Extended étale homotopy groups from profinite Galois categories

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Abstract
In this note we show that the protruncated shape of a spectral co-topos is a delocalization of its profinite stratified shape. This gives a way to reconstruct the extended étale homotopy groups (i.e., the non-profitely complete étale homotopy groups) of a coherent scheme from its profinite Galois category.

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Introduction

Let $X$ be a coherent (i.e., quasicompact quasiseparated) scheme. In recent work with Clark Barwick and Saul Glasman [3], we constructed a delocalization of the profinite completion of the Artin–Mazur–Friedlander étale homotopy type of $X$ [1; 5]. We call this delocalization the profinite Galois category $Gal(X)$ of $X$. The profinite Galois category $Gal(X)$ is pro-object in finite categories, or, equivalently, a category object in profinite topological spaces [2; 3, p. 5 & Construction 13.5]. The underlying category of $Gal(X)$ has objects geometric points of $X$ and morphisms specialization in the étale topology (i.e., is the category of points of the étale topos of $X$). Concretely, given geometric points $x \to X$ and $y \to X$, a morphism $x \to y$ in $Gal(X)$ is a lift $y \to X_{(x)}$ of the geometric point $y \to X$ to the strict localization $X_{(x)}$ of $X$ at $x$. The topology on $Gal(X)$
globalizes the profinite topology on the absolute Galois group $\text{Gal}(\kappa(x_0)^{\text{sep}}/\kappa(x_0))$ of the residue field $\kappa(x_0)$ at each point $x_0 \in X$.

From the profinite category $\text{Gal}(X)$ we can extract a prospace $H(\text{Gal}(X))$ by formally inverting all morphisms. Our delocalization result \cite[Examples 11.6 & 13.6]{3} says that $H(\text{Gal}(X))$ and the étale homotopy type of $X$ become (canonically) equivalent after profinite completion. In this note we provide a stronger relationship between the prospace $H(\text{Gal}(X))$ and the étale homotopy type: they agree up to protruncation. Morphisms in the $\infty$-category $\text{Pro}(\text{Spc})$ of prospaces that induce equivalences after protruncation are precisely those morphisms that become $\sharp$-isomorphisms in the category $\text{Pro}(\text{hSpc})$, in the terminology of Artin–Mazur \cite[Definition 4.2]{1}.

**A Theorem.** Let $X$ be a coherent scheme and write $\Pi^\text{ét}_\infty(X) \in \text{Pro}(\text{Spc})$ for the étale homotopy type of $X$. Then there is a natural natural map of prospaces

$$\theta_X : \Pi^\text{ét}_\infty(X) \to H(\text{Gal}(X)).$$

Moreover, $\theta_X$ induces an equivalence on protruncations. As a consequence:

- For each integer $n \geq 1$ and geometric point $x \to X$, we have canonical isomorphisms of progroups

$$\pi^\text{ét}_n(X, x) \cong \pi_n(H(\text{Gal}(X)), x),$$

where $\pi^\text{ét}_n(X, x)$ is the $n^\text{th}$ homotopy progroup of the étale homotopy type of $X$.

- For any ring $R$, there is an equivalence of $\infty$-categories between local systems of $R$-modules on $X$ that are uniformly bounded both below and above and continuous functors $\text{Gal}(X) \to D^b(R)$ that carry every morphism to an equivalence.

The progroups $\pi^\text{ét}_n(X, x)$ are what we call the extended étale homotopy groups of $X$. Note that the progroup $\pi^\text{ét}_1(X, x)$ is the *groupe fondamentale élargi* of \cite[Exposé X, §6]{sga3}; the usual étale fundamental group of \cite[Exposé V, §7]{sga1} is the profinite completion of $\pi^\text{ét}_1(X, x)$.

While the protruncated étale homotopy type of a connected Noetherian geometrically unibranch scheme is already profinite \cite[Theorem 11.1; 5, Theorem 7.3; DAG xiii, Theorem 3.6.5]{1}, in general Theorem A provides more refined information about the étale homotopy type, as illustrated in the following example.

**B Example.** Consider the nodal cubic curve

$$C = \text{Spec}(\mathbb{C}[x, y]/(y^2 - x^2(x + 1)))$$

over the complex numbers. The Riemann Existence Theorem \cite[Theorem 12.9; 4, Proposition 4.12; 5, Theorem 8.6]{1} implies that the profinite completion of the étale homotopy type of $C$ is equivalent to the profinite completion of the circle $S^1$. It is well-known that, in fact, the protruncation of the étale homotopy type of $C$ is $S^1$; Theorem A provides an easy "categorical" explanation of this fact.

