimes \Omega M))$, where $\pi_L, \pi_R : M \times M \to M$ are the projections onto the left/right factor, so $\pi_L(p, q) = p$, $\pi_R(p, q) = q$. This correspondence is given by assigning to $K$ the operator $O_K : C^\infty_c(X; E) \to \mathcal{D}'(X; F)$, defined as

$$(O_K \phi)(\psi) = \langle K, \pi_L^* \psi \otimes \pi_R^* \phi \rangle, \quad \phi \in C^\infty_c(M; E), \ \psi \in C^\infty_c(M; F^* \otimes \Omega M).$$

(5.37)

5.3. Differential operators on manifolds. Let $M$ be a smooth $n$-dimensional manifold. Before we talk about ps.d.o.s on $M$, let us think about differential operators.

Definition 5.17. The space of smooth vector fields on $M$ is $\mathcal{V}(M) := C^\infty_c(M; TM)$.

An element $V \in \mathcal{V}(M)$ can be regarded as a differential operator by assigning

$$C^\infty(M) \ni f \mapsto Vf \in C^\infty(M), \quad (Vf)(p) = df(p)(V(p))$$

(5.38)

Definition 5.18. (1) We define $\text{Diff}^0(M) = C^\infty(M)$.

(2) We define $\text{Diff}^1(M)$ as the space of all operators $A : C^\infty(M) \to C^\infty(M)$ of the form $Au = Vf + fu$ with $V \in \mathcal{V}(M), \ f \in C^\infty(M)$.

(3) Let $m \in \mathbb{N}_0$. Then $\text{Diff}^m(M)$ is the space of all operators $A : C^\infty(M) \to C^\infty(M)$ which are of the form

$$Au = \sum_{k=1}^K A_{k1} \cdots A_{kJ_k} u, \quad A_{kj} \in \text{Diff}^1(M), \ K \in \mathbb{N}, \ N_k \leq m.$$ 

(5.39)

(Check that this agrees with the standard local coordinate definition.) Of course, differential operators also map $C^\infty_c(M) \to C^\infty_c(M), \ \mathcal{D}'(M) \to \mathcal{D}'(M), \ \mathcal{E}'(M) \to \mathcal{E}'(M)$. What
are the Schwartz kernels of differential operators? The Schwartz kernel \( K_I \) of the identity operator \( I \in \text{Diff}^0(M) \) should be
\[
K_I(x, y) = \delta(x - y). \tag{5.40}
\]
To make precise sense of this, define the projections
\[
\pi_L: M^2 \to M, \quad (p, q) \mapsto p,
\]
\[
\pi_R: M^2 \to M, \quad (p, q) \mapsto q, \tag{5.41}
\]
and define the right density bundle by
\[
\Omega_R := \pi^*_R(\Omega M) \tag{5.42}
\]
Thus, integration in the second variable is a well-defined map
\[
\mathcal{D}'(M^2; \Omega_R) \times C^\infty_c(M) \ni (K, u) \mapsto \int_M K(\cdot, y)u(y) \in \mathcal{D}'(M). \tag{5.43}
\]
By the Schwartz kernel theorem, every continuous map \( C^\infty_c(M) \to \mathcal{D}'(M) \) is of this type! Thus, (5.40) is well-defined as an element
\[
K_I \in \mathcal{D}'(M^2; \Omega_R). \tag{5.44}
\]
Remark 5.19. As a check, recall that \( K_I \) acts on elements of\(^6\)
\[
C^\infty_c(M^2; \Omega R) \otimes (\Omega R)^* = C^\infty_c(M^2; \Omega L), \tag{5.45}
\]
and indeed maps \( u \in C^\infty_c(M^2; \Omega_L) \) into \( \int_M u(x, x) \), defined by Proposition 5.13. (Note that restriction to the diagonal gives a map \( C^\infty_c(M^2; \Omega_L) \to C^\infty_c(M; \Omega M) \) by Lemma 5.12.)

Given \( A \in \text{Diff}^m(M) \), its Schwartz kernel \( K_A \in \mathcal{D}'(M^2; \Omega_R) \) is then given by
\[
K_A = (\pi^*_L A)K_I, \tag{5.46}
\]
where \( \pi^*_L A \) denotes the lift of \( A \) to the first factor, i.e. differentiating only in the first factor of \( M^2 \). Check that this is well-defined in the following general context: if \( \pi: E \to M \) is a vector bundle, then
\[
(\pi^*_L A)K \in \mathcal{D}'(M^2; \pi^*_R E), \quad ((\pi^*_L A)K)(\cdot, y) = (AK)(\cdot, y), \quad y \in M, \quad K \in \mathcal{D}'(M^2; \pi^*_R E), \tag{5.47}
\]
is well-defined.

5.4. Definition of \( \Psi^m(M) \). We continue to denote by \( M \) a smooth \( n \)-dimensional manifold, and use the notation (5.41). The following definition captures what we would like pseudodifferential operators on a manifold (not necessarily compact) to be: their Schwartz kernels should, near the diagonal, be Schwartz kernels of ps.d.o.s on \( \mathbb{R}^n \) in local coordinates, while away from the diagonal they are simply smooth.

**Definition 5.20.** Let \( M \) be a smooth \( n \)-dimensional manifold. Let \( m \in \mathbb{R} \). Then \( \Psi^m(M) \) is the space of linear operators
\[
A: C^\infty_c(M) \to C^\infty(M) \tag{5.48}
\]
with the following properties:

\(^6\text{This uses that } \Omega(M^2) = \Omega_L \otimes \Omega_R.\)
(1) if \( \phi, \psi \in \mathcal{C}^\infty(M) \) have \( \text{supp} \phi \cap \text{supp} \psi = \emptyset \), then there exists \( K \in \mathcal{C}^\infty(M^2; \Omega_R) \) such that
\[
\phi A(\psi u) = \int_M K(\cdot, y)u(y), \quad u \in \mathcal{C}^\infty_c(M). \tag{5.49a}
\]

(2) if \( F: U \to \mathbb{R}^n \) is a diffeomorphism from an open set \( \emptyset \neq U \subset M \) to \( F(U) \), and if \( \psi \in \mathcal{C}^\infty_c(U) \), then there exists \( B \in \Psi^m(F(U)) \subset \Psi^m(\mathbb{R}^n) \) (see Definition 5.1) such that on \( U \), we have
\[
\psi A(\psi u) = F^*(B((F^{-1})^*(\psi u))), \quad u \in \mathcal{C}^\infty_c(M). \tag{5.49b}
\]

Remark 5.21. Taking as the smooth manifold \( M = \mathbb{R}^n \), the space \( \Psi^m(M) \) defined here is much larger than the space \( \Psi^m(\mathbb{R}^n) \) of uniform pseudodifferential operators defined in §4. (One reason is that we do not constrain the size of Schwartz kernels away from the diagonal \( \Delta_M = \{(p, p) : p \in M\} \).) To avoid confusion, one should denote the latter space by \( \Psi^m_\infty(\mathbb{R}^n) \). When working on \( \mathbb{R}^n \), we shall, in these notes, only ever employ operators in the uniform algebra, hence we shall right away drop the \( \infty \) subscript again!

Ps.d.o.s act on distributions with compact support. We give a direct proof here, and defer a ‘better’ proof in the spirit of (4.31) to later; see Corollary 5.40.

**Proposition 5.22.** Let \( A \in \Psi^m(M) \). Then \( A \) extends by continuity from \( \mathcal{C}^\infty_c(M) \) to a bounded linear operator
\[
A: \mathcal{E}'(M) \to \mathcal{D}'(M). \tag{5.50}
\]

**Proof.** Fix a cover of \( M \) by coordinate systems \( F_i: U_i \to F_i(U_i) \subset \mathbb{R}^n \) with \( \overline{U_i} \) compact, and let \( \{\phi_i\}, \phi_i \in \mathcal{C}^\infty_c(U_i) \), be a subordinate partition of unity. Fix \( \tilde{\phi}_i \in \mathcal{C}^\infty_c(U_i) \) with \( \tilde{\phi}_i = 1 \) near \( \text{supp} \phi_i \). By (5.49b), we can write
\[
\tilde{\phi}_i A \phi_i = \tilde{\phi}_i A \phi_i \phi_i = F_i^* B_i (F_i^{-1})^* \phi_i, \quad B_i \in \Psi^m_c(F_i(U_i)). \tag{5.51}
\]

Let now \( u \in \mathcal{E}'(M) \), then \( \phi_i u \neq 0 \) only for finitely many \( i \). We then set
\[
\tilde{A} u := \sum_i F_i^* B_i (F_i^{-1})^* (\phi_i u) + \sum_i (1 - \tilde{\phi}_i) A \phi_i u. \tag{5.52}
\]

Each one of the finitely many non-zero summands in the first sum is a pullback from \( \mathbb{R}^n \) of a tempered distribution with compact support, hence lies in \( \mathcal{E}'(M) \). The second (also finite) sum lies in \( \mathcal{C}^\infty(M) \) by (5.49a).

For \( u \in \mathcal{C}^\infty_c(M) \), we clearly have \( \tilde{A} u = A u \). Since \( \mathcal{C}^\infty_c(M) \subset \mathcal{E}'(M) \) is dense, (5.52) defines the unique continuous extension of \( A \) to \( \mathcal{E}'(M) \) (which, of course, we call \( A \) simply, rather than \( \tilde{A} \)). \qed

We also note the following, which follows directly from the definition:

**Lemma 5.23.** The space \( \Psi^{-\infty}(M) \) consists of all operators which have a Schwartz kernel in \( \mathcal{C}^\infty(M^2; \Omega_R) \). Equivalently, \( \Psi^{-\infty}(M) \) is the space of all bounded linear operators \( \mathcal{E}'(M) \to \mathcal{C}^\infty(M) \).

To get a more manageable characterization of \( \Psi^m(X) \), we first prove:
Lemma 5.24. Let $M$ be an $n$-dimensional manifold, and let $F: U \to F(U) \subset \mathbb{R}^n$ be a coordinate patch. If $B \in \Psi^m_c(F(U)) \subset \Psi^m(\mathbb{R}^n)$, then the operator $A: C^\infty_c(M) \to C^\infty(M)$ defined by
\[
Au = F^*B(F^{-1})^*(u|_U), \quad u \in C^\infty_c(M),
\]
on $U$, and $Au = 0$ in $M \setminus U$, defines an element $A \in \Psi^m(M)$.

Proof. We first check (5.49a): given $\phi, \psi \in C^\infty(M)$ with $\text{supp}\, \phi \cap \text{supp}\, \psi = \emptyset$, we have for $u \in C^\infty_c(M)$
\[
\phi A(\psi u) = F^*B'(F^{-1})^*(u|_U), \quad B' := ((F^{-1})^*\phi)B((F^{-1})^*\psi) \in \Psi^{-\infty}(F(U)) \subset \Psi^{-\infty}(\mathbb{R}^n),
\]
where we used that $\text{supp}((F^{-1})\phi) \cap \text{supp}((F^{-1})\psi) = \emptyset$. Since the Schwartz kernel of $B'$ is smooth, we obtain (5.49a). (In more detail, if $K_{B'} \in C^\infty_c(F(U) \times F(U))$ denotes the Schwartz kernel of $B'$, then (5.49a) holds for $K(x, y) := F^*(K_{B'}(x, y)|dy|$.)

As for (5.49b), suppose $G: V \to G(V) \subset \mathbb{R}^n$ is another coordinate patch, and let $\chi \in \Psi^m_c(M)$, $\text{supp}\, \chi \subset V$. Then
\[
B_1 := ((F^{-1})^*\chi)B((F^{-1})^*\chi) \in \Psi^m_c(F(U \cap V)).
\]
Denote the change of coordinates by $\kappa = F \circ G^{-1}: G(U \cap V) \to F(U \cap V)$, then
\[
B_2 := (B_1)_\kappa = \kappa^*B_1(\kappa^{-1})^* \in \Psi^m_c(G(U \cap V))
\]
by Theorem 5.2. Therefore,
\[
\chi A(\chi u) = F^*B_1(F^{-1})^*u|_{U \cap V} = G^*\kappa^*B_1(\kappa^{-1})^*(G^{-1})^*u|_{U \cap V} = G^*B_2(G^{-1})^*u|_{U \cap V},
\]
as desired. \hfill \qed

This already implies that there are lots of pseudodifferential operators on $M$, given by locally finite sums of operators of the form (5.53). This gives almost (namely, up to operators with smooth integral kernels) all of $\Psi^m(X)$:

Theorem 5.25. Let $M$ be an $n$-dimensional manifold, and let $M = \bigcup_i U_i$ be a locally finite open cover by coordinate charts $F_i: U_i \to F_i(U_i) \subset \mathbb{R}^n$ with $U_i$ compact. Let $A: C^\infty_c(M) \to C^\infty(M)$ be a linear operator. Then $A \in \Psi^m(M)$ if and only if there exist operators $B_i \in \Psi^m_c(F_i(U_i))$ and a section $K \in C^\infty(M^2; \Omega_R)$ such that
\[
A = K + \sum_i F_i^*B_i(F_i^{-1})^*.
\]

Proof. If $A$ is of the form (5.58), then $A \in \Psi^m(X)$ by the previous lemma. Conversely, suppose $A \in \Psi^m(X)$. Let $\{\phi_i\}$, $\phi_i \in C^\infty_c(U_i)$, be a partition of unity subordinate to $\{U_i\}$, i.e. $\text{supp}\, \phi_i \subset U_i$, and $\sum_i \phi_i(x) = 1$ for all $x \in M$. Choose $\tilde{\phi}_i \in C^\infty_c(U_i)$ with $\text{supp}\, \tilde{\phi}_i \subset U_i$ and $\tilde{\phi}_i \equiv 1$ near $\text{supp}\, \phi_i$. For $u \in C^\infty_c(M)$, we then have
\[
Au = \sum_i \tilde{\phi}_i A\phi_i u + \sum_i (1 - \tilde{\phi}_i) A\phi_i u.
\]
By definition, $(1 - \tilde{\phi}_i)A\phi_i$ has a smooth Schwartz kernel $K_i \in C^\infty(M^2; \Omega_R)$; since $\text{supp}\, K_i$ is locally finite, we can define
\[
K := \sum_i K_i \in C^\infty(M^2; \Omega_R).
\]
Considering a term $\tilde{\phi}_i A \phi_i$ in the first sum in (5.59), note that
\[ \phi_i A(\phi_i u) = \tilde{\phi}_i A \phi_i (\phi_i u). \] (5.61)
But $\tilde{\phi}_i A \phi_i = F_i^* B_i'(F_i^{-1})^* \phi_i$ for some $B_i' \in \Psi^m(F_i(U_i))$, and therefore
\[ \tilde{\phi}_i A \phi_i = F_i^* B_i'(F_i^{-1})^*, \quad B_i u := B'_i((F_i^* \phi_i) u), \quad u \in C_c^\infty(M), \] with $B_i \in \Psi^m(F_i(U_i))$, as desired. \hfill \Box

When $M$ is not compact, one can in general not compose two ps.d.o.s, even when both are of order $-\infty$, since they only act on $C_c^\infty(M)$, but not on $C^\infty(M)$ in general, the problem being potential growth of the Schwartz kernel away from the diagonal. The simplest cure is to place an additional assumption on the Schwartz kernels:

**Definition 5.26.** We say that $A \in \Psi^m(M)$, with Schwartz kernel $K \in \mathcal{D}'(M^2; \Omega_R)$, is **properly supported** if the projection maps $\pi_i: \text{supp } K \to M$ and $\pi_R: \text{supp } K \to M$ are proper, i.e. preimages of compact sets are compact.

