EXERCISES AROUND GIRAUD’S AXIOMS

PETER J. Haine

Exercise 1.
(a) Prove that the \( \omega \)-compact objects of \( \text{Set} \) are the finite sets. Use this to prove that \( \text{Set} \) is \( \omega \)-presentable.
(b) Let \( C \) be a small category. Prove that the functor category \( \text{Set}^C \) is \( \omega \)-presentable.
(c) Let \( D \) be a \( \kappa \)-presentable category and \( i : D' \hookrightarrow D \) a localization of \( D \) with localization functor \( L : D \to D' \). Prove that if \( i \) preserves \( \kappa \)-filtered colimits, then \( D' \) is \( \kappa \)-presentable.

Definition. Let \( D \) be a category with finite coproducts, and let \( \emptyset \) denote the initial object of \( D \) (i.e., the empty coproduct). We say that coproducts are disjoint in \( D \) if for any objects \( X, Y \in D \), the square

\[
\begin{array}{ccc}
\emptyset & \longrightarrow & Y \\
\downarrow & & \downarrow \iota_Y \\
X & \longrightarrow & X \sqcup Y
\end{array}
\]

is a pullback square in \( D \).

Exercise 2.
(a) Prove that coproducts in \( \text{Set} \) are disjoint
(b) Let \( I \) be a small category and \( D \) a category admitting finite coproducts and \( I \)-shaped colimits. Prove that if coproducts are disjoint in \( D \), then coproducts are disjoint in \( D^I \).
(c) Let \( D \) be a category admitting finite coproducts and \( D' \hookrightarrow D \) a localization of \( D \) with localization functor \( L : D \to D' \). Prove that if coproducts are disjoint in \( D \), then coproducts are disjoint in \( D' \).

Exercise 3. Let \( C \) be a category, and \( F \in \text{PShv}(C) \). Prove that the slice category \( \text{PShv}(C)/F \) is (equivalent to) a presheaf category.

Hint: The category of elements.