There is a continuous functor from $\text{Gal}(C)$ to the poset category $\{0 < 1\}$ given by sending the node point to 0 and every other geometric point to 1. The local ring $O_{C, (x, y)}$
at the node point has two prime ideals and the strict Henselization of $O_{C,(x,y)}$ is isomorphic to the strict Henselization of 

$$(C[u,v]/(uv))_{(u,v)}.$$ Using this one sees that there are two lifts of the generic geometric point of $C$ to the strict localization of $C$ at the node. Hence the continuous functor $\text{Gal}(C) \to \{0 < 1\}$ factors through the category $D$ with two objects $0$ and $1$ and two distinct morphisms $0 \Rightarrow 1$. Moreover, the functor $\text{Gal}(C) \to D$ induces an equivalence on underlying homotopy types: the prospace $H(\text{Gal}(C))$ is equivalent to $H(D) = S^1$. Theorem A now shows that the protruncation of the étale homotopy type of the nodal cubic is $S^1$.

We relate the étale homotopy type and profinite Galois category of a coherent scheme by situating the problem in a more general context. In [3] we provided an equivalence of $\infty$-categories

$$\text{Pro}(\text{Str}_\pi) \Rightarrow \text{StrTop}^{\text{spec}}_{\infty}$$

between the $\infty$-category of profinite stratified spaces (on the left) and the $\infty$-category of spectral stratified $\infty$-topoi (on the right) [3, Theorem 10.10]. The primary example of a spectral stratified $\infty$-topos is the étale $\infty$-topos $X_{et}$ of a coherent scheme $X$ with its natural stratification by the Zariski space of $X$ [3, Example 10.6]. The corresponding profinite stratified space is the profinite Galois category $\text{Gal}(X)$ [3, Construction 13.5].

The equivalence $\text{Pro}(\text{Str}_\pi) = \text{StrTop}^{\text{spec}}_{\infty}$ provides a way to reconstruct the prospace given by the shape of the étale $\infty$-topos of a coherent scheme $X^1$ from its profinite Galois category $\text{Gal}(X)$, via the composite

$$\text{Pro}(\text{Str}_\pi) \longrightarrow \text{StrTop}^{\text{spec}}_{\infty} \longrightarrow \text{Top}_{\infty} \overset{H_{\infty}}{\longrightarrow} \text{Pro}(\text{Spc}) ,$$

where the middle functor forgets the stratification, and $H_{\infty}$ is the shape (see Definition 1.3). There's another functor $H : \text{Pro}(\text{Str}_\pi) \to \text{Pro}(\text{Spc})$ that doesn't require the use of $\infty$-topoi, namely, the extension to pro-objects of the composite

$$\text{Str}_\pi \longrightarrow \text{Cat}_{\infty} \overset{H}{\longrightarrow} \text{Spc} ,$$

where the first functor forgets the stratification and the second functor sends an $\infty$-category $C$ to the homotopy type $H(C)$ obtained by inverting every morphism in $C$. It follows formally that these two functors agree on $\text{Str}_\pi$. Moreover, the extension to pro-objects of a functor $\text{Str}_\pi \to \text{Spc}$, the functor $H : \text{Pro}(\text{Str}_\pi) \to \text{Pro}(\text{Spc})$ preserves inverse limits. Thus we have a map

$$\theta_C : H_{\infty}(\widetilde{C}) \to H(C)$$

natural in $C \in \text{Pro}(\text{Str}_\pi)$. In this note we prove that this map is an equivalence after protruncation:

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\[ \text{This is, up to protruncation, the Artin–Mazur–Friedlander étale homotopy type of } X; \text{ see [6, §5], which we recall in Examples 1.6 and 1.9.} \]
C Theorem (Theorem 2.5). Let $\text{Spc}_{\infty} \subset \text{Spc}$ denote the $\infty$-category of truncated spaces, and write $\tau_{\text{cof}} : \text{Pro}(\text{Spc}) \to \text{Pro}(\text{Spc}_{\text{cof}})$ for the left adjoint to the inclusion. For any profinite stratified space $C$, the natural map
\[
\tau_{\text{cof}} \theta_C : \tau_{\text{cof}} \Pi_{\text{cof}}(\tilde{C}) \to \tau_{\text{cof}} H(C)
\]
of protruncated spaces is an equivalence.