Thus, properly supported operators are bounded on $C_c^\infty(M)$, $\mathcal{E}'(M)$, $C^\infty(M)$, $\mathcal{D}'(M)$.

**Remark 5.27.** Complementing Remark 5.21, the subspace of $\Psi^m(M)$, $M = \mathbb{R}^n$, consisting of properly supported operators does not have a simple relationship with $\Psi_m^\infty(\mathbb{R}^n)$: on the one hand, Schwartz kernels of elements of $\Psi^m(\mathbb{R}^n)$ may even have full support in $\mathbb{R}^n \times \mathbb{R}^n$, hence are not properly supported; on the other hand, properly supported elements of $\Psi^m(M)$ may not be elements of $\Psi^m(\mathbb{R}^n)$ since membership in the latter space requires uniform bounds off the diagonal, see e.g. Exercise 4.2.

**Theorem 5.28.** Let $A \in \Psi^m(M)$ and $B \in \Psi^{m'}(M)$, and assume at least one of $A$ and $B$ is properly supported. Then $A \circ B: C_c^\infty(M) \to C^\infty(M)$ is a ps.d.o., $A \circ B \in \Psi^{m+m'}(M)$. If both $A$ and $B$ are properly supported, then so is $A \circ B$.

We will use the description of Theorem 5.25 for a particular kind of open cover:

**Lemma 5.29.** Let $M$ be a smooth manifold. There exists a locally finite open cover $\{U_i\}$ of $M$ such that whenever $U_i \cap U_j \neq \emptyset$, then there exists a local coordinate chart $F: U \to F(U) \subset \mathbb{R}^n$ with $U \supset U_i \cup U_j$.

**Proof.** $M$ is metrizable; this follows either by Urysohn’s metrization theorem, or from basic Riemannian geometry. Denote a fixed metric on $M$ by $d$, and denote metric balls by $B(p, r) = \{q \in M: d(p, q) < r\}$. For each $p \in M$, let
\[ r_0(p) := \sup\{r \in [0, 1]: B(p, r) \text{ is contained in a coordinate chart}\}. \] (5.63)
Since $M$ is a manifold, we have $r_0(p) > 0$ for all $p \in M$. For $p \in M$, define the open set
\[ V_p := B\left(p, \frac{r_0(p)}{10}\right). \] (5.64)
Suppose $V_p \cap V_q \neq \emptyset$; then $d(p, q) \leq \frac{1}{10}(r_0(p) + r_0(q)) \leq \frac{1}{5} \max(r_0(p), r_0(q))$. By symmetry, we may assume $r_0(p) \geq r_0(q)$. If $z \in V_p \cup V_q$, then
\[ d(p, z) < \max\left(\frac{r_0(p)}{10}, d(p, q) + \frac{r_0(q)}{10}\right) \leq \max\left(\frac{r_0(p)}{10}, \frac{r_0(p)}{5} + \frac{r_0(p)}{10}\right) < \frac{1}{2}r_0(p). \] (5.65)
Therefore, $V_p \cup V_q \subset B(p, \frac{\rho_0(p)}{2})$ is contained in a coordinate chart. Any locally finite refinement $\{U_i : p \in M\}$ of $M$ satisfies the conditions of the lemma. \qed

Proof of Theorem 5.28. By the previous lemma, we can fix an open cover $M = \bigcup_i U_i$ of $M$ by coordinate charts $F_i : U_i \to F_i(U_i) \subset \mathbb{R}^n$, with $\overline{U_i}$ compact, and so that for any $i, j$ with $U_i \cap U_j \neq \emptyset$, the union $U_i \cup U_j$ is contained in a coordinate chart $F_{ij} : U_{ij} \to F_{ij}(U_{ij}) \subset \mathbb{R}^n$.

Let us assume that $A$ is properly supported. (The case that $B$ is properly supported is handled similarly.) Write

$$A = K + \sum_i F_i^* A_i (F_i^{-1})^*, \quad A_i \in \Psi_c^m(F_i(U_i)),
B = K' + \sum_i F_i^* B_i (F_i^{-1})^*, \quad B_i \in \Psi_c^m(F_i(U_i)), \quad K' \in \mathcal{C}^\infty(M^2; \Omega_R).$$

Since $A$ and all the $F_i^* A_i (F_i^{-1})^*$ are properly supported, so is $K \in \mathcal{C}^\infty(M^2; \Omega_R)$.

We consider the composition $A \circ B$ term by term and keep track of the support of the Schwartz kernels of the various pieces.

We first prove that $K \circ K'(x,y) = \int K(x,z)K'(z,y)\,dz$; for any compact set $K_1 \subset M$ there exists $K_2 \subset M$ such that in fact

$$(K \circ K')(x,y) = \int_{K_2} K(x,z)K'(z,y)\,dz, \quad x \in K_1.$$ (5.67)

Indeed, this holds for $K_2 = \pi_L(supp K \cap \pi_R^{-1}(K_1))$. Thus, the Schwartz kernel of $K \circ K'$ lies in $\mathcal{C}^\infty(M^2; \Omega_R)$.

Consider next the composition $K_i := K \circ F_i^* B_i (F_i^{-1})^*$. This maps $u \in \mathcal{E}'(M)$ into $\mathcal{C}^\infty(M)$ (in fact, into $\mathcal{C}^\infty_c(M)$); and if $supp u \cap U_i = \emptyset$, then $K_i u = 0$. Thus, by Lemma 5.23,

$$K_i \in \mathcal{C}^\infty(M^2; \Omega_R), \quad supp K_i \subset M \times U_i.$$ (5.68)

(In fact, $supp K_i \subset \pi_L(supp K \cap \pi_R^{-1}(U_i)) \times U_i$ is compact, but we do not need this information.) Similarly, one shows that

$$K_i' := F_i^* A_i (F_i^{-1})^* \circ K' \in \mathcal{C}^\infty(M^2; \Omega_R), \quad supp K_i' \subset U_i \times M.$$ (5.69)

(Note that its Schwartz kernel is not compactly supported since $K'$ is not properly supported.) Finally, we need to consider the composition

$$C_{ij} := F_i^* A_i (F_i^{-1})^* \circ F_j^* B_j (F_j^{-1})^* : \mathcal{C}^\infty_c(M) \to \mathcal{C}^\infty(M).$$ (5.70)

When $U_i \cap U_j = \emptyset$, this composition is the 0 operator. When $U_i \cap U_j \neq \emptyset$, we can use Lemma 5.24 and write (5.70) equivalently as

$$F_i^* A_i (F_i^{-1})^* \circ F_j^* B_j (F_j^{-1})^* = F_i^* (A_{ij} \circ B_{ij}) (F_i^{-1})^*,$$ (5.71)

where $A_{ij} \in \Psi_c^m(F_{ij}(U_i))$, $B_{ij} \in \Psi_c^{m'}(F_{ij}(U_j))$. But then $A_{ij} \circ B_{ij} \in \Psi_c^{m+m'}(F_{ij}(U_i \cup U_j))$, hence (5.71) lies in $\Psi_c^{m+m'}$, with Schwartz kernel supported in $U_{ij} \times U_{ij}$.

The proof is complete once we show that the supports of the Schwartz kernels of $K_i$, $K_i'$, and $C_{ij}$ are locally finite. Take a point $(p, q) \in M^2$, and choose $i_0, j_0$ such that $p \in U_{i_0}$ and $q \in U_{j_0}$. Then $U := U_{i_0} \times U_{j_0}$ has non-trivial intersection with only finitely many of these supports, as follows immediately from the local finiteness of $\{U_i\}$. \qed
Since every operator on a compact manifold is properly supported, we deduce:

**Corollary 5.30.** If $M$ is a compact manifold, then $\Psi^m(M) \circ \Psi^m(M) \subset \Psi^{m+m}(M)$.

5.5. Principal symbol. Motivated by Theorem 5.2, in particular formula (5.2), we want to define the principal symbol of $A \in \Psi^m(M)$ as an equivalence class of symbols on $T^*M$.

**Definition 5.31.** Let $M$ be a manifold and $\pi: E \to M$ a rank $k$ vector bundle. For $m \in \mathbb{R}$, we define $S^m(E) \subset C^\infty(E)$ as the subspace of all $a \in C^\infty(E)$ having the following property: for each coordinate chart $F: U \to F(U) \subset \mathbb{R}^n$ on $M$ on which $E$ is trivial with trivialization $\tau: \pi^{-1}(U) \to U \times \mathbb{R}^k \xrightarrow{F \times \text{Id}} F(U) \times \mathbb{R}^k$, set

$$b(x,v) := \tau^*(a|_{\pi^{-1}(U)})(x,v) = a(\tau^{-1}(x,v)) \in C^\infty(F(U) \times \mathbb{R}^k).$$

(5.72)

Then for any $\phi \in C^\infty_c(F(U)) \subset C^\infty_c(\mathbb{R}^n)$, we have $\phi(x)b(x,v) \in S^m(\mathbb{R}^n; \mathbb{R}^k)$.

The key to making this a useful definition is the analogue of Lemma 5.24.

**Lemma 5.32.** In the notation of Definition 5.31, suppose $\phi \in C^\infty_c(F(U))$, $b \in S^m(\mathbb{R}^n; \mathbb{R}^k)$. Then $a := (\tau^{-1})^*(\phi b) \in S^m(E)$.

**Proof.** The expression for $a$ in another coordinate system $F': U' \to F'(U') \subset \mathbb{R}^n$ on $M$ and a trivialization of $E$ on $U'$ is

$$b'(x,v) = \phi(\kappa(x))b(\kappa(x), \Phi(x)v), \quad x \in F'(U \cap U')$$

(5.73)

for some diffeomorphism $\kappa: F'(U \cap U') \to F(U \cap U')$ and a smooth map $\Phi: U' \to U$. Let $\psi \in C^\infty_c(F'(U'))$. Then $\chi(x) := \psi(x)\phi(\kappa(x)) \in C^\infty_c(F'(U') \cap \kappa^{-1}(F(U))) = C^\infty_c(F'(U \cap U'))$, and we need to check that

$$\chi(x)b(\kappa(x), \Phi(x)v) \in S^m(\mathbb{R}^n; \mathbb{R}^k).$$

(5.74)

This however follows from the same type of calculation as (5.14). □

In analogy with Theorem 5.25, we have:

**Corollary 5.33.** Let $M = \bigcup_i U_i$ be a locally finite open cover by coordinate charts $F_i: U_i \to F_i(U_i) \subset \mathbb{R}^n$, with $\overline{U_i}$ compact, over which $E$ has a trivialization $\tau_i: \pi^{-1}(U_i) \to F_i(U_i) \times \mathbb{R}^k$. Let $a \in C^\infty(E)$. Then $a \in S^m(E)$ if and only if there exist symbols $b_i \in S^m(\mathbb{R}^n; \mathbb{R}^k)$ and $\chi_i \in C^\infty_c(F_i(U_i))$ such that

$$a = \sum_i (\tau_i^{-1})^*(\chi_i b_i).$$

(5.75)

Now, given an operator $A \in \Psi^m(M)$, we expect its principal symbol to be an element of the quotient space $S^m(T^*M)/S^{m-1}(T^*M)$. We first define it locally. Let $F: U \to F(U) \subset \mathbb{R}^n$ be a coordinate chart with $U$ compact, and let $V \subset U$ be open with $V \subset U$. Denote by $\tau: T^*_U M \to F(U) \times \mathbb{R}^n$ the trivialization induced by $F$. Let $\chi, \tilde{\chi} \in C^\infty_c(U)$ be such that $\chi = 1$ on $V$, and $\tilde{\chi} = 1$ on supp $\chi$. Then we put

$$a_V := (\tau^{-1})^* \left( \sigma_L((F^{-1})^* \tilde{\chi} A \chi f^*) \right) |_{T^*_V M} \in S^m(T^*_V M).$$

(5.76)

By the local ($\mathbb{R}^n$) theory, and in particular by Theorem 5.2, the equivalence class

$$[a_V] \in S^m(T^*_V M)/S^{m-1}(T^*_V M)$$

(5.77)

is independent of the choice of $\chi, \tilde{\chi}$, and of the coordinate system $F$ covering a neighborhood of $V$. Moreover, if $V' \subset V$, then restriction to $V'$ gives $[a_V]|_{V'} = [a_{V'}]$. 

The principal symbol map gives a short exact sequence

\[ \Psi^m(M) \to \Psi^m(T^*M) \xrightarrow{\sigma_m} S^m(T^*M) / S^{m-1}(T^*M) \rightarrow 0. \]  

We only prove surjectivity of \( \sigma_m \).

**Proof.** We check that this satisfies the property required in Definition 5.34. Given \( V \) open as above, it suffices to show that for \( \phi \in C_\infty^c(V) \), we have \( [\phi a]_V = [\phi a_V] \). Now \( \phi V_i = \phi V_i + e_i \) for some \( e_i \in S^{m-1}(T^*M) \) with support in \( T^*_V \cap M \). Let \( \tilde{\phi}_i \in C_\infty^c(V_i) \) be equal to 1 on \( \text{supp } \phi_i \), and with \( \tilde{\phi}_i \) locally finite; then

\[ \phi a|_V = \sum_i \tilde{\phi}_i (\phi_i a V_i) = \sum_i \tilde{\phi}_i (\phi_i a V + e_i) = \phi a V + \sum_i \tilde{\phi}_i e_i, \]  

as desired (since \( \sum_i \tilde{\phi}_i e_i \in S^{m-1}(T^*M) \)).

