EXERCISES ON ADJUNCTIONS

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Notation. We denote an adjunction by an opposing pair of arrows: $F : C \rightleftarrows D : G$. In this notation, $F$ will always be the left adjoint and $G$ the right adjoint.

Exercise 1. Let $F : C \rightleftarrows D : G$ be an adjunction. Prove that the right adjoint $G$ is fully faithful if and only if the counit $\varepsilon : FG \Rightarrow id_D$ is a natural isomorphism. Notice that in this case the left adjoint $F$ is essentially surjective on objects. Dually, $F$ is fully faithful if and only if the unit $\eta : id_C \Rightarrow GF$ is a natural isomorphism.

Exercise 2. Let $C$ and $D$ be categories and suppose that we are given adjunctions

$$
\begin{array}{c}
C \\
\downarrow U \\
\downarrow R
\end{array}
\quad \quad
\begin{array}{c}
D \\
\downarrow U \\
\downarrow R
\end{array}
$$

Prove that $L$ is fully faithful if and only if $R$ is fully faithful.