NOTES ON ÉTALE COHOMOLOGY

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Abstract. These notes outline the "fundamental theorems" of étale cohomology, following [4, Ch. vi], as well as briefly discuss the Weil conjectures.

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1. COHOMOLOGICAL DIMENSION

1.1. Definition. Let $X$ be a scheme, $\tau$ a topology on $X$, and $\ell$ a prime. A $\tau$-sheaf $\mathcal{F}$ of abelian groups on $X$ is $\ell$-torsion if for every object $U \to X$ of the site $X_\tau$ such that $U$ is quasicompact, the abelian group $\mathcal{F}(U)$ is $\ell$-torsion. The $\ell$-torsion sheaves are defined in the obvious way.

1.2. Definition. Let $X$ be a scheme, $\tau$ a topology on $X$, and $\ell$ a prime. The $\ell$-cohomological dimension $\text{cd}_\ell(X_\tau)$ of $X_\tau$ is defined to be the smallest integer so that $H^r_{et}(X; \mathcal{F}) = 0$ for all $r > \text{cd}_\ell(X_\tau)$ and $\ell$-torsion sheaves $\mathcal{F}$ on $X_\tau$, and $\infty$ if no such integer exists.

The cohomological dimension of $X_\tau$ is defined by

$$\text{cd}(X_\tau) := \sup_{\ell \text{ prime}} \text{cd}_\ell(X_\tau)$$

1.3. Warning. This definition of cohomological dimension is highly nonstandard and it is not obvious whether or not it agrees with the standard one from topos theory.

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1.4. **Theorem** (Tate [3] Thm. 15.2). If \( k \subset K \) is a field extension, then
\[
\text{cd}_\ell(K) \leq \text{cd}_\ell(k) + \text{trdeg}_k(K),
\]
where \( \text{cd}_\ell \) refers to the étale \( \ell \)-cohomological dimension of the field.

1.5. **Theorem** ([4], Ch. vi Thm. 1.1). If \( X \) is a scheme of finite type over a separably closed field, then
\[
\text{cd}_\ell(X_{\text{et}}) \leq 2 \dim(X).
\]

1.6. **Corollary** ([4], Ch. vi Cor. 1.4). Let \( X \) be a scheme of finite type over a field \( k \). For all primes \( \ell \neq \text{char}(k) \) we have
\[
\text{cd}_\ell(X_{\text{et}}) \leq \text{cd}_\ell(k) + 2 \dim(X).
\]

2. **The Proper Base Change Theorem**

2.1. **Theorem** ([4], Ch. vi Thm. 2.1). If \( \pi : Y \to X \) is a proper morphism of schemes and \( F \in \text{Cnstr}(X) \), then the sheaves
\[
R^i \pi_* (F)
\]
are constructible for all \( i \geq 0 \).

2.2. **Corollary** (proper base change [4], Ch. vi Cor. 2.3). Let
\[
\begin{array}{ccc}
Y' & \xrightarrow{f} & Y \\
\downarrow{\pi'} & & \downarrow{\pi} \\
X' & \xrightarrow{f} & X
\end{array}
\]
be a pullback square of schemes. If \( \pi \) is proper and \( F \) is a torsion étale sheaf on \( Y \), then the base change morphism
\[
(f^{-1} \circ R^i \pi_*)(F) \to (R^i \pi_* \circ f^{-1})(F)
\]
is an isomorphism for all \( i \geq 0 \).

2.3. **Corollary** ([4], Ch. vi Cor. 2.5). Let \( \pi : Y \to X \) be a proper morphism of schemes and \( \mathfrak{x} \to X \) a geometric point. If \( F \in \text{Shv}_{\text{et}}(Y) \) is torsion, then there is a canonical isomorphism
\[
R^i \pi_* (F) \cong H^i_{\text{et}}(Y_{\mathfrak{x}}; F|_{\mathfrak{x}})
\]
for all \( i \geq 0 \).

If, moreover, all fibers of \( \pi \) have dimension at most \( n \), then \( R^i \pi_* (F) = 0 \) for \( i > 2n \). If all fibers of \( \pi \) have dimension at most \( n \), \( X \) is a characteristic \( p \) scheme, and \( F \) is \( p \)-torsion, then \( R^i \pi_* (F) = 0 \) for \( i > n \).

2.4. **Corollary** ([4], Ch. vi Cor. 2.6). Let \( k \subset K \) be separably closed fields and \( X \) a proper \( k \)-scheme. Write \( f : X_K \to X \) for the basechange projection. If \( F \in \text{Shv}_{\text{et}}(X) \) is torsion, then the natural map
\[
H^i_{\text{et}}(X; F) \to H^i_{\text{et}}(X_K; f^{-1}F)
\]
is an isomorphism for all \( i \geq 0 \).

2.5. **Corollary** ([4], Ch. vi Cor. 2.7). Let \( A \) be a Henselian ring and \( s_0 \in \text{Spec}(A) \) the closed point. Let \( \pi : X \to \text{Spec}(A) \) be a proper morphism and \( X_0 := X_{s_0} \) the closed fiber. If \( F \in \text{Shv}_{\text{et}}(X) \) is torsion, then the natural map
\[
H^i_{\text{et}}(X; F) \to H^i_{\text{et}}(X_0; F|_{X_0})
\]
is an isomorphism for all \( i \geq 0 \).
3. Higher Direct Images with Compact Support

3.1. Definition. A morphism of schemes \( \pi : X \to S \) is **compactifiable** if there exists a factorization

\[
\begin{array}{ccc}
X & \xrightarrow{j} & \overline{X} \\
\downarrow{\pi} & & \downarrow{\overline{\pi}} \\
S & & \overline{S},
\end{array}
\]

where \( j \) is an open immersion and \( \overline{\pi} \) is proper.

A **compactification** of a compactifiable morphism \( \pi \) is a choice of a factorization \( \pi = \overline{\pi} j \), where \( j \) is an open immersion and \( \overline{\pi} \) is proper.

3.2. Definition. Let \( \pi : X \to S \) be a compactifiable morphism and \( \pi = \overline{\pi} j \) a compactification of \( \pi \). Define \( \mathcal{R}_c \pi_* \) and for \( r \geq 0 \) define \( \mathcal{R}_c^{r} \pi_* \).

3.3. Warning. The notations of Definition 3.2 are abusive as they depend on the choice of compactification and it is not generally possible to make a functorial choice of compactifications. In general \( R_c \) is not even functorial in proper maps of compactifiable \( S \)-schemes.

3.4. Theorem ([4, Ch. vi Thm. 3.1]). If \( \pi : X \to S \) is a compactifiable morphism and \( F \in \text{Shv}^{\text{ét}}_c(X) \) is torsion, then the sheaves \( \mathcal{R}_c^{r} \pi_* (F) \) are independent of the choice of compactification of