**Proposition 5.35.** The principal symbol map gives a short exact sequence

\[ 0 \to \Psi^{m-1}(M) \to \Psi^m(M) \xrightarrow{\sigma_m} S^m(T^*M) / S^{m-1}(T^*M) \to 0. \]  

**Proof.** We only prove surjectivity of \( \sigma_m \). Take a partition of unity \( \{ \phi_i \} \) subordinate to a locally finite cover of \( M \) by coordinate charts \( F_i : U_i \to F_i(U_i) \subset \mathbb{R}^n \) with \( U_i \) compact. Writing any \( a \in S^m(T^*M) \) as \( a = \sum \phi_i a \), it suffices to show that there exists an operator \( A_i \in \Psi^m(M) \) with Schwartz kernel supported in \( U_i \times U_i \) such that \( \sigma_m(A_i) = [\phi_i a] \), as we can then take \( A = \sum A_i \) (which is a locally finite sum). This is easy: if \( \tilde{\phi}_i \in C_\infty^c(U_i) \), \( \tilde{\phi}_i = 1 \) on \( \text{supp } \phi_i \), then simply take

\[ A_i = F_i^* \text{Op} \left( \left( (F_i^{-1})^* \phi_i \right)(x) a(x, \xi) \left( (F_i^{-1})^* \tilde{\phi}_i \right)(y) \right) (F_i^{-1})^*. \]

An immediate consequence of the \( \mathbb{R}^n \) result, Proposition 4.21, is:

**Proposition 5.36.** The principal symbol map is multiplicative: if \( A \in \Psi^m(M) \), \( B \in \Psi^{m'}(M) \), with at least one of them properly supported, then

\[ \sigma_{m+m'}(A \circ B) = \sigma_m(A) \sigma_{m'}(B). \]  

The analogue of Proposition 4.22 concerning the principal symbol of commutators will be discussed in §5.12.

**5.6. Operators acting on sections of vector bundles.** The reader might ask why we have not discussed adjoints of \( A \in \Psi^m(M) \) (or even \( A \in \text{Diff}^m(M) \)) yet. Since we do not have an invariant way of integrating functions on \( M \), the only sensible way to define \( A^* \) is by

\[ \int_M (Au)(x)v(x) = \int_M u(x)A^*v(x), \quad u \in C_\infty^c(M), \ v \in C_\infty^c(M; \Omega M), \]  

with the following property: if \( a \in S^m(T^*M) \) is any representative of \( \sigma_m(A) \), and \( V \) is as above, then \( [a|_V] = [aV] \in S^m(T^*_V M) / S^{m-1}(T^*_V M) \).
that is, $A^*$ should be an operator acting on sections of $\Omega M$. We leave it to the reader to define the space of $m$-th order differential operators $\text{Diff}^m(M; E, F)$ mapping sections of $E$ to section of $F$, and go straight for the pseudodifferential version.

**Definition 5.37.** Let $M$ be a smooth manifold, and let $\pi_E: E \to M$, $\pi_F: F \to M$ denote two real vector bundles of rank $k_E, k_F$. Then $\Psi^m(M; E, F)$ is the space of linear operators

$$A: C^\infty_c(M; E) \to C^\infty(M; F)$$

(5.85)

with the following properties:

1. if $\phi, \psi \in C^\infty(M)$ have $\text{supp } \phi \cap \text{supp } \psi = \emptyset$, then there exists $K \in C^\infty(M^2; \pi^*_E F \otimes \pi^*_F (E^* \otimes \Omega M))$ such that $\phi A \psi = K$.

2. suppose $U \subset M$ is open, $G: U \to G(U) \subset \mathbb{R}^n$ is a diffeomorphism, and $\tau_E: \pi^{-1}_E(U) \to G(U) \times \mathbb{R}^{k_E}$, $\tau_F: \pi^{-1}_F(U) \to G(U) \times \mathbb{R}^{k_F}$ are local trivializations of $E, F$. Using $\tau_E$, identify smooth sections of $E$ over $U$ with $k_E$-tuples of smooth functions on $U$, likewise for sections of $F$. If $\psi \in C^\infty(U)$, then there exists a $k_F \times k_E$ matrix $B = (B_{ij})$ of ps.d.o.s $B_{ij} \in \Psi^m_c(G(U))$ such that on $U$

$$\psi A(\psi u)(x)_i = \sum_{j=1}^{k_E} G^*(B_{ij}((G^{-1})^*u))_j, \quad u \in C^\infty_c(M; E), \quad i = 1, \ldots, k_F. \quad (5.86)$$

In the special case $F = E$, we write $\Psi^m(M; E) = \Psi^m(M; E, E)$.

In local coordinates and trivializations, the symbol of $A \in \Psi^m(M; E, F)$ is a symbol with values in linear maps from $\mathbb{R}^{k_E}$ to $\mathbb{R}^{k_F}$. The invariant definition is as follows. Denote by $\pi: T^* M \to M$ the projection. Given a vector bundle $G \to M$, we can consider its pullback $\pi^* G \to T^* M$; a trivialization of $G$ over an open set $U \subset M$, so $G|_U \cong U \times \mathbb{R}^{k_G}$, then induces a trivialization of $\pi^* G$ over $T^*_U M$ which is ‘constant in the fibers of $T^* M$’, namely

$$(\pi^* G)|_{T^*_U M} \cong T^* U \times \mathbb{R}^{k_G}, \quad (5.87)$$

by identifying $(\pi^* G)_{(x, \xi)} = G_x \cong \mathbb{R}^{k_G}$ using the local trivialization. We then denote by

$$S^m(T^* M; \pi^* G) \subset C^\infty(T^* M; \pi^* G) \quad (5.88)$$

the space of all smooth functions which in local coordinates and in a trivialization of $G$ (which induces a trivialization of $\pi^* G$ as in (5.87)) are $k_G$-vectors of symbols on $\mathbb{R}^n$ of order $m$. Invariantly then,

$$\sigma_m(A) \in (S^m/S^{m-1})(T^* M; \pi^* \text{Hom}(E, F)), \quad \pi: T^* M \to M. \quad (5.89)$$

This means that a representative of $\sigma_m(A)$ is a map assigning to $(x, \xi) \in T^* M$ an element of $\text{Hom}(E_x, F_x)$. We have a short exact sequence

$$0 \to \Psi^{-1}(M; E, F) \to \Psi^m(M; E, F) \xrightarrow{\partial} (S^m/S^{m-1})(T^* M; \pi^* \text{Hom}(E, F)) \to 0. \quad (5.90)$$

The results from §§5.4–5.5 carry over; moreover, one can define adjoints:

**Proposition 5.38.** Let $M$ be a smooth manifold, and let $E, F, G \to M$ denote three vector bundles.
Remark 5.39. \(7\) Let \(A \in \Psi^m(M; F, G)\) and \(B \in \Psi^{m'}(M; E, F)\), with at least one of them properly supported. Then \(A \circ B \in \Psi^{m+m'}(M; E, G)\), and \(\sigma_{m+m'}(A \circ B) = \sigma_m(A) \circ \sigma_{m'}(B)\).

(2) Let \(A \in \Psi^m(M; E, F)\). Then the (real) adjoint \(A^T\), defined by

\[
\int_M (Au)v = \int_M u(A^Tv), \quad u \in C^\infty_c(M; E), \quad v \in C^\infty_c(M; F^* \otimes \Omega M),
\]

(5.91)
is a pseudodifferential operator,

\[
A^T \in \Psi^m(M; F^* \otimes \Omega M, E^* \otimes \Omega M).
\]

(5.92)
It is properly supported if \(A\) is.

**Corollary 5.40.** Let \(A \in \Psi^m(M; E, F)\). Then \(A\) extends to a bounded linear operator \(A: \mathscr{E}'(M; E) \to \mathscr{D}'(M; F)\). If \(A\) is properly supported, then \(A\) also maps \(\mathscr{D}'(M; E) \to \mathscr{E}'(M; F)\), and by restriction \(C^\infty(M; E) \to C^\infty(M; F)\).

**Proof.** \(A^T\) is a bounded map \(C^\infty_c(M; F^* \otimes \Omega M) \to C^\infty_c(M; E^* \otimes \Omega M)\). Therefore we can define \(A: \mathscr{E}'(M; E) \to \mathscr{D}'(M; F)\) by duality using (5.92); this agrees with the original operator \(A\) when restricted to \(C^\infty(M; E)\).

If \(A\) is properly supported, then \(A^T\) maps \(C^\infty_c(M; F^* \otimes \Omega M) \to C^\infty_c(M; E^* \otimes \Omega M)\), hence we can now define \(A: \mathscr{D}'(M; E) \to \mathscr{E}'(M; F)\) by duality. Now \(C^\infty(M; E) \subset \mathscr{D}'(M; E)\). It remains to check that \(Au \in C^\infty(M; F)\) when \(u \in C^\infty(M; E)\); but using a partition of unity \(1 = \sum_i \phi_i\), with \(\text{supp} \phi_i\) compact, we have

\[
Au = A \left( \sum_i \phi_i u \right) = \sum_i A(\phi_i u),
\]

(5.94)
with convergence in \(\mathscr{D}'(M; F)\). But since \(A\) is properly supported, the final sum is a locally finite sum of smooth terms, hence smooth.

TBC as an example: half-densities (simplest, most symmetric version)

TBC subprincipal symbol (exercise?)

---

\(7\) Note that for operators acting between bundles, composition is no longer commutative on the level of principal symbols.
5.7. Special classes of operators. Let $M$ be a manifold, and let $E, F \to M$ denote two vector bundles of rank $k_E, k_F$.

**Definition 5.41.** Let $m \in \mathbb{C}$. The subspace $\Psi^m_{\text{cl}}(M; E, F) \subset \Psi^m_{\text{Re}}(M; E, F)$ of *classical pseudodifferential operators* consists of those operators whose full symbol in a local coordinate chart and in local trivializations of $E, F$ is a $k_F \times k_E$ matrix of classical symbols of order $m$. The principal symbol map on $\Psi^m_{\text{cl}}(M; E, F)$ records the leading order homogeneous part,

$$\sigma_m : \Psi^m_{\text{cl}}(M; E, F) \to S^m_{\text{hom}}(T^*M \setminus \pi^* \text{Hom}(E, F)).$$

(5.95)

The reason this is a sensible definition is that classicality is preserved under local coordinate transformations; this follows from the proof of Theorem 5.2, in particular equation (5.13). Using the $\mathbb{R}^n$ results such as Proposition 4.24, one easily checks that the composition of two classical ps.d.o.s (at least one of which is properly supported) is again a classical ps.d.o., and that taking adjoints preserves classicality as well. A class of a classical ps.d.o.s is of course given by differential operators:

$$\text{Diff}^m(M; E, F) \subset \Psi^m_{\text{cl}}(M; E, F).$$

(5.96)

Moreover, parametrices of classical operators are again classical.

Often, operators arising in geometric problems are Laplace operators to leading order, such as the Hodge Laplacian (5.114). A very useful generalization of this is the following.

**Definition 5.42.** Let $m \in \mathbb{R}$ and $A \in \Psi^m(M; E)$. Then $A$ is *principally scalar* if its principal symbol is multiplication by scalars on the fibers of $E$, that is, if there exists a symbol $a \in S^m(T^*M)$ such that $\sigma_m(A)(x, \xi) = a(x, \xi) \text{Id}_{E_x}$.

Principally scalar operators behave similarly to operators acting on scalar functions; we shall see examples of this in §8.

5.8. Elliptic operators on compact manifolds, Fredholm theory. Let $M$ be a compact $n$-dimensional manifold (without boundary), and let $E, F \to M$ denote two vector bundles.

**Definition 5.43.** We say that $A \in \Psi^m(M; E, F)$, with principal symbol $a = \sigma_m(A)$, is *elliptic* if there exists a symbol $b \in S^{-m}(\pi^* \text{Hom}(F, E))$ such that $ab - 1 \in S^{-1}(\pi^* \text{End}(F))$, $ba - 1 \in S^{-1}(\pi^* \text{End}(E))$.

**Remark 5.44.** By ‘abstract group theory’ as in (4.86), the following seemingly more general definition is in fact equivalent to the ellipticity of $A$: there exist $b, b' \in S^{-m}(\pi^* \text{Hom}(F, E))$ such that $ab - 1 \in S^{-1}(\pi^* \text{End}(F))$ and $b' a - 1 \in S^{-1}(\pi^* \text{End}(E))$.

**Theorem 5.45.** Let $A \in \Psi^m(M; E, F)$ be an elliptic operator.

1. Then $A : C^\infty(M; E) \to C^\infty(M; F)$ is Fredholm, that is, ker $A$ is finite-dimensional, and ran $A$ is closed and has finite codimension.

2. Let $\nu \in C^\infty(M; \Omega M)$ be a smooth non-vanishing measure on $M$, and suppose $E, F$ are equipped with positive definite fiber metrics. Define the generalized inverse of $A$ by

$$Gf = u \quad \text{if } f \in \text{ran } A, \quad Au = f, \quad u \perp \ker A \text{ in } L^2(M; E; \nu),$$

$$Gf = 0 \quad \text{if } f \perp \text{ran } A \text{ in } L^2(M; F; \nu).$$

(5.97)
Then $G \in \Psi^{-m}(M; F, E)$, and
\[ GA = I - \pi_N, \quad AG = I - \pi_R, \] (5.98)
where $\pi_N \in \Psi^{-\infty}(M; E)$ is the $L^2(M; E; \nu)$-orthogonal projection to the finite-dimensional space $\ker A$, and $\pi_R \in \Psi^{-\infty}(M; F)$ is the $L^2(M; F; \nu)$-orthogonal projection to the finite-dimensional space $(\text{ran } A) \perp$. In particular, if $A$ is invertible, then $G = A^{-1} \in \Psi^{-m}(M; F, E)$.