In light of [3, Construction 13.5], Theorem A is immediate from Theorem C. Since the functor $H$ and the shape $\Pi_{\text{cof}}$ agree on $\text{Str}_{\pi}$ and both $H$ and $\tau_{\text{cof}}$ preserve inverse limits, by the universal property of the $\infty$-category of pro-objects, Theorem C follows once we know that the protruncated shape $\tau_{\text{cof}} \Pi_{\text{cof}}$ preserves inverse limits. The forgetful functor $\text{StrTop}_{\text{cof}} \to \text{Top}_{\text{cof}}$ factors through the subcategory $\text{Top}_{\text{cof}}^{bc} \subset \text{Top}_{\text{cof}}$ of bounded coherent $\infty$-topoi and coherent geometric morphisms. Theorem C thus reduces to the following fact.

D Theorem (Proposition 2.2). The protruncated shape
\[
\tau_{\text{cof}} \Pi_{\text{cof}} : \text{Top}_{\text{cof}}^{bc} \to \text{Pro}(\text{Spc}_{\text{cof}})
\]
preserves inverse limits.

In § 1 we review the necessary background on pro-objects and shape theory. The familiar reader should skip straight to § 2 where we prove Theorems C and D.

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1 Preliminaries on shapes & protruncated spaces

In this section we review $\infty$-categories of pro-objects and shape theory for $\infty$-topoi. We then record some facts about protruncations that we’ll need.

Review of shape theory

1.1. We say that a small $\infty$-category $I$ is inverse if the opposite $\infty$-category $I^{\text{op}}$ is filtered. An inverse system in an $\infty$-category $C$ is a functor $I \to C$, where $I$ is an inverse $\infty$-category. An inverse limit is a limit of an inverse system.

Let $C$ be an $\infty$-category. We write $\text{Pro}(C)$ for the $\infty$-category of pro-objects in $C$ obtained by freely adjoining inverse limits to $C$, and $j : C \to \text{Pro}(C)$ for the Yoneda embedding. We say that a pro-object $X \in \text{Pro}(C)$ is constant if $X$ lies in the essential image of $j : C \to \text{Pro}(C)$. If $X : I \to C$ is an inverse system, we write $[X]_{\text{rel}} = \lim_{i \in I} j(X_i)$ for the pro-object it defines.

If $C$ is accessible and admits finite limits, then $\text{Pro}(C)$ is equivalent to the full subcategory of $\text{Fun}(C, \text{Spc})^{\text{op}}$ spanned by the left exact accessible functors [SAG, Proposition A.8.1.6]. Let $f : C \to D$ be a left exact accessible functor between accessible $\infty$-categories which admit small limits. Then the functor $f : \text{Pro}(C) \to \text{Pro}(D)$ admits a
left adjoint $L : \text{Pro}(D) \to \text{Pro}(C)$ [SAG, Example A.8.1.8]. We refer to $L \circ j : D \to \text{Pro}(C)$ as the pro-left adjoint of $f$.

1.2 Notation. We write $\text{Cat}_{\infty}$ for the $\infty$-category of $\infty$-categories and $\text{Spc} \subset \text{Cat}_{\infty}$ for the full subcategory spanned by the $\infty$-groupoids, i.e., the $\infty$-category of spaces.

We write $\text{Top}_{\infty} \subset \text{Cat}_{\infty}$ for the $\infty$-category of $\infty$-topoi and geometric morphisms. For any $\infty$-topos $X$, we write $\Gamma_X$, or $\Gamma_X^*$ for the global sections geometric morphism, which is the essentially unique geometric morphism $X \to \text{Spc}$.

1.3 Definition. The shape $\Pi_{\infty} : \text{Top}_{\infty} \to \text{Pro}(\text{Spc})$ is the left adjoint to the extension to pro-objects of the fully faithful functor $\text{Spc} \hookrightarrow \text{Top}_{\infty}$ given by $K \mapsto \text{Fun}(K, \text{Spc})$ [SAG, §E.2.2]. The shape admits two other very useful descriptions:

- Let $X$ be an $\infty$-topos, and write $\Gamma_X : \text{Spc} \to X$ for the pro-left adjoint of $\Gamma_X^* : \text{Spc} \to X$. The shape of $X$ is equivalent to the prospace $\Gamma_X(1)$, where $1 \in X$ denotes the terminal object [HA, Remark A.1.10; §2].

