EXERCISES ON LIMITS & COLIMITS

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Exercise 1. Prove that pullbacks of epimorphisms in $\text{Set}$ are epimorphisms and pushouts of monomorphisms in $\text{Set}$ are monomorphisms. Note that these statements cannot be deduced from each other using duality. Now conclude that the same statements hold in $\text{Top}$.

Exercise 2. Let $X$ be a set and $A, B \subset X$. Prove that the square

$$
\begin{array}{ccc}
A \cap B & \rightarrow & A \\
\downarrow & & \downarrow \\
B & \rightarrow & A \cup B
\end{array}
$$

is both a pullback and pushout in $\text{Set}$.

Exercise 3. Let $R$ be a commutative ring. Prove that every $R$-module can be written as a filtered colimit of its finitely generated submodules.

Exercise 4. Let $X$ be a set. Give a categorical definition of a topology on $X$ as a subposet of the power set of $X$ (regarded as a poset under inclusion) that is stable under certain categorical constructions.

Exercise 5. Let $X$ be a space. Give a categorical description of what it means for a set of open subsets of $X$ to form a basis for the topology on $X$.

Exercise 6. Let $C$ be a category. Prove that if the identity functor $\text{id}_C : C \rightarrow C$ has a limit, then $\text{lim}_C \text{id}_C$ is an initial object of $C$.

Definition. Let $C$ be a category and $X \in C$. If the coproduct $X \sqcup X$ exists, the codiagonal or fold morphism is the morphism $\gamma_X : X \sqcup X \rightarrow X$ induced by the identities on $X$ via the universal property of the coproduct.

If the product $X \times X$ exists, the diagonal morphism $\Delta_X : X \rightarrow X \times X$ is defined dually.

Exercise 7. In $\text{Set}$, show that the diagonal $\Delta_X : X \rightarrow X \times X$ is given by $\Delta_X(x) = (x, x)$ for all $x \in X$, so $\Delta_X$ embeds $X$ as the diagonal in $X \times X$, hence the name.

The codiagonal $\gamma_X : X \sqcup X \rightarrow X$ is a bit more mysterious. Give a description of $\gamma_X$ (still in the category $\text{Set}$).

Exercise 8. Let $C$ be a category and $X, Y \in C$. Suppose that the coproducts $X \sqcup X$ and $Y \sqcup Y$ exist, and let $\gamma_X : X \sqcup X \rightarrow X$ and $\gamma_Y : Y \sqcup Y \rightarrow Y$ denote the codiagonals. Show that for any morphism $f : X \rightarrow Y$ we have $f \circ \gamma_X = \gamma_Y \circ (f \sqcup f)$.

What is the dual statement?

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Exercise 9. Let $C$ be a category with pullbacks and

$$
\begin{array}{ccc}
X_1 & \longrightarrow & X_0 & \longleftarrow & X_2 \\
\downarrow & & \downarrow & & \downarrow \\
Z_1 & \longrightarrow & Z_0 & \longleftarrow & Z_2 \\
\uparrow & & \uparrow & & \uparrow \\
Y_1 & \longrightarrow & Y_0 & \longleftarrow & Y_2
\end{array}
$$

a commutative diagram in $C$. Prove that we have a natural isomorphism

$$(X_1 \times_{Z_1} Y_1) \times_{X_0 \times_{Y_0} Z_0} (X_2 \times_{Z_2} Y_2) \cong (X_1 \times_{X_0} X_2) \times_{Z_1 \times_{Z_0} Z_2} (Y_1 \times_{Y_0} Y_2).$$

Definition. Let $C$ be a category with pullbacks and $f: X \rightarrow Y$ a morphism in $C$. The diagonal of $f$ is the morphism $\Delta_f: X \rightarrow X \times_Y X$ induced via the universal property of the pullback by the square

$$
\begin{array}{ccc}
X & \longrightarrow & X \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

Remark. Note that $\Delta_{id_X} = \Delta_X$.

Exercise 10 (magic square). Let $C$ be a category and let $f_1: X_1 \rightarrow Y$, $f_2: X_2 \rightarrow Y$, and $g: Y \rightarrow Z$ morphisms in $C$. Assuming that $C$ has pullbacks, prove that the square

$$
\begin{array}{ccc}
X_1 \times_Y X_2 & \longrightarrow & X_1 \times_Z X_2 \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y \times_Z Y
\end{array}
$$

is a pullback square in $C$ (where the unlabeled morphisms are the morphisms naturally induced by the universal property of the pullback).

Definition. Let $C$ be a category with pullbacks, $Z \in C$, and $f: X \rightarrow Y$ a morphism in the slice category $C_{/Z}$. The graph morphism of $f$ is the morphism $f: X \rightarrow X \times_Z Y$ induced via the universal property of the pullback by the square

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & Z
\end{array}
$$

Exercise 11. Show that if $f: X \rightarrow Y$ is a map of sets, then the graph $\Gamma_f: X \rightarrow X \times Y$ is given by $x \mapsto (x, f(x))$. 
Exercise 12. Let $C$ be a category with pullbacks, $Z \in C$, and $f : X \to Y$ a morphism in the slice category $C_{/Z}$. Write $g : Y \to Z$ for the structure morphism. Prove that the square

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X \times_Z Y \\
\downarrow f & & \downarrow f \times_Z id_Y \\
Y & \xrightarrow{\Delta_g} & Y \times_Z Y
\end{array}
$$

is a pullback in $C$.