Proof. The elliptic parametrix construction, see Theorem 4.26, works in this setting as well: all one uses is (1) the multiplicativity of the symbol map in Proposition 5.38, and (2) the short exact sequence (5.90). Thus, there exists $B \in \Psi^{-m}(M; F; E)$ such that
\[ R_1 = AB - I \in \Psi^{-\infty}(M; F), \quad R_2 = BA - I \in \Psi^{-\infty}(M; E). \] (5.99)
We show that $\dim \ker A < \infty$. First, note that
\[ u \in \mathcal{D}'(M; E), \quad Au = 0 \implies u = (BA - R_2)u = -R_2u \in \mathcal{C}^\infty(M; E). \] (5.100)
Let us look at this from the point of view that the identity map on $\ker A$ can be written as $I = BA - R_2 - R_2$. Now $R_2 : L^2(M; E; \nu) \to \mathcal{C}^\infty(M; E)$, hence is compact as a map $L^2(M; \nu) \to L^2(M; \nu)$ by Arzelà–Ascoli. Therefore, the unit ball in the closed subspace $\ker A \subset L^2(M; \nu)$ is compact, thus $\ker A \subset L^2(M; \nu)$ is finite-dimensional, as desired.

Next, we show that $\text{ran } A$ is closed. Suppose $f_j = Au_j \to f \in \mathcal{C}^\infty(M; F)$, $u_j \in \mathcal{C}^\infty(M; E)$. We may change $u_j$ by an element of $\ker A$ to ensure that $u_j \perp \ker A$. We have
\[ u_j = BAu_j - R_2u_j = Bf_j - R_2u_j. \] (5.101)
Suppose that, along some subsequence, $\|u_j\|_{L^2} \to \infty$. Then
\[ \frac{u_j}{\|u_j\|} = B \left( \frac{f_j}{\|u_j\|} \right) - R_2 \left( \frac{u_j}{\|u_j\|} \right). \] (5.102)
This is bounded in $\mathcal{C}^\infty(M; E)$, hence we can pass to a subsequence which converges in $L^2(M; E)$, say $u_j/\|u_j\| \to u \in L^2(M; E; \nu)$. Then $Au = \lim_{j \to \infty} f_j/\|u_j\| = 0$, so $u \in \ker A$, but also $u \perp \ker A$ by construction. Since $\|u\|_{L^2} = 1$, this is a contradiction.

Therefore, $\|u_j\|_{L^2}$ is bounded. Equation (5.101) then shows that $u_j$ is bounded in $\mathcal{C}^\infty(M; E)$, hence has a subsequence converging to $u \in \mathcal{C}^\infty(M; E)$, and
\[ Au = \lim_{j \to \infty} Au_j = \lim_{j \to \infty} f_j = f. \] (5.103)

Finally, we prove that $\text{ran } A$ has finite codimension: but this follows from $(\text{ran } A) \perp = \ker A^*$ and the ellipticity of $A^* \in \Psi^m(M; F, E)$,
\[ \pi_R = \pi_R(AB - R_1) = -\pi_R R_1 \in \Psi^{-\infty}(M; F). \] (5.104)
which implies that the unit ball in $\pi_R = (\text{ran } A) \perp \subset L^2(M; F; \nu)$ is compact, hence $(\text{ran } A) \perp$ is finite-dimensional.

Note that (5.104) and the analogous calculation $\pi_N = (BA - R_2)\pi_N = -R_2\pi_N$ imply that $\pi_N, \pi_R$ have smooth Schwartz kernels, hence lie in $\Psi^{-\infty}$. The statement $G \in \Psi^{-m}(M; F, E)$
for the generalized inverse (5.97) then follows by writing
\[
G = G(AB - R_1) \\
= B - GR_1 \\
= (I - \pi_N)B - (BA - R_2)GR_1 \\
= (I - \pi_N)B - B(I - \pi_R)R_1 + R_2GR_1.
\]
(5.105)

Indeed, the first summand lies in \(\Psi^m(M; F, E)\), the second in \(\Psi^{-\infty}(M; F, E)\), and the last one is a smoothing operator, hence lies in \(\Psi^{-\infty}(M; F, E)\) as well. \(\square\)

As a typical example, we discuss the Laplace operator on a compact \(n\)-dimensional manifold \(M\), which we assume to be connected for convenience. Denote by \(S^2T^*M\) the second symmetric tensor product of \(T^*M\) with itself. Let \(g \in C^\infty(M; S^2T^*M)\) be a Riemannian metric, so in local coordinates
\[
g = \sum_{i,j=1}^n g_{ij}(x) \cdot \frac{1}{2} (dx^i \otimes dx^j + dx^j \otimes dx^i).
\]
(5.106)
Write \(g^{ij}(x) = g^{-1}(x)_{ij}\) and \(|g| = |\det(g_{ij})|\). Then the (scalar) Laplace operator is
\[
\Delta_g u = \sum_{i,j=1}^n |g|^{-1/2} D_{x_i}(|g|^{1/2} g^{ij}(x) D_{x_j} u)
\]
in local coordinates. Thus, \(\Delta_g \in \Psi^2(M)\), with
\[
\sigma_2(\Delta_g)(x, \xi) = \sum_{i,j=1}^n g^{ij}(x) \xi_i \xi_j =: |\xi|_{g^{-1}(x)}^2.
\]
(5.108)
Thus, \(\Delta_g\) is elliptic. By Theorem 5.45, the kernel and cokernel of \(\Delta_g\) are finite-dimensional.

Moreover, \(\Delta_g\) is a symmetric operator with respect to the inner product on \(L^2(M; |dg|)\), where \(|dg| \in C^\infty(M; \Omega M)\) is defined in local coordinates by
\[
|dg| = |g(x)|^{1/2} dx.
\]
(5.109)
Thus, \(\ker \Delta_g = (\ran \Delta_g)^\perp\); and if \(u \in \ker \Delta_g\), then
\[
0 = \int_M (\Delta_g u) \bar{u} |dg| = \int_M |\nabla u|^2_g |dg|,
\]
(5.110)
so \(u\) is constant. In the notation of Theorem 5.45, we thus have
\[
\pi_N = \frac{1}{\text{vol}(M)} \langle \cdot, 1 \rangle 1 = \pi_R
\]
(5.111)
(projection onto constants).

Let us study
\[
\Delta_g u = f, \quad f \in \mathcal{D}'(M).
\]
(5.112)
Let \(G \in \Psi^{-2}(M)\) denote the generalized inverse of \(\Delta_g\). In the notation of Theorem 5.45, we then have
\[
u = (G\Delta_g + \pi_N)u = Gf + \pi_N u.
\]
(5.113)
This solves (5.112) if and only if \(f = \Delta_g u = \Delta_g Gf + \Delta_g \pi_N u = (I - \pi_R)f\). This shows:
Proposition 5.46. The equation (5.112) has a solution \( u \in H^r(M) \) if and only if \( \langle f, 1 \rangle = 0 \), and in this case \( u \) is unique up to additive constants. If \( f \in C^\infty(M) \), then \( u \in C^\infty(M) \).

Example 5.47. For the operator \( A = \Delta_g + 1 \in \Psi^2(M) \) on a compact Riemannian manifold, one finds \( \ker A = 0 = (\operatorname{ran} A)^\perp \), thus one can always solve \( Au = f \) for \( f \in C^\infty(M) \) or \( f \in H^r(M) \), with solution \( u \in C^\infty(M) \) or \( u \in H^r(M) \).

Example 5.48. One can define natural generalizations of \( \Delta_g \) which act on vector bundles rather than functions. Let \( d_k \in \operatorname{Diff}^1(M; \Lambda^k T^* M; \Lambda^{k+1} T^* M) \) denote the exterior derivative, and denote by \( \delta_k \in \operatorname{Diff}^1(M; \Lambda^k T^* M; \Lambda^{k-1} T^* M) \) the adjoint of \( d_{k-1} \). Let \( d_n = 0 \) and \( \delta_0 = 0 \). Then the Hodge Laplacian in degree \( k \) is

\[
\Delta_k := \delta_{k+1} d_k + d_{k-1} \delta_k \in \operatorname{Diff}^2(M; \Lambda^k T^* M).
\]

(5.114)

Its principal symbol is scalar, i.e. at each \( (x, \xi) \in T^* M \) a multiple of the identity operator on \( (\pi^* \Lambda^k T^* M)(x, \xi) \); in fact \( \sigma_2(\Delta_k)(x, \xi) = |\xi|^2 |\gamma^{-1}(x)| \text{ Id} \). (The expression (5.107) is the special case \( k = 0 \).) Again \( \Delta_k \) is symmetric with respect to the fiber inner product and volume density induced by \( g \). Its kernel and orthocomplement of the range are finite-dimensional, and can be identified with the singular cohomology group \( H^k(M; \mathbb{C}) \) by Hodge theory. For a general version of this, see Exercise 5.7.

5.9. Sobolev spaces on manifolds. We need two key facts about Sobolev spaces \( H^s(\mathbb{R}^n) \) for the generalization of Sobolev spaces to manifolds. For an open set \( \Omega \subseteq \mathbb{R}^n \), we define

\[
H^s_\ell(\Omega) := \{ u \in H^s(\mathbb{R}^n) : \text{ supp } u \subseteq \Omega \}.
\]

(5.115)

Lemma 5.49. Sobolev spaces on \( \mathbb{R}^n \) have the following properties.

1. Let \( a \in C^\infty_0(\mathbb{R}^n) \). Then multiplication by \( a \) defines a bounded linear map \( a : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n) \) for all \( s \in \mathbb{R}^n \).

2. Suppose \( \kappa : \Omega \to \Omega' \) is a diffeomorphism of precompact open subsets \( \Omega, \Omega' \subseteq \mathbb{R}^n \).

Then \( \kappa^* : H^s_\ell(\Omega') \to H^s_\ell(\Omega) \).

Proof. The ‘standard’ proof of the first claim proceeds by proving it for \( s \in \mathbb{N}_0 \) using the Leibniz rule, then for all real \( s \geq 0 \) by complex interpolation, and then for all \( s \in \mathbb{R} \) by duality. With the machinery of §4 at hand, we can instead just observe that \( a \in \Psi^0(\mathbb{R}^n) \), and appeal to Corollary 4.32.

The second claim is clear for \( s = 0 \). We shall prove it for general \( s \in \mathbb{R} \) using our ps.d.o. machinery. Indeed, given \( u \in H^s_\ell(\Omega') \subseteq E'(\Omega') \), we certainly have \( \kappa^* u \in E'(\Omega) \). Let \( A \in \Psi^s(\mathbb{R}^n) \) be elliptic, and let \( \phi, \tilde{\phi} \in C^\infty_0(\Omega) \) be such that \( \phi = \tilde{\phi} = 1 \) on \( \text{ supp } (\kappa^* u) \), and such that \( \tilde{\phi} = 1 \) in a neighborhood of \( \text{ supp } \phi \). By choosing \( A \) carefully (localizing its Schwartz kernel near the diagonal—which does not affect its ellipticity property), we may arrange that

\[
A(\kappa^* u) = \tilde{\phi} A \phi \kappa^* u.
\]

(5.116)

Note that \( \tilde{\phi} A \phi \in \Psi^s_\ell(\Omega) \). Therefore, by Theorem 5.2,

\[
A(\kappa^* u) = \kappa^* (A' u), \quad A' = (\kappa^{-1})^* \tilde{\phi} A \phi \kappa^* \in \Psi^s_\ell(\Omega').
\]

(5.117)

Therefore \( A' u \in L^2(\Omega') \), hence \( \kappa^* (A' u) \in L^2(\Omega) \), so \( A(\kappa^* u) \in L^2(\mathbb{R}^n) \). Since \( A \) is elliptic, Corollary 4.34 implies that \( \kappa^* u \in H^s(\mathbb{R}^n) \), as desired.

The ‘local coordinate’ definition of Sobolev spaces on manifolds is then:
Definition 5.50. Let $M$ be an $n$-dimensional manifold, $s \in \mathbb{R}$. Then:

1. We define $H^s_{\text{loc}}(M)$ as the space of all $u \in \mathcal{D}'(M)$ such that for all coordinate charts $F: U \to F(U) \subset \mathbb{R}^n$ on $M$, and all $\phi \in C_c^\infty(U)$, we have $(F^{-1})^*\phi u \in H^s(\mathbb{R}^n)$.
2. We define $H^s_c(M) = \{ u \in H^s_{\text{loc}}(M) : \text{supp } u \subset M \text{ is compact} \}$.

If $M$ is compact, we write

$$H^s(M) = H^s_{\text{loc}}(M) = H^s_c(M). \quad (5.118)$$

Lemma 5.49 shows that if $u \in H^s_{\text{loc}}(F(U))$ for some coordinate chart $F: U \to F(U) \subset \mathbb{R}^n$ on $M$, then $F^*u \in H^s_c(M)$.

Remark 5.51. One can equip $H^s_{\text{loc}}(M)$ with the structure of a Fréchet space by using the seminorms $\| (F_i^{-1})^*\phi_i u \|_{H^s(\mathbb{R}^n)}$ for any fixed countable cover of $M$ by coordinate charts $F_i: U_i \to F_i(U_i) \subset \mathbb{R}^n$ and a subordinate partition of unity $\{ \phi_i \}$, $\phi_i \in C_c^\infty(U_i)$. The resulting topology is independent of the cover and the partition of unity.

The proof of Lemma 5.49 suggests a more intrinsic definition of Sobolev spaces on $M$. Note first that the spaces $L^2_{\text{loc}}(M)$ and $L^2_c(M)$ are well-defined, independently of a choice of integration measure on $M$. (On the other hand, the space $L^2(M)$, even as a set, is not well-defined when $M$ is non-compact without specified integration measure.)

Proposition 5.52. Let $u \in \mathcal{D}'(M)$.

1. Suppose $u \in H^s_c(M)$. Then $Au \in L^2_{\text{loc}}(M)$ for all $A \in \Psi^s(M)$. If $A$ is properly supported, then $A: H^s_c(M) \to L^2_c(M)$.
2. If $Au \in L^2_{\text{loc}}(M)$ for some properly supported elliptic operator $A \in \Psi^s(M)$, then $u \in H^s_{\text{loc}}(M)$.

Proof. Suppose that $F: U \to F(U) \subset \mathbb{R}^n$ is a coordinate system on $M$, and let $\phi, \tilde{\phi} \in C_c^\infty(U)$ with $\tilde{\phi} = 1$ in a neighborhood of $\text{supp } \phi$.

The first claim follows by writing

$$\phi Au = \phi A\tilde{\phi} u + \phi A(1 - \tilde{\phi}) u. \quad (5.119)$$

Indeed, the second summand lies in $C_c^\infty(M) \subset L^2_{\text{loc}}(M)$. The first summand can be evaluated in local coordinates, and lies in $L^2_{\text{loc}}(M)$.