- As a left exact accessible functor $\text{Spc} \to \text{Spc}$, the prospace $\Pi_{\infty}(X)$ is the composite $\Gamma_X^* \Gamma_X^*$ [HTT, §7.1.6; §2].

1.4 Notation. We write $H : \text{Cat}_{\infty} \to \text{Spc}$ for the left adjoint to the inclusion. The $\infty$-groupoid $H(C)$ is given by the colimit $H(C) = \text{colim}_C \Sigma_{\text{Spc}}$ of the constant diagram $C \to \text{Spc}$ at the terminal object $1_{\text{Spc}} \in \text{Spc}$.

1.5 Example. If $C$ is a small $\infty$-category, then $\Gamma^* : \text{Spc} \to \text{Fun}(C, \text{Spc})$ admits a genuine left adjoint $\Gamma_C : \text{Fun}(C, \text{Spc}) \to \text{Spc}$ given by taking the colimit of a diagram $C \to \text{Spc}$. The shape of the $\infty$-topos $\text{Fun}(C, \text{Spc})$ is thus given by the colimit of the constant diagram at the terminal object of $\text{Spc}$:

$$\Pi_{\infty}(\text{Fun}(C, \text{Spc})) = \Gamma_C(1_{\text{Fun}(C, \text{Spc})}) = \text{colim}_C \Sigma_{\text{Spc}} = H(C).$$

Moreover, the functor $H : \text{Cat}_{\infty} \to \text{Spc}$ is equivalent to the composite

$$\text{Cat}_{\infty} \xrightarrow{\text{Fun}(-, \text{Spc})} \text{Top}_{\infty} \xrightarrow{\Pi_{\infty}} \text{Spc}.$$

1.6 Example ([6, Corollary 5.6]). If $X$ is a locally Noetherian scheme, then the Artin–Mazur–Friedlander étale homotopy type of $X$ corepresents the shape of the hypercomplete étale co-topos $X^{\text{hTop}}_\text{et}$ of $X$.

The shape of the étale co-topos $X_\text{et}$ of $X$ agrees with the Artin–Mazur–Friedlander étale homotopy type up to protruncation (Example 1.9), to which we now turn.

**Protruncated objects**

In this subsection, we recall some facts about protruncated objects and record an interesting observation (Lemma 1.11) that we couldn’t locate in the literature.

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5 See [HTT, §6.5.2] for a treatment of hypercomplete $\infty$-topoi.
1.7 Notation. Let $C$ be a presentable ∞-category. For each integer $n \geq -2$, write $C_{≤n} \subset C$ for the full subcategory spanned by the $n$-truncated objects, and $τ_{≤n} : C \to C_{≤n}$ for the $n$-truncation functor, which is left adjoint to the inclusion $C_{≤n} \subset C$ [HTT, Proposition 5.5.6.18]. Write $C_{≤∞} \subset C$ for the full subcategory spanned by those objects which are $n$-truncated for some integer $n \geq -2$.

The pro-$n$-truncation functor $τ_{≤n} : \text{Pro}(C) \to \text{Pro}(C_{≤n})$ is the extension of the $n$-truncation functor $τ_{≤n} : C \to C_{≤n}$ to pro-objects.

1.8. Let $C$ be a presentable ∞-category. Then the extension to pro-objects of the functor $C \to \text{Pro}(C_{≤∞})$ given by sending an object $X \in C$ to the inverse system given by its Postnikov tower $\{τ_{≤n}(X)\}_{n≥-2}$ is left adjoint to the inclusion $\text{Pro}(C_{≤∞}) \hookrightarrow \text{Pro}(C)$. We call this left adjoint $τ_{≤∞} : \text{Pro}(C) \to \text{Pro}(C_{≤∞})$ pro-truncation.

A morphism of pro-objects $f : X \to Y$, regarded as left exact accessible functors $C \to \text{Spc}$, is an equivalence after truncation if and only if for every truncated object $K \in C_{≤∞}$, the induced morphism $f(K) : X(K) \to Y(K)$ is an equivalence.