**Definition.** Let $C$ be a category. We say that a collection of morphisms $P \subset \text{Mor}(C)$ is *stable under pullback* if for any pullback square

$$
\begin{array}{ccc}
X \times_Z Y & \xrightarrow{p} & Y \\
\downarrow q & & \downarrow q \\
X & \xrightarrow{p} & Z
\end{array}
$$

in $C$, if $p \in P$ then $\bar{p} \in P$.

Exercise 13. Let $C$ be a category with pullbacks and $P \subset \text{Mor}(C)$ a collection of morphisms in $C$ stable under composition and pullback. Prove that given any commutative triangle

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow p & & \downarrow g \\
Z &
\end{array}
$$

in $C$, if $p \in P$ and $\Delta_g \in P$, then $f \in P$.

**Definition.** A functor $F : C \to D$ is *left cofinal* if for all categories $E$ and diagrams $G : D \to E$, the colimit $\text{colim}_D G$ exists if and only if $\text{colim}_C GF$ exists, in which case the natural morphism

$$
\text{colim}_C GF \to \text{colim}_D G
$$

is an isomorphism.

Dually, $F : C \to D$ is *right cofinal* if $F^{op} : C^{op} \to D^{op}$ is left cofinal.

**Remark.** The “co” in “cofinal” uses the non-mathematical English prefix meaning “jointly” — there’s no duality involved here.

Exercise 14. Show that equivalences of categories are both left and right cofinal.

Exercise 15. Show that if a category $C$ has a terminal object $*$, then the inclusion $\{ * \} \hookrightarrow C$ of the full subcategory of $C$ spanned by $*$ is left cofinal.

Exercise 16. Let $F : C \to D$ be a functor. Show that $F$ is left cofinal if and only if for all diagrams $G : D \to \text{Set}$ the natural morphism

$$
\text{colim}_C GF \to \text{colim}_D G
$$

is an isomorphism.

**Notation.** For a positive integer $n$, write $\Delta_{\leq n}$ for the full subcategory of $\Delta$ spanned by those sets of cardinality at most $n$. 
Exercise 17. Show that the inclusion $\Delta_{\leq 2} \hookrightarrow \Delta$ is right cofinal.

Notation. Write $\Delta^{inj}_n \subset \Delta$ for the wide subcategory, i.e., subcategory containing all of the objects, of $\Delta$ where the morphisms are injective maps of linearly ordered finite sets. For a positive integer $n$, write $\Delta^{inj}_{\leq n}$ for the full subcategory of $\Delta^{inj}_n$ spanned by those sets of cardinality at most $n$.

Exercise 18. Show that the inclusion $\Delta^{inj}_{\leq 2} \hookrightarrow \Delta^{inj}$ is right cofinal. Now deduce that the inclusion $\Delta^{inj}_{\leq 2} \hookrightarrow \Delta$ is right cofinal.

Exercise 19. Let $C$ be a category with pullbacks and $f : X \to Y$ a morphism in $C$. Construct a functor $\tilde{C}(f) : \Delta^\text{op} \to C$ whose value on $[n] = \{0 < \cdots < n\}$ is the $(n + 1)$-fold iterated pullback $X \times_Y \cdots \times_Y X$ (so that the value on $[0]$ is simply $X$). The simplicial object $\tilde{C}(f)$ is called the Čech nerve of $f$.

Exercise 20. Let $X$ be a topological space, $U \subset X$ an open set, $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ an open cover of $U$, and $\mathcal{F}$ a presheaf on $X$. Choose a well-ordering of $A$ (this does not really matter, but is necessary to make the next step well-defined.) Extend the usual "sheaf condition diagram"

$$\prod_{\alpha_0 \in A} \mathcal{F}(U_{\alpha_0}) \longrightarrow \prod_{\alpha_0, \alpha_1 \in A} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1})$$

to a diagram $\tilde{C}(U; \mathcal{F}) : \Delta^{inj} \to \text{Set}$ of the form

$$\prod_{\alpha_0 \in A} \mathcal{F}(U_{\alpha_0}) \longrightarrow \prod_{\alpha_0, \alpha_1 \in A} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1}) \longrightarrow \prod_{\alpha_0, \alpha_1, \alpha_2 \in A} \mathcal{F}(U_{\alpha_0} \cap U_{\alpha_1} \cap U_{\alpha_2}) \longrightarrow \cdots .$$

Now reformulate the sheaf condition for a presheaf $\mathcal{F}$ in terms of the diagrams $\tilde{C}(U; \mathcal{F})$. 