Turning to the second claim, we need to show that $(F^{-1})^*\phi u \in \mathcal{E}'(F(U))$ lies in $H^s(\mathbb{R}^n)$. Let $B \in \Psi^s(\mathbb{R}^n)$ be elliptic. We can arrange for its Schwartz kernel to be supported so close to the diagonal that

$$(1 - (F^{-1})^*\tilde{\phi}) B((F^{-1})^*\phi) = 0. \quad (5.120)$$

By elliptic regularity, we need to establish $B(F^{-1})^*\phi u \in L^2(\mathbb{R}^n)$, which by (5.120) is equivalent to

$$B' u \in L^2_c(M), \quad B' := F^* B (F^{-1})^* \phi \in \Psi^s(M). \quad (5.121)$$

Since $A$ is elliptic, there exists a properly supported parametrix $Q \in \Psi^{-s}(M)$ with $I = QA + R$, where $R \in \Psi^{-\infty}(M)$ is then also properly supported. Therefore,

$$B' u = B'(QA + R) u = (B'Q)(Au) + B' R u. \quad (5.122)$$

Now $B'Q \in \Psi^0(M)$ is bounded on $L^2_{\text{loc}}(M)$, so $(B'Q)(Au) \in L^2_{\text{loc}}(M)$, while $B'R \in \Psi^{-\infty}(M)$, so $B' R u \in C_c^\infty(M)$. Therefore, $B'u \in L^2_{\text{loc}}(M)$. \[ \square \]
Corollary 5.53. Let $A \in \Psi^m(M)$. Then $A$ is a bounded linear operator

$$A : H^s_c(M) \to H^s_{loc}^m(M).$$

(5.123)

If $A$ is properly supported, then $A : H^s_c(M) \to H^s_{loc}^m(M)$, $H^s_{loc}(M) \to H^s_{loc}^m(M)$.

Proof. We only prove (5.123). Let $\Lambda \in \Psi^{-m}(M)$ be properly supported and elliptic. By the second part of Proposition 5.52, it suffices to show that $\Lambda \circ A : H^s_c(M) \to L^2_{loc}(M)$; but this follows from $\Lambda \circ A \in \Psi^s(M)$ and the first part of Proposition 5.52. □

On a compact manifold $M$, the space $H^s(M)$ can be given the structure of a Hilbert space:

Proposition 5.54. Let $M$ be compact, and let $s \in \mathbb{R}$. Fix a smooth positive volume density on $M$. Then there exists $A \in \Psi^s(M)$ such that

$$\langle u, v \rangle_{H^s(M)} := \langle Au, Av \rangle_{L^2(M)}, \quad \|u\|_{H^s(M)} := \langle u, u \rangle_{H^s(M)},$$

(5.124)

gives $H^s(M)$ the structure of a Hilbert space. The topology on $H^s(M)$ is equal to the norm topology of $(H^s(M), \| \cdot \|_{H^s(M)})$.

Proof. Let $s \geq 0$. Fix a smooth fiber metric $\| \cdot \|$ on $T^*M$, and let $\Lambda' \in \Psi^{s/2}(M)$ be an operator with $\sigma_{s/2}(\Lambda')(x, \xi) = \|\xi\|^{s/2}$. Then $\Lambda'$ is elliptic, and so is

$$\Lambda_s := I + (\Lambda')^*\Lambda' \in \Psi^s(M).$$

(5.125)

By Theorem 5.45, $\Lambda_s : C^\infty(M) \to C^\infty(M)$ is Fredholm. We claim that $\Lambda_s$ is invertible on $C^\infty(M)$. Indeed, $\Lambda_s u = 0$ implies $\|u\|^2_{L^2(M)} + \|\Lambda' u\|^2_{L^2(M)} = 0$, hence $u = 0$. Since $\Lambda_s$ is symmetric (that is, $\langle \Lambda_s u, f \rangle_{L^2(M)} = \langle u, \Lambda_s f \rangle_{L^2(M)}$ for $u, f \in C^\infty(M)$), this also shows that $\Lambda_s$ is surjective. The second part of Theorem 5.45 then implies that

$$\Lambda_{-s} := \Lambda_s^{-1} \in \Psi^{-s}(M).$$

(5.126)

Using Proposition 5.52, we conclude that $\Lambda_s : H^s(M) \to L^2(M)$ and $\Lambda_{-s} : H^{-s}(M) \to L^2(M)$ are isomorphisms.

For $s \in \mathbb{R}$, we can thus take $A = \Lambda_s$. □

Remark 5.55. For $s = 2k$, $k \in \mathbb{N}$, one can take $\Lambda_{2k} = (\Delta + 1)^k$ for any Riemannian metric $g$ on $M$. (In fact, this is true for any $k \in \mathbb{R}$ by a theorem of Seeley which states, as a special case, that $(\Delta + 1)^s \in \Psi^{2s}(M)$ for any $s \in \mathbb{R}$. This operator is defined using the functional calculus for self-adjoint operators.)

Adding vector bundles to this discussion requires only notational changes. Namely, if $E \to M$ is a real/complex rank $k$ vector bundle, we say that $u \in \mathcal{D}'(M; E)$ lies in $H^s_{loc}(M; E)$ if and only if in local trivializations of $E$ over coordinate charts on $M$, $u$ is a $k$-vector of real-valued/complex-valued elements of $H^s(\mathbb{R}^n)$. We let $H^s_c(M; E) = H^s_{loc}(M; E) \cap \mathcal{D}'(M; E)$ as usual. We leave the statements and proofs of the generalizations of Proposition 5.52, Corollary 5.53, and Proposition 5.54 to the reader.

Example 5.56. If $M$ is $n$-dimensional and $p \in M$, then $\delta_p \in H^s(M; \Omega M)$ for all $s < -n/2$; cf. Example (5.15).

---

8Strictly speaking, one should smooth the right hand side out near $\xi = 0$ to get a smooth symbol; but principal symbols only care about behavior for large $\xi$, hence we do not do this here.
5.10. **Elliptic operators on compact manifolds, revisited.** Throughout this section, we denote by $M$ a compact manifold.

**Lemma 5.57.** Let $s' < s$. Then the inclusion $H^s(M) \hookrightarrow H^{s'}(M)$ is compact.

We leave the proof to the reader; it can be proved by localizing in coordinate charts and using a suitable analogue on $\mathbb{R}^n$—in fact, one can use a special case of the first part of Exercise 4.18.

We can now refine Theorem 5.45:

**Theorem 5.58.** Let $A \in \Psi^m(M; E, F)$ be an elliptic operator. Then for any $s \in \mathbb{R}$,

$$A : H^s(M; E) \to H^{s-m}(M; F)$$

is Fredholm. Its kernel $\ker A$ is independent of $s$, and $\ker A \subset C^\infty(M; E)$. Moreover, the cokernel $\text{coker } A$ can be identified with a subset $Y \subset C^\infty(M; F)$ independent of $s$, in the sense that $f \in H^{s-m}(M; F)$ lies in $\text{ran } A$ if and only if $\langle f, g \rangle = 0$ for all $g \in Y$.

**Proof.** If $B \in \Psi^m(M; F, E)$ denotes an elliptic parametrix, then $AB = I + R_1$ and $BA = I + R_2$ with $R_1, R_2 \in \Psi^{-\infty}$ as in (5.99). By Lemma 5.57, the errors $R_1 : H^s(M; F) \to C^\infty(M; F) \hookrightarrow H^s(M; F)$ and $R_2 : H^{s-m}(M; E) \to C^\infty(M; E) \hookrightarrow H^{s-m}(M; E)$ are compact operators. Therefore, $A$ is Fredholm. The regularity statement $\ker A \subset C^\infty(M; E)$ is a special case of (5.100). The solvability claim follows from Theorem 5.45 and elliptic regularity.

In fact, this theorem as a converse: if $A \in \Psi^m(M; E, F)$ is such that (5.127) is Fredholm for some $s \in \mathbb{R}$, then $A$ is elliptic. See Exercise 5.5.

**Remark 5.59.** Theorem 5.58 also shows that the index $\text{ind } A = \dim \ker A - \dim \text{coker } A$ is independent of $s$. Simple functional analytic arguments show that $\text{ind } A = \text{ind } (A + B)$ for any $B \in \Psi^{m-1}(M; E, F)$; thus, $\text{ind } A$ only depends on the principal symbol $\sigma_m(A)!$ The Atiyah–Singer index theorem gives a formula to compute $\text{ind } A$ in terms of $\sigma_m(A)$.

An interesting application concerns the spectral theory of symmetric ps.d.o.s.

**Theorem 5.60.** Fix a smooth positive volume density on $M$, and a positive definite fiber inner product on $E \to M$. Let $m > 0$, and let $A \in \Psi^m(M; E)$ be symmetric, that is, $\langle Au, v \rangle = \langle u, Av \rangle$ for $u, v \in C^\infty(M; E)$, where $\langle \cdot, \cdot \rangle$ is the inner product on $L^2(M; E)$. Then $A$ is an unbounded self-adjoint operator on $L^2(M; E)$ with domain $H^m(M; E)$. Its spectrum $\text{spec } A \subset \mathbb{R}$ is discrete and accumulates only at $\infty$.

**Proof.** By [RS72, Theorem VIII.3], we need to show that $A \pm i : H^m(M; E) \to L^2(M; E)$ is surjective. By Theorem 5.58, its range is closed, and any element $u \in (\text{ran } (A \pm i))\perp$ lies in $\ker (A \mp i) \subset C^\infty(M; E)$, so

$$0 = \text{Im}((A \mp i)u, u) = \mp i \|u\|_{L^2(M; E)}^2 \implies u = 0. \quad (5.128)$$

This proves self-adjointness.

---

9Note that the inclusion $H^s(\mathbb{R}^n) \hookrightarrow H^{s'}(\mathbb{R}^n)$ is not compact!
One can also argue directly: if $A$ is given the domain $\mathcal{D}(A) = H^m(M; E)$, then $v \in L^2(M; E)$ lies in $\mathcal{D}(A^*)$ if and only if $\mathcal{D}(A) \ni u \mapsto \langle Au, v \rangle$ satisfies a bound $\|Au, v\| \leq C\|u\|_{L^2}$ for some $C$. But $\langle Au, v \rangle = \langle u, Av \rangle$, hence we conclude that $A^*v \in L^2(M; E)$, and by elliptic regularity $v \in H^m(M; E)$; thus $\mathcal{D}(A^*) \subset \mathcal{D}(A)$. The converse is clear since $A$ is symmetric.

To prove the discreteness of the spectrum, note simply that $(A + i)^{-1} : L^2(M; E) \to H^m(M; E) \hookrightarrow L^2(M; E)$ is a compact operator, hence its spectrum is discrete and can only accumulate at 0. But $A\phi = \lambda\phi$ implies $(A + i)^{-1}\phi = (\lambda + i)^{-1}\phi$, hence spec $A$ can only accumulate at $\infty$.

**Example 5.61.** This applies to the Laplacian $\Delta_g$ on any compact Riemannian manifold $(M, g)$, acting on functions or differential forms.

**Example 5.62.** There exist elliptic operators whose spectrum is the entire complex plane. In fact, there exists an elliptic operator $A \in \Psi^1(S^1)$ with index 1 (or any other integer). By Remark 5.59, $A - \lambda$ is never invertible for any $\lambda \in \mathbb{C}$.

### 5.11. A simple nonlinear example

As a simple (and naive, weak, and wasteful, but instructive) nonlinear application of the elliptic theory developed thus far, we shall solve a non-linear elliptic equation on a compact 2-dimensional manifold $M$. If $g$ is a Riemannian metric on $M$, we denote the Gauss curvature of $M$ by $K_g \in \mathcal{C}^\infty(M)$. If $\phi \in \mathcal{C}^\infty(M)$, then the metric $g'(x) = e^{2\phi(x)}g(x)$ is said to be conformal to $g$. The Gauss curvature of $g'$ is given by

$$K_{g'} = e^{-2\phi}(K_g - \Delta_g \phi). \quad (5.129)$$

**Proposition 5.63.** Suppose $(M, g)$ has constant Gauss curvature $K_g \equiv -1$. Let $\tilde{g} \in \mathcal{C}^\infty(M; S^2 T^* M)$ be a Riemannian metric with $\|g - \tilde{g}\|_{H^4} < \epsilon$, $\epsilon > 0$ small. Then there exists $\phi \in \mathcal{C}^\infty(M)$ such that $e^{2\phi}\tilde{g}$ has constant Gauss curvature $-1$.

This is a local version of the uniformization theorem; the conclusion holds for any metric $\tilde{g}$, not necessarily close to $g$. The assumptions require that $M$ is a manifold of genus at least 2 (that is, a donut with at least two holes). For $M \cong S^2$, one can always find a conformal multiple with constant curvature $+1$, and for $M \cong T^2$, one can always find one with constant curvature 0.

**Proof of Proposition 5.63.** We have $H^4(M) \subset C^2(M)$ (in fact $C^{2,\alpha}(M)$ for all $\alpha < 1$); and moreover the map $H^4(M; S^2 T^* M) \ni g \mapsto K_g \in H^2(M; S^2 T^* M)$ is smooth.

We want to solve the equation

$$-1 = K_{e^{2\phi}} - e^{-2\phi}(K_g + \Delta_g \phi), \quad (5.130)$$

or equivalently

$$\Delta_g \phi + e^{2\phi} = K_g. \quad (5.131)$$

Since $\|K_g - K_{\tilde{g}}\|_{H^2}$ is small, we expect $\phi \in H^4(M)$ to be small; this suggests Taylor expanding:

$$A\phi = E(\phi) - N(\phi), \quad A = \Delta_g + 2, \quad E(\phi) = K_{\tilde{g}} - (\Delta_g - \Delta_\tilde{g})\phi, \quad N(\phi) = e^{2\phi} - 1 - 2\phi. \quad (5.132)$$

We solve this using the contraction mapping principle, i.e. by iterating the map

$$T : H^4(M) \ni \phi \mapsto A^{-1}(E(\phi) - N(\phi)) \in H^4(M). \quad (5.133)$$
Now \(\|\Delta_{\bar{g}} - \Delta_g\|_{L^1(H^4(M),H^2(M))} \leq C\varepsilon\) for some constant \(C\), thus \(\|E(\phi)\|_{H^2} \leq C\varepsilon(1 + \|\phi\|_{H^4(M)})\). Moreover, \(\|N(\phi)\|_{H^2} \leq C\|\phi\|^2_{H^4(M)}\) for \(\|\phi\|_{H^4(M)} \leq 1\). Therefore, if \(\|\phi\|_{H^4(M)} \leq \delta\) for some \(\delta > 0\), then

\[
\|T\phi\|_{H^4(M)} \leq C'(C\varepsilon(1 + \delta) + C\delta^2) \leq \delta, \quad C' = \|A^{-1}\|_{L^2(H^2(M),H^4(M))},
\]

provided we take \(\varepsilon = \varepsilon(\delta) := \delta/(10CC')\) and \(\delta\) small enough. The map \(T\) is then a contraction on the \(\delta\)-ball in \(H^4(M)\), since

\[
\begin{align*}
\|T\phi - T\psi\|_{H^4(M)} &\leq C'(C\varepsilon\|\phi - \psi\|_{H^4(M)} + C\|\phi - \psi\|_{H^4(M)}(\|\phi\|_{H^4(M)} + \|\psi\|_{H^4(M)})) \\
&\leq C'(C\varepsilon + C\delta)\|\phi - \psi\|_{H^4(M)} \\
&\leq \frac{1}{2}\|\phi - \psi\|_{H^4(M)}
\end{align*}
\]

for small enough \(\delta > 0\) and \(\varepsilon = \varepsilon(\delta)\).