1.9 Example. Since truncated objects are hypercomplete, for any ∞-topos $X$, the inclusion $X^{byp} \hookrightarrow X$ of the ∞-topos of hypercomplete objects of $X$ induces an equivalence

$$τ_{≤∞} \Pi_{≤∞}(X^{byp}) \simeq τ_{≤∞} \Pi_{≤∞}(X)$$

on protruncated shapes. In light of Example 1.6, the shape of the étale ∞-topos of a locally Noetherian scheme $X$ agrees with the Artin–Mazur–Friedlander étale homotopy type of $X$ after truncation.

For an arbitrary scheme $X$, we simply refer to the shape $Π_{≤∞}(X_{ét})$ of the étale ∞-topos $X_{ét}$ of $X$ as the étale homotopy type of $X$.

1.10. Let $C$ be a presentable ∞-category. The essentially unique functor $\text{Pro}(C) \to C$ that perserves inverse limits and restricts to the identity $C \hookrightarrow C$ is right adjoint to the Yoneda embedding $j : C \hookrightarrow \text{Pro}(C)$ [SAG, Example A.8.1.7]. Hence we have adjunctions

$$C \leftrightarrow \text{Pro}(C) \xrightarrow{τ_{≤∞}} \text{Pro}(C_{≤∞}).$$

If Postnikov towers converge in $C$, i.e., $C$ is a Postnikov complete presentable ∞-category [SAG, Definition A.7.2.1], then the composite right adjoint is also fully faithful:

1.11 Lemma. Let $C$ be a Postnikov complete presentable ∞-category (e.g., a Postnikov complete ∞-topos). Then the pro-truncation functor $τ_{≤∞} : C \to \text{Pro}(C_{≤∞})$ is fully faithful. Moreover, the essential image of $τ_{≤∞} : C \hookrightarrow \text{Pro}(C_{≤∞})$ is the full subcategory spanned by those protruncated objects $X$ such that for each integer $n \geq -2$, the pro-$n$-truncation $τ_{≤n}(X) \in \text{Pro}(C_{≤n})$ is a constant pro-object.

1.12. Composing the fully faithful functor $τ_{≤∞} : \text{Spc} \hookrightarrow \text{Pro}(\text{Spc}_{≤∞})$ with the inclusion $\text{Pro}(\text{Spc}_{≤∞}) \hookrightarrow \text{Pro}(\text{Spc})$ gives another embedding of spaces into prospaces: for a space $K$, the natural morphism of prospaces $j(K) \to τ_{≤∞}(K)$ is an equivalence if and only if $K$ is truncated. Unlike the Yoneda embedding, the functor $τ_{≤∞} : \text{Spc} \hookrightarrow \text{Pro}(\text{Spc})$ is neither a left nor a right adjoint.
2 Limits & the protruncated shape

The shape does not preserve inverse limits, even of bounded coherent $\infty$-topoi. In this section we prove that, nevertheless, the protruncated shape preserves inverse limits of bounded coherent $\infty$-topoi. Our main theorem (Theorem 2.5) is an easy consequence.

2.1 Notation. Write $\text{Top}_{\infty}^{bc} \subset \text{Top}_{\infty}$ for the subcategory of bounded coherent $\infty$-topoi and coherent geometric morphisms [SAG, Definitions A.2.0.12 & A.7.1.2; 3, Definition 5.28].

2.2 Proposition. The protruncated shape

$$\tau_{<\infty} \Pi_{\infty} : \text{Top}_{\infty}^{bc} \to \text{Pro}(\text{Spc}_{<\infty})$$

preserves inverse limits.

Proof. Let $X : I \to \text{Top}_{\infty}^{bc}$ be an inverse system of bounded coherent $\infty$-topoi and coherent geometric morphisms. For each $i \in I$, the forgetful functor $I / i \to I$ is limit-cofinal [HTT, Example 5.4.5.9 & Lemma 5.4.5.12], so we may without loss of generality assume that $I$ admits a terminal object $1$. For each $i \in I$, write $\pi_i : \lim_{j \in I} X_j \to X_i$ for the projection, $\varGamma_i : \lim_{j \in I} X_j \to \text{Spc}$ for the global sections geometric morphism.

We want to show that the natural morphism

$$\colim_{i \in I} \pi_i \varGamma_i^* \to \varGamma_1^*$$

in $\text{Fun}(\text{Spc}, \text{Spc})$ is an equivalence when restricted to truncated spaces (1.8). By [3, Lemma 8.11] the natural morphism

$$\colim_{i \in I} f_i \pi_i^* \to \pi_1 \pi_1^*$$

is an equivalence in $\text{Fun}(X_1, X_1)$. Since $X_1$ is bounded coherent, the global sections functor $\varGamma_1^* : X_1 \to \text{Spc}$ preserves filtered colimits of uniformly truncated objects [SAG, Proposition A.2.3.1; 3, Corollary 5.55]. Thus for any truncated space $K$ we see that

$$\colim_{i \in I} \varGamma_i^* (K) = \colim_{i \in I} \varGamma_i^* f_i^* f_i^* \varGamma_1^* (K)$$

$$= \varGamma_1^* \left( \colim_{i \in I} f_i \pi_i^* \right)$$

$$= \varGamma_1^* \circ \left( \colim_{i \in I} f_i^* \right) \circ \varGamma_1^* (K)$$

$$= \varGamma_1^* \circ \varGamma_1^* (K)$$

$$= \varGamma_1^* (K). \quad \square$$
Proof of the Main Theorem

We now prove the main result of this note. Recall that we write

\[ \tilde{(-)} : \text{Pro}(\text{Str}) \Rightarrow \text{StrTop}_{\infty}^{\text{spec}} \]

for the equivalence of \(\infty\)-categories of [3, Theorem 10.10].

2.3 Lemma. The square

\[
\begin{array}{ccc}
\text{Str}_\pi & \xrightarrow{(-)} & \text{StrTop}_{\infty}^{\text{spec}} \\
\downarrow H & & \downarrow \Pi_{\infty} \\
\text{Spc} & \xleftarrow{f} & \text{Pro}(	ext{Spc})
\end{array}
\]

commutes.

Proof. By the definition of the equivalence \(\text{Pro}(\text{Str}_\pi) \Rightarrow \text{StrTop}_{\infty}^{\text{spec}}\) of [3, Theorem 10.10], the following square commutes

\[
\begin{array}{ccc}
\text{Str}_\pi & \xrightarrow{(-)} & \text{StrTop}_{\infty}^{\text{spec}} \\
\downarrow & & \downarrow \\
\text{Cat}_{\infty} & \xrightarrow{\text{Fun}(-, \text{Spc})} & \text{Top}_{\infty}
\end{array}
\]

where the vertical functors forget stratifications. Combining this with Example 1.5 proves the claim.

2.4. Since the extension of \(H : \text{Str}_\pi \rightarrow \text{Spc}\) to pro-objects preserves inverse limits, Lemma 2.3 shows that we have a morphism of prospaces

\[ \theta_C : \Pi_{\infty}(\tilde{C}) \rightarrow H(C) \]

natural in \(C \in \text{Pro}((\text{Str}_\pi))\).

2.5 Theorem. For any profinite stratified space \(C\), the natural map

\[ \tau_{\infty}C \theta_C : \tau_{\infty} \Pi_{\infty}(\tilde{C}) \rightarrow \tau_{\infty}H(C) \]

of protruncated spaces is an equivalence.

Proof. Since the forgetful functor \(\text{StrTop}_{\infty}^{\text{spec}} \rightarrow \text{Top}_{\infty}^{bc}\) preserves inverse limits, Proposition 2.2 implies that the protruncated shape \(\tau_{\infty} \Pi_{\infty} : \text{StrTop}_{\infty}^{\text{spec}} \rightarrow \text{Pro}(\text{Spc}_{\infty})\) preserves inverse limits. Both \(\tau_{\infty}\) and \(H\) preserve inverse limits, hence their composite \(\tau_{\infty}H : \text{Pro}((\text{Str}_\pi)) \rightarrow \text{Pro}(\text{Spc}_{\infty})\) preserves inverse limits. The claim now follows from the fact that \(\theta_C\) is an equivalence for \(C \in \text{Str}_\pi\) (Lemma 2.3) and the universal property of the \(\infty\)-category \(\text{Pro}((\text{Str}_\pi))\) of profinite stratified spaces.

2.6. Note that Theorem A from the introduction is immediate from Theorem 2.5, [3, Construction 13.5], and the definition of the étale homotopy type in terms of shape theory (Examples 1.6 and 1.9).
References