Let now \(\phi \in H^4(M)\), \(\|\phi\|_{H^4(M)} \leq \delta\), denote the unique fixed point of \(T\); then \(\phi\) solves (5.131). We rewrite this one last time as

\[
\Delta_{\bar{g}}\phi = K_{\bar{g}} - e^{2\phi}.
\]

Suppose we already know \(\phi \in H^k(M), k \geq 4\). Then the right hand side of this equation lies in \(H^k(M)\), so by elliptic regularity, we conclude that \(\phi \in H^{k+2}(M)\). Therefore, \(\phi \in \bigcap_k H^k(M) = C^\infty(M)\), finishing the proof.

5.12. Commutators and symplectic geometry. We tie up a loose end and describe, invariantly, the principal symbol of the commutator of two ps.d.o.s. Key is the symplectic structure of the cotangent bundle \(T^*M\).

**Definition 5.64.** Let \(M\) be an \(n\)-dimensional manifold. The **canonical 1-form** on \(T^*M\) is the section \(\alpha \in C^\infty(TM; T^*(T^*M))\) defined by

\[
\alpha_{(x,\xi)}(v) := \xi(\pi_*v), \quad x \in M, \ \xi \in T^*_xM, \ v \in T_{(x,\xi)}(T^*M),
\]

where \(\pi: T^*M \to M\) is the projection. The **canonical symplectic form** on \(T^*M\) is

\[
\omega := d\alpha \in C^\infty(T^*M; \Lambda^2(T^*M)).
\]

In local coordinates \(x \in \mathbb{R}^n\) and corresponding canonical coordinates \(\xi \in \mathbb{R}^n\) on the fibers of \(T^*M\), we have \(\pi_*(\sum_k a_k\partial_{x_k} + b_k\partial_{\xi_k}) = \sum_k a_k\partial_{x_k}\), and therefore

\[
\alpha = \sum_{k=1}^n \xi_k dx_k, \quad \omega = \sum_{k=1}^n d\xi_k \wedge dx_k.
\]

This is a non-degenerate 2-form: contraction \(T(T^*M) \ni v \mapsto v \hook \omega = \omega(v, -) \in T^*(T^*M)\) is an isomorphism, and identifies vector fields and 1-forms on \(T^*M\):

\[
\sum_{k=1}^n a_k\partial_{x_k} + b_k\partial_{\xi_k} \mapsto \sum_{k=1}^n b_k dx_k - a_k d\xi_k.
\]

**Definition 5.65.** Let \(p \in C^\infty(T^*M)\). Then the **Hamilton vector field** of \(p\) is the unique \(H_p \in \mathcal{V}(T^*M)\) such that

\[
H_p \hook \omega = dp.
\]
The Poisson bracket of $p, q \in C^\infty(T^*M)$ is defined as

$$\{p, q\} := H_p q = -H_q p.$$  \hfill (5.142)

In local coordinates, we deduce from (5.140) that

$$H_p = \sum_{k=1}^{n} (\partial_{\xi_k} p) \partial_{x_k} - (\partial_{x_k} p) \partial_{\xi_k}.$$  \hfill (5.143)

Thus, the ‘ad hoc’ definition (4.75) in fact makes invariant sense on $T^*M$! As a consequence of the local $\mathbb{R}^n$ theory, Proposition 4.22, we thus deduce:

**Corollary 5.66.** Let $A \in \Psi^m(M), B \in \Psi^{m'}(M)$, at least one of which is properly supported. Then $[A, B] \in \Psi^{m+m'-1}(M)$, and

$$\sigma_{m+m'-1}(i[A, B]) = \{\sigma_m(A), \sigma_{m'}(B)\}.$$  \hfill (5.144)

### 5.13. Exercises.

**Exercise 5.1.** Let $M$ denote a smooth manifold, and denote by

$$d: C^\infty(M; \Lambda^k T^*M) \to C^\infty(M; \Lambda^{k+1} T^*M)$$  \hfill (5.145)

the exterior derivative. Show that $d \in \text{Diff}^1(M; \Lambda^k T^*M, \Lambda^{k+1} T^*M)$, and compute its principal symbol.

**Exercise 5.2.** Let $\Gamma \subset \mathbb{C}$ be a smooth, simple, closed curve. Let $K \in C^\infty(\Gamma \times \Gamma)$. Prove that

$$Au(t) := \lim_{\epsilon \to 0} \int_{|t-s| \geq \epsilon} \frac{K(t, s)}{t-s} u(s) \, ds, \quad u \in C^\infty(\Gamma)$$  \hfill (5.146)

is well-defined and defines an element $A \in \Psi^0_{cl}(\Gamma)$. Compute its principal symbol.

**Exercise 5.3.** Let $(X, \| \cdot \|_X), (Y, \| \cdot \|_Y)$ be two Banach spaces, and suppose $A: X \to Y$ is a bounded linear map.

1. Suppose $Z$ is another Banach space, and there is an inclusion (continuous injective map) $X \hookrightarrow Z$ which is compact. Suppose there exists a constant $C > 0$ such that

$$\|u\|_X \leq C (\|Au\|_Y + \|u\|_Z).$$  \hfill (5.147)

Show that $\ker A \subset X$ is finite-dimensional, and that $\text{ran} A \subset Y$ is closed.

2. Suppose that, in addition, to (1), there exists a Banach space $\tilde{Z}$ and an inclusion $Y^* \hookrightarrow \tilde{Z}$ which is compact. Suppose there exists $C > 0$ such that

$$\|v\|_{Y^*} \leq C \left(\|A^* v\|_{X^*} + \|v\|_{\tilde{Z}}\right).$$  \hfill (5.148)

Show that if $f \in Y$ is such that $v(f) = 0$ for all $v \in \ker A^*$, then there exists $u \in X$ with $Au = f$. Deduce that $A$ is a Fredholm operator.

TBC analytic Fredholm theorem

**Exercise 5.4.** TBC principal symbol by oscillatory testing (diff ops, and ps.d.o.s, generalizing $\mathbb{R}^n$ version)

**Exercise 5.5.** TBC elliptic iff Fredholm: estimates imply ellipticity (using oscillatory testing)
Exercise 5.6. Let $M$ be a compact manifold, let $E, F \to M$ denote two vector bundles, and let $A \in \Psi^m(M; E, F)$.

1. Suppose $\sigma_m(A)$ is injective; that is, there exists $b \in S^m(T^*M; \text{Hom}(F, E))$ such that $b\sigma_m(A) - 1 \in S^{-1}(T^*M; \text{End}(E))$. Show that $A: H^s(M; E) \to H^{s-m}(M; F)$ has finite-dimensional kernel and closed range.

2. Suppose $\sigma_m(A)$ is surjective; that is, there exists $b \in S^m(T^*M; \text{Hom}(F, E))$ such that $\sigma_m(A) b - 1 \in S^{-1}(T^*M; \text{End}(F))$. Show that $A: H^s(M; E) \to H^{s-m}(M; F)$ has closed range and finite-dimensional cokernel.

Exercise 5.7. Let $M$ be a compact manifold, let $E_i \to M$, $i = 0, \ldots, N$, be complex vector bundles, and suppose $d_i \in \text{Diff}^1(M; E_i, E_{i+1})$, $i = 0, \ldots, N - 1$. Suppose they form a complex of differential operators

$$C^\infty(M; E_0) \xrightarrow{d_0} C^\infty(M; E_1) \xrightarrow{d_1} \cdots \xrightarrow{d_{N-1}} C^\infty(M; E_N);$$

that is, for each $i < N$,

$$d_{i+1} \circ d_i = 0 \in \text{Diff}^2(M; E_i, E_{i+2}).$$

Assume moreover that this complex is elliptic, meaning that the symbol complex

$$C^\infty(T^*M \setminus o; \pi^*E_0) \xrightarrow{\sigma_1(d_0)} C^\infty(T^*M \setminus o; \pi^*E_1) \xrightarrow{\sigma_1(d_1)} \cdots \xrightarrow{\sigma_1(d_{N-1})} C^\infty(T^*M \setminus o; \pi^*E_N)$$

is exact (that is, ran $\sigma_1(d_{i+1})(x, \xi) = \ker \sigma_1(d_i)$ for all $i < N$). The goal of this exercise is to study the cohomology of the complex

$$H^i(E_\bullet) := (\ker d_i)/(\text{ran } d_{i-1}), \quad i = 1, \ldots, N - 1,$$

using PDE theory.

1. Equip $M$ with a volume density and the $E_i$ with Hermitian fiber inner products; define $\delta_i \in \text{Diff}^1(M; E_i, E_{i-1})$ to be the adjoint of $d_{i-1}$. Show that the ‘Laplacian’

$$\Delta_i := d_{i-1} \circ \delta_i + \delta_{i+1} \circ d_i \in \text{Diff}^2(M; E_i), \quad 1 \leq i \leq N - 1,$$

is elliptic and symmetric.

2. Show that

$$\ker \Delta_i = \{u \in C^\infty(M; E_i): d_i u = 0, \delta_i u = 0\}.$$  

3. Show that the inclusion $\ker \Delta_i \hookrightarrow \ker d_i$ induces an isomorphism of vector spaces

$$\ker \Delta_i \cong H^i(E_\bullet).$$

4. Prove the Hodge theorem: if $(M, g)$ is a compact Riemannian manifold, and $\Delta_k \in \text{Diff}^2(M; \Lambda^kT^*M)$ is the Hodge Laplacian on $k$-forms, then $\ker \Delta_k \cong H^k(M)$, where $H^k(M)$ denotes the $k$-th de Rham cohomology group (with complex coefficients) of $M$.

TBC Helmholtz decomposition

TBC tensor-vector-scalar decomposition on $n$-sphere (bonus problem)
6. Microlocalization

We now turn to the second part of these lecture notes: finer properties of distributions, and, closely related, non-elliptic phenomena. We develop the notion of distributional wave front from the observation about the local nature of full symbolic expansions that we observed e.g. after the statement of Theorem 4.16.

From now on, all ps.d.o.s shall either be properly supported, or elements of the uniform ps.d.o. algebra on $\mathbb{R}^n$.

6.1. Operator wave front set. Recall from (4.55) and (4.57) the full symbols, modulo $S^{-\infty}$, for adjoints and compositions: if $A = \text{Op}_L(a)$, $B = \text{Op}_L(b)$ are ps.d.o.s on $\mathbb{R}^n$, then

$$
\sigma_L(A^*)(x, \xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_\xi^\alpha D_x^\alpha a(x, \xi),
$$

$$
\sigma_L(A \circ B)(x, \xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi) \cdot D_x^\alpha b(x, \xi). \tag{6.1}
$$

A key feature, which so far we have only exploited at the principal symbol level, is that these formulas are \textit{local} in $(x, \xi)$. We would like to say that if $A$ is ‘trivial’ at or near $(x, \xi)$, in the sense that $a$ vanishes there, then $A^*$ and $A \circ B$ (for any $B$) are trivial there as well. Unfortunately, since the expressions (6.1) are \textit{asymptotic} sums, thus have no content at any fixed point $(x, \xi)$, the meaning of this is not immediately clear.

The correct notion of ‘triviality’ must depend on the behavior of symbols as $|\xi| \to \infty$, and must be insensitive to modifications by symbols of order $-\infty$. This leads to the following definition:

**Definition 6.1.** Let $a \in S^m(\mathbb{R}^n; \mathbb{R}^N)$. Then a point $(x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^N \setminus \{0\})$ does not lie in the \textit{essential support}

$$
\text{ess supp } a \subset \mathbb{R}^n \times (\mathbb{R}^N_\xi \setminus \{0\}) \tag{6.2}
$$

if and only if $a$ is a symbol of order $-\infty$ near $x_0$ and in a conic neighborhood of $\xi_0$; that is, there exists $\epsilon > 0$ such that for all $\alpha, \beta \in \mathbb{N}_0^n$, $k \in \mathbb{R}$, we have

$$
|\partial_\xi^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha \beta k} |\xi|^{-k} \quad \forall (x, \xi), \quad |\xi| \geq 1, \quad |x - x_0| + \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \epsilon. \tag{6.3}
$$

It suffices, in fact, to assume (6.3) only for $\alpha = \beta = 0$; the estimates for general $\alpha, \beta$ are then automatic. See Exercise 6.1.

**Remark 6.2.** By definition, ess supp $a$ is a \textit{closed} subset of $\mathbb{R}^n \times (\mathbb{R}^N \setminus \{0\})$. Moreover, ess supp $a$ is \textit{conic} in $\xi$, that is, $(x, \xi) \in \text{ess supp } a$ implies $(x, \lambda \xi) \in \text{ess supp } a$ for all $\lambda > 0$.

**Definition 6.3.** Let $A = \text{Op}_L(a)$. Then we define the \textit{operator wave front set} of $A$ as the closed, conic set

$$
WF'(A) := \text{ess supp}(a) \subset \mathbb{R}^n_\xi \times (\mathbb{R}^N_\xi \setminus \{0\}). \tag{6.4}
$$

The following follows immediately from (6.1), the formulas for left/right reductions in (4.34)–(4.35), as well as formula (5.13) in the proof of the local coordinate invariance of ps.d.o.s:
Proposition 6.4. The operator wave front set for operators $A, B \in \Psi(\mathbb{R}^n)$ has the following properties:

1. We have $\text{WF}'(A) = \emptyset$ if and only if $A \in \Psi^{-\infty}(\mathbb{R}^n)$.
2. Let $A = \text{Op}_R(a')$. Then $\text{WF}'(A) = \text{ess supp } a'$.
3. $\text{WF}'(A + B) \subset \text{WF}'(A) \cup \text{WF}'(B)$.
4. $\text{WF}'(A \circ B) \subset \text{WF}'(A) \cap \text{WF}'(B)$.
5. $\text{WF}'(A^*) = \text{WF}'(A)$.
6. If $\Omega, \Omega' \subset \mathbb{R}^n$, $\kappa: \Omega \to \Omega'$ is a diffeomorphism, $A \in \Psi_c(\Omega')$, and $A_\kappa = \kappa^* A(\kappa^{-1})^*$, then

$$\text{WF}'(A_\kappa) = \kappa^* \text{WF}'(A),$$

where we define $\kappa^*(x, \xi) = (\kappa^{-1}(x), \kappa'(x)^T \xi)$.

Of these, properties (4) and (5) were our motivation for the introduction of $\text{ess supp }$ above. Property (6) implies that the operator wave front set can be defined invariantly for operators on manifolds:

Definition 6.5. Let $M$ be a manifold and $A \in \Psi^m(M)$. Then $\text{WF}'(A) \subset T^*M \setminus o$ is the closed conic subset (i.e. invariant under dilations in the fibers of $T^*M$) given near $T^*_pM$, $p \in M$, by $\text{WF}'(A_0)$ where $A_0 \in \Psi^m(\mathbb{R}^n)$ is the expression for $A$ in a local coordinate system near $p$.

Properties (1) and (4)–(6) in Proposition 6.4 thus hold for ps.d.o.s on manifolds as well. We leave the details of the definition of $\text{WF}'(A)$ for operators $A \in \Psi^m(M; E, F)$ acting between sections of vector bundles over $M$ to the reader; in local coordinates and trivializations, a point is in $\text{WF}'(A)$ if it is in the operator wave front set of just one entry of the matrix of ps.d.o.s on $\mathbb{R}^n$ representing $A$ locally.

One thinks of $\text{WF}'(A) \subset T^*M \setminus o$ as the set in phase space where $A$ is microlocally non-trivial. This is a much weaker notion than having a (microlocally) elliptic principal symbol, see §??.

Example 6.6. If $A \in \Psi^m(M)$ is elliptic, then $\text{WF}'(A) = T^*M \setminus o$.

Example 6.7. Let $A = \sum_{|\alpha| \leq m} a_\alpha(x) D_x^\alpha \in \text{Diff}^m(\mathbb{R}^n)$. Then

$$\text{WF}'(A) = \left( \bigcup_{|\alpha| \leq m} \text{supp } a_\alpha \right) \times (\mathbb{R}^n \setminus \{0\}).$$

(6.6)

Thus, differential operators never have ‘interesting’ operator wave front set.

Example 6.8. Let $\chi \in \mathcal{S}^m(\mathbb{R}^n_\xi)$, and consider the Fourier multiplier $A = \chi(D) := \text{Op}(\chi)$. Then $\text{WF}'(A) = \mathbb{R}^n_\xi \times \text{ess supp } \chi$.

Example 6.9. We combine Examples 6.7 (for $m = 0$) and 6.8. Let $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $\chi \in \mathcal{S}^0(\mathbb{R}^n_\xi)$. Then

$$A := \chi(D) \circ \phi(x) = \text{Op}_R(\chi(\xi) \phi(y)) \in \Psi^0(\mathbb{R}^n)$$

has $\text{WF}'(A) \subset (\text{supp } \phi) \times (\text{ess supp } \chi)$. 

(6.7)
Working with conic sets is a bit tedious. In most circumstances, one can simplify notation by working on the cosphere bundle

\[ S^* M := (T^* M \setminus o)/\mathbb{R}^+, \]  

(6.8)

with fibers given by \( S^*_p M = (T^*_p M \setminus o)/\mathbb{R}^+ \), where \( \mathbb{R}^+ \) acts by dilations in the fibers. Thus, \( S^* M \) is a fiber bundle with typical fiber \( S^{n-1} \). We can identify conic subsets of \( T^* M \setminus o \) with their image in \( S^* M \). For instance, if \( A \in \Psi^m(M) \), then for \( \alpha \in S^*_p M \), the condition \( \alpha \in WF'(A) \) means that \( (p, \xi) \in WF'(A) \) where \( \alpha = [\xi] \) (i.e. \( \alpha = R_+ \xi \)). Note that a compact subset of \( S^* M \) is identified with a conic subset of \( T^* M \setminus o \) whose cross section (i.e. intersection with \( |\xi| = 1 \) for some choice of fiber metric on \( T^* M \)) is compact. The projection map is denoted

\[ \pi: S^* M \rightarrow M. \]  

(6.9)

The following technical result states that one can construct partitions of unity microlocally:

**Lemma 6.10.** Suppose \( S^* M = \bigcup_i U_i \) is an open cover. Then there exist operators \( A_i \in \Psi^0(M) \) such that

1. the supports of the Schwartz kernels of \( A_i \) are locally finite,
2. \( WF'(A_i) \subset U_i \),
3. \( \sum_i A_i = I \).

**Proof.** It suffices to prove this for a locally finite refinement of the cover, which we shall denote by \( \{U_i\} \) still, for which moreover each \( U_i \) lies over a coordinate chart, i.e. \( U_i \subset S^*_i M \) with \( F_i: V_i \rightarrow F_i(V_i) \subset \mathbb{R}^n \) a chart, and with \( \{V_i\} \) locally finite.

Pick partitions of unity \( \{\chi\} \) subordinate to the cover \( \{U_i\} \); fix \( \psi_i, \tilde{\psi}_i \in C_c^\infty(F_i(V_i)) \) such that \( \psi_i \equiv 1 \) near \( F_i(\pi(\text{supp} \, \phi_i)) \), and \( \tilde{\psi}_i \equiv 1 \) near \( \text{supp} \, \psi_i \). We then put

\[ A'_i := F^*_i \left( \tilde{\psi}_i \text{Op}(\chi_i) \psi_i \right) (F^{-1}_i)^* \]  

(6.10)

Then (1) and (2) are satisfied for \( A'_i \), but rather than (3) we only have \( \sum_i A'_i = I - R', \)

\( R' \in \Psi^{-1}(M) \). Thus, simply let \( B \sim \sum_{j=0}^{\infty}(R')^j \) and put \( A_i := A'_i B \). (This still satisfies (1) and (2), in the former case since \( B \) is properly supported, and in the latter case by part (4) of Proposition 6.4.) then \( \sum_i A_i = I - R, R \in \Psi^{-\infty}(M) \). Replacing any single one of the \( A_i \) by \( A_i + R \), we are done. \( \square \)

**Corollary 6.11.** Let \( K \subset U \subset S^* M \), with \( U \) open. Then there exists \( A \in \Psi^0(M) \) such that \( WF'(A) \subset U \) and \( WF'(I - A) \cap K = \emptyset \).

We say that \( A \) is microlocally equal to \( I \) on \( K \).

**Proof of Corollary 6.11.** \( S^* M = U \cup (S^* M \setminus K) \) is an open cover of \( S^* M \), hence there exists a partition of unity \( I = A + B \) with \( WF'(A) \subset U \) and \( WF'(I - A) \cap K = WF'(B) \cap K = \emptyset \). \( \square \)

### 6.2. Elliptic set, characteristic set

We next refine the notion of ellipticity of operators and symbols in a microlocal manner analogous to \( \text{ess supp} \) and \( WF' \).

**Definition 6.12.** Let \( A \in \Psi^m(M) \). Then the elliptic set of \( A \),

\[ \text{Ell}(A) \subset T^* M \setminus o \]  

(6.11)
The complement of $\text{Ell}(A)$ is the \textit{characteristic set}
\[
\text{Char}(A) := (T^*M \setminus o) \setminus \text{Ell}(A).
\] (6.13)

An equivalent definition of $\text{Ell}(M)$, closer to Definition 3.8, is that there exists $b \in S^{-m}(T^*M)$ such that $\sigma_m(A)b - 1$ is a symbol of order $-1$ in a conic neighborhood of $(x_0, \xi_0)$. Note that $\text{Ell}(A)$ is automatically \textit{open}. Moreover,
\[
\text{Ell}_{m+m'}(A \circ B) = \text{Ell}(A) \cap \text{Ell}_{m'}(B), \quad A \in \Psi^m(M), \ B \in \Psi^{m'}(M).
\] (6.14)

Example 6.13. Elliptic operators on $M$ have elliptic set equal to $T^*M \setminus o$, and empty characteristic set.

Example 6.14. On $\mathbb{R}^{1+n} = \mathbb{R}_x \times \mathbb{R}_\xi^n$, with canonical momentum variables $\sigma \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, the wave operator $\Box = D_x^2 - D_\xi^2$ has principal symbol $\sigma_2(\Box) = \sigma^2 - \xi^2$, hence its characteristic set is the double cone
\[
\text{Char}(\Box) = \{(t, x, \sigma, \xi) : \sigma^2 - |\xi|^2 = 0, \ (\sigma, \xi) \neq (0, 0)\}.
\] (6.15)

The elliptic parametrix construction, Theorem 4.26, can be microlocalized:

\textbf{Proposition 6.15.} Let $A \in \Psi^m(M)$, and suppose $K \subset \text{Ell}(A)$ is a closed subset. Then there exists a microlocal parametrix for $A$ on $K$, namely, an operator $B \in \Psi^{-m}(M)$ such that
\[
K \cap \text{WF}'(AB - I) = \emptyset, \quad K \cap \text{WF}'(BA - I) = \emptyset.
\] (6.16)

\textit{Proof.} By Corollary 6.11, we can pick $Q, \tilde{Q} \in \Psi^0(M)$ with
\[
\text{WF}'(I - Q) \cap K = \emptyset, \ \text{WF}'(Q), \ \text{WF}'(\tilde{Q}) \subset \text{Ell}(A), \ \text{WF}'(I - \tilde{Q}) \cap \text{WF}'(Q) = \emptyset.
\] (6.17)

Let then $B_0 \in \Psi^{-m}(M), \ \sigma_m(B_0) = \sigma_0(\tilde{Q})/\sigma_m(A)$, and write
\[
AB_0 = I - QR - (I - Q)R, \quad R \in \Psi^0(M).
\] (6.18)

Now $\text{WF}'((I - Q)R) \cap K = \emptyset$, while $\sigma_0(QR) = \sigma_0(Q)(1 - \sigma_0(\tilde{Q})) = 0$, so $QR \in \Psi^{-1}(M)$. We then improve the situation near $K$ using a Neumann series argument; that is, let
\[
B' \sim \sum_{j=0}^{\infty} (QR)^j \in \Psi^{-1}(M),
\] (6.19)

and put $B := B_0B' \in \Psi^{-m}(M)$. Then
\[
AB = I - R' - (I - Q)RB', \quad R' \in \Psi^{-\infty}(M), \ \text{WF}'((I - Q)RB') \cap K = \emptyset,
\] (6.20)
as desired.

A microlocal left parametrix, say $\tilde{B}$, can be constructed similarly. Then, modulo operators with $\text{WF}'$ disjoint from $K$, we have
\[
B \equiv (\tilde{B}A)B = \tilde{B}(AB) \equiv \tilde{B}.
\] (6.21)
Since (6.16) is invariant under addition to $B$ of an operator with $WF'$ disjoint from $K$, this proves that any microlocal left parametrix is also a right parametrix, and vice versa. □

One would like to use this to sharpen elliptic regularity theory, Proposition 4.27, by saying that if $Au = f$, then on $\text{Ell}(A)$, $u$ is smooth when $f$ is.

6.3. Wave front set of distributions. Let $u \in \mathcal{S}'(\mathbb{R}^n)$ denote a distribution, and $A \in \Psi^m(\mathbb{R}^n)$. Then $Au \in \mathcal{S}'(\mathbb{R}^n)$ is ‘trivial’ outside of $WF'(A)$: all information about singularities of $u$ is lost. Indeed, if $B \in \Psi^0(\mathbb{R}^n)$ is such that $WF'(B) \cap WF'(A)$, we have $B(Au) \in C^\infty(\mathbb{R}^n)$ by part (1) of Proposition 6.4. The precise notion of ‘triviality’ here is, directly stated on manifolds:

**Definition 6.16.** Let $u \in \mathcal{D}'(M)$. Then $\alpha \in S^*M$ does not lie in the wave front set, 

$$\alpha \notin WF(u) \subset S^*M \quad (6.22)$$

if and only if there exists a neighborhood $U \subset S^*M$ of $\alpha$ such that

$$A \in \Psi^0(M), \ WF'(A) \subset U \Rightarrow Au \in C^\infty(M). \quad (6.23)$$

By definition, $WF(u) \subset S^*M$ is closed. We leave to the reader the simple verification that for $M = \mathbb{R}^n$, $WF(u)$ for $u \in \mathcal{S}'(\mathbb{R}^n)$ can be defined equivalently by testing with uniform ps.d.o.s $A \in \Psi^0(\mathbb{R}^n)$.

We will give equivalent conditions which are easier to verify. We begin by reducing the number of operators for which one needs to check (6.23) to one.

**Lemma 6.17.** Let $u \in \mathcal{D}'(M)$. Then $\alpha \notin WF(u)$ if and only if there exists $A \in \Psi^0(M)$, elliptic at $\alpha$, such that $Au \in C^\infty(M)$.

**Proof.** The direction ‘$\Rightarrow$’ is obvious. To prove ‘$\Leftarrow$’, we take $U := Ell(A)$, which by assumption is a neighborhood of $\alpha$. Let $B \in \Psi^0(M)$, $WF'(B) =: K \subset U$; we claim that $Bu \in C^\infty(M)$. By Proposition 6.15, there exists a microlocal parametrix $Q \in \Psi^0(M)$ of $A$ with $QA = I - R$, $R \in \Psi^0(M)$, $WF'(R) \cap K = \emptyset$. Therefore,

$$Bu = B(QA + R)u = (BQ)(Au) + BRu \in C^\infty(M). \quad (6.24)$$

Indeed, $Au \in C^\infty(M)$, hence the first summand is smooth; and $WF'(BR) \subset WF'(B) \cap WF'(R) = \emptyset$, hence $BR \in \Psi^{-\infty}(M)$ and so $BRu \in C^\infty(M)$ as well. □

**Corollary 6.18.** Let $u \in \mathcal{D}'(M)$. Then

$$WF(u) = \bigcap_{A \in \Psi^0(M), \ Au \in C^\infty(M)} \text{Char}(A). \quad (6.25)$$

**Proof.** If $\alpha \notin WF(u)$, then $Au \in C^\infty(M)$ for some $A \in \Psi^0(M)$ with $\alpha \in \text{Ell}(A)$, so $\alpha \notin \text{Char}(A)$. Conversely, if $\alpha \in \text{Ell}(A)$ for some $A \in \Psi^0(M)$ with $Au \in C^\infty(M)$, then $\alpha \notin WF(u)$ by Lemma 6.17. □

This leads to the following very concrete description of the wave front set, which we state directly on $\mathbb{R}^n$; it is the same on manifolds upon localizing in a chart and transferring to $\mathbb{R}^n$. 
Proposition 6.19. Let \( u \in \mathcal{D}'(\mathbb{R}^n) \), and let \((x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})\). Then \((x_0, \xi_0) \notin \text{WF}(u)\) if and only if there exist \( \phi \in C_c^\infty(\mathbb{R}^n) \), \( \phi(x_0) \neq 0 \), and \( \epsilon > 0 \) such that

\[
|\hat{\phi}u(\xi)| \leq C_N |\xi|^{-N}, \quad \xi \in \mathbb{R}^n, \ |\xi| \geq 1, \quad \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \epsilon. \tag{6.26}
\]

Remark 6.20. The converse direction can be strengthened slightly: to show that \((x_0, \xi_0) \in \text{WF}(u)\) does lie in the wave front set, it suffices to show that for any \( \epsilon > 0 \) there exists \( \phi \in C_c^\infty(\mathbb{R}^n) \), \( \phi(x_0) \neq 0 \), \( \text{supp} \phi \subset B(x_0, \epsilon) \), such that the estimate (6.27) fails. (That is, as witnesses one can take any convenient cutoffs \( \phi \), as long as their support can be taken to be arbitrarily close to \( x_0 \).) Indeed, this follows from the definition of WF and the proof of Corollary 6.19.

An advantage of our invariant approach to WF is that it implies ‘for free’ that the hands-on condition (6.27) gives a well-defined notion of wave front set as a subset of the cotangent bundle. We encourage the reader to try and give a direct proof of this fact, based on the characterization (6.27).

Proof of Corollary 6.19. Suppose (6.27) holds. Let \( \psi \in C_*^\infty(S^{n-1}) \) have support in an \( \epsilon \)-ball around \( \xi_0/|\xi_0| \), with \( \psi\left( \frac{\xi_0}{|\xi_0|} \right) \neq 0 \). Let moreover \( \chi \in C_*^\infty(\mathbb{R}^n) \), \( \chi(\xi) = 0 \) for \(|\xi| \leq 1 \) and \( \chi(\xi) = 1 \) for \(|\xi| \geq 2 \). Then

\[
a(\xi, y) := \chi(\xi)\psi\left( \frac{\xi}{|\xi|} \right)\phi(y) \in S^0(\mathbb{R}^n \times \mathbb{R}^n) \tag{6.27}
\]

and \( |\mathcal{F}(\text{Op}_R(a)u)(\xi)| \leq C_N |\xi|^{-N} \); therefore \( \text{Op}_R(a)u \in C_*^\infty(\mathbb{R}^n) \). Since \( a \) is elliptic at \((x_0, \xi_0)\), this implies \((x_0, \xi_0) \notin \text{WF}(u)\).

Conversely, if \((x_0, \xi_0) \notin \text{WF}(u)\), pick \( B \in \Psi^0(\mathbb{R}^n) \), elliptic at \((x_0, \xi_0)\), such that \( Bu \in C_*^\infty(\mathbb{R}^n) \). We can then choose \( \phi \in C_c^\infty(\mathbb{R}^n) \), \( \phi(x_0) \neq 0 \), and \( \psi \in C_*^\infty(S^{n-1}) \), \( \psi\left( \frac{\xi_0}{|\xi_0|} \right) \neq 0 \), such that for \( a(\xi, y) \) defined in (6.27), and \( A := \text{Op}_R(a) \), we have

\[\text{WF}'(A) \subset \text{Ell}(B).\tag{6.28}\]

(Cf. Example 6.9.) By the proof of Lemma 6.17, we thus have \( Au \in C_*^\infty(\mathbb{R}^n) \). We claim that in fact

\[Au \in \mathcal{S}(\mathbb{R}^n),\tag{6.29}\]

which proves (6.27) upon taking the inverse Fourier transform of \( Au \). To prove (6.29), let \( \tilde{\phi} \in C_c^\infty(\mathbb{R}^n) \) be identically 1 near \( \text{supp} \phi \). Then \( \tilde{\phi}(Au) \in C_c^\infty(\mathbb{R}^n) \), while \((1 - \tilde{\phi})Au = \text{Op}(a')u\) where

\[a'(x, y, \xi) = (1 - \tilde{\phi}(x))a(\xi, y) \in S^0(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n).\tag{6.30}\]

But \( a'(x, y, \xi) \) near \( x = y \). In fact, \( a' \) is a scattering symbol of order \((0, 0, 0)\) vanishing near the diagonal, hence \( \text{Op}(a') \in \Psi_{sc}^{\infty, -\infty}(\mathbb{R}^n) \) has Schwartz kernel in \( \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \) by Exercises 4.12–4.13, which implies \( \text{Op}(a')u \in \mathcal{S}(\mathbb{R}^n) \).\(\square\)

\footnote{In fact, the Fourier transform of \( \hat{\phi}u \in \mathcal{E}'(\mathbb{R}^n) \) is analytic and polynomially bounded. Using Cauchy’s integral formula, or Exercise 6.1 if one wants to stick to real methods, one then shows that the estimate (6.27) holds for all derivatives \( \partial_\xi^\alpha \hat{\phi}u(\xi) \), \( \alpha \in \mathbb{N}_0^3 \), as well. Thus \( \text{Op}_R(a)u \in \mathcal{S}(\mathbb{R}^n) \).}

\footnote{A direct proof proceeds by writing \( \text{Op}(a') = \text{Op}(|x - y|^{-2N} \Delta_\xi^N a') \) and noting that \( |x - y|^{-2N} \lesssim \langle x \rangle^{-N} \langle y \rangle^{-N} \) on \( \text{supp} a' \) as well as \( \Delta_\xi^N a' \in S^{-2N}(\mathbb{R}^n \times \mathbb{R}^n) \). Thus, mimicking the proof of Proposition 4.10 gives \( \text{Op}(a') \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \).}
Another important consequence of Lemma 6.17 and Corollary 6.18 is the following result which shows that WF is a significant refinement of sing supp:

**Theorem 6.21.** Let \( u \in \mathcal{D}'(M) \), and denote by \( \pi : T^*M \to M \) the projection. Then

\[
\pi(WF(u)) = \text{sing supp } u. \tag{6.31}
\]

**Proof.** If \( x_0 \notin \text{sing supp } u \), then there exists \( \chi \in C^\infty_c(M) \) with \( \chi(x_0) \neq 0 \) such that \( \chi u \in C^\infty_c(M) \). But \( \chi \) is elliptic at \( (x_0, \xi_0) \) for any \( 0 \neq \xi_0 \in T_{x_0}^*M \); hence \( T_{x_0}^*M \cap WF(u) = \emptyset \).

To prove the converse, suppose \( x_0 \notin \pi(WF(u)) \). Then for each \( \xi \in S_{x_0}^*M \), there exists \( A_\xi \in \Psi^0(M) \), elliptic at \( \xi \), such that \( Au \in C^\infty(M) \). Let \( U_\xi := \text{Ell}(A_\xi) \cap S_{x_0}^*M \). Then \( U_\xi \) is an open cover of the compact set \( S_{x_0}^*M \); thus we can pick a finite subcover,

\[
S_{x_0}^*M = \bigcup_{i=1}^N U_{\xi_i}, \quad \xi_i \in S_{x_0}^*M, \ i = 1, \ldots, N. \tag{6.32}
\]

But then the operator

\[
A := \sum_{i=1}^N A_{\xi_i}^* \in \Psi^0(M) \tag{6.33}
\]

is elliptic on \( S_{x_0}^*M \), and satisfies \( Au \in C^\infty(M) \). If \( \chi \in C^\infty_c(M) \), \( \chi(x_0) \neq 0 \), is chosen to have support so close to \( x_0 \) such that \( S_{x_0}^*M \cap \text{Ell}(A) \), then \( \chi u \in C^\infty(M) \) by the proof of Lemma 6.17. \( \square \)

**Corollary 6.22.** Let \( u \in \mathcal{D}'(M) \). Then \( WF(u) = \emptyset \) if and only if \( u \in C^\infty(M) \).

We next study the relationship of wave front sets and PDE. We first prove that pseudodifferential operators are microlocal:

**Proposition 6.23.** Let \( A \in \Psi^m(M) \) and \( u \in \mathcal{D}'(M) \). Then

\[
WF(Au) \subset WF(A) \cap WF(u). \tag{6.34}
\]

In view of Theorem 6.21, this is a significant strengthening of the pseudolocality property of ps.d.o.s, see Proposition 4.17 (which holds on manifolds as well).

**Proof of Proposition 6.23.** Suppose \( \alpha \notin WF(A) \), then there exists \( B \in \Psi^0(M) \), elliptic at \( \alpha \), but with \( WF(B) \cap WF(A) = \emptyset \). Thus \( B(Au) = (BA)u \in C^\infty(M) \) since \( BA \in \Psi^{-\infty}(M) \).

If \( \alpha \notin WF(u) \), then there exists \( B \in \Psi^0(M) \), elliptic at \( \alpha \), such that \( Bu \in C^\infty(M) \). Let \( \tilde{B} \in \Psi^0(M) \) be elliptic at \( \alpha \) and with \( WF(\tilde{B}) \subset \text{Ell}(B) \). Let \( Q \in \Psi^0(M) \) be a microlocal parametrix of \( B \) on \( WF(\tilde{B}) \), that is, \( QB = I - R, \ R \in \Psi^0(M), \ WF(\tilde{B}) \cap WF(R) = \emptyset \). Then

\[
\tilde{B}(Au) = \tilde{B}A(QB + R)u = \tilde{B}AQ(Bu) + (\tilde{B}AR)u. \tag{6.35}
\]

The first summand is smooth since \( Bu \) is; the second summand is smooth since \( \tilde{B}AR \in \Psi^{-\infty}(M) \). \( \square \)

Moreover, we have the following microlocal elliptic regularity result:

**Proposition 6.24.** Let \( u \in \mathcal{D}'(M) \) and \( A \in \Psi^m(M) \). Then

\[
WF(u) \subset WF(Au) \cup \text{Char}(A). \tag{6.36}
\]

In particular, if \( A \) is elliptic, then \( WF(u) = WF(Au) \).
Proposition 6.26. The analogue of Proposition 6.19 is the following: results analogous to those for the \( C^\infty \) wave front set, as it measures the lack of smoothness of a distribution. In applications, a more refined notion is much more useful:

**Definition 6.25.** Let \( s \in \mathbb{R}, u \in \mathcal{D}'(M) \). Then the \( H^s \) wave front set of \( u \) is

\[
WF^s(u) := \bigcap_{A \in \Psi^0(M)} \text{Char}(A).
\]

(6.37)

That is, its complement is the set of all \( \alpha \in S^*M \) for which there exists \( A \in \Psi^0(M) \), elliptic at \( \alpha \), such that \( Au \in H^s_{\text{loc}}(M) \).

This is equivalent to the alternative definition parallelling Definition 6.16. We collect results analogous to those for the \( C^\infty \) wave front set; the proofs are left to the reader. (They are the same as those for the \( C^\infty \) wave front set, except one now one needs to keep track of Sobolev orders.) We have

\[
WF^s(u) = \emptyset \iff u \in H^s_{\text{loc}}(M).
\]

(6.38)

The analogue of Proposition 6.19 is the following:

**Proposition 6.26.** Let \( u \in \mathcal{D}'(\mathbb{R}^n), (x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \). Then \( (x_0, \xi_0) \notin WF^s(u) \) if and only if there exists \( \phi \in C^\infty_c(\mathbb{R}^n) \), \( \phi(x_0) \neq 0 \), and \( \psi \in C^\infty(\mathbb{S}^{n-1}) \), \( \psi(\xi_0/|\xi_0|) \neq 0 \), such that

\[
\langle \xi \rangle^s \psi(\xi) \widehat{\phi u}(\xi) \in L^2(\mathbb{R}_x^n).
\]

(6.39)

The sharpening of Propositions 6.23 and (6.24) is:

**Proposition 6.27.** Let \( A \in \Psi^m(M), u \in \mathcal{D}'(M) \). Then

\[
WF^{s-m}(Au) \subset WF^s(A) \cap WF^s(u),
\]

(6.40)

\[
WF^s(u) \subset WF^{s-m}(Au) \cup \text{Char}(A).
\]

(6.41)

In particular, if \( A \) is elliptic, then \( WF^s(u) = WF^{s-m}(Au) \).

6.4. Calculus of wave front sets. TBC products, pushforwards, pullbacks

6.5. Exercises.

**Exercise 6.1.** Let \( a \in S^m(\mathbb{R}^n; \mathbb{R}^N) \). Show that \( (x_0, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^N \setminus \{0\}) \) does not lie in \( \text{esssupp } a \) if and only if there exist \( \epsilon > 0 \) such that for all \( k \in \mathbb{R} \), we have

\[
|a(x, \xi)| \leq C_k |\xi|^{-k} \quad \forall (x, \xi), \ |\xi| \geq 1, \ |x - x_0| + \frac{|\xi|}{|\xi|} < \varepsilon.
\]

(6.42)

**Exercise 6.2.** TBC wave front set of real-valued distributions is symmetric around \( \xi = 0 \)

**Exercise 6.3.** TBC WF(\(1_\Omega\))

**Exercise 6.4.** TBC distribution with WF a point

**Exercise 6.5.** TBC polarization set?

TBC prove directly the invariance of the wave front set under coordinate changes
7. **Summary of the Pseudodifferential Calculus on Compact Manifolds**

TBC Wunsch? Add references to relevant theorems etc to all statements made here

8. **Real Principal Type Propagation of Singularities**

TBC point out that everything works for principally scalar operators on vector bundles

8.1. **Exercises.** TBC

\[ |g'| \lesssim g^{1/2} \text{ uniformly on compacts for } C^2 \ni g \geq 0. \text{ Optimality in general?} \]

9. **Propagation of Singularities at Radial Points**

10. **Meromorphic Continuation of the Resolvent on Asymptotically Hyperbolic Spaces**

**References**


Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139-4307, USA

Email address: phintz@mit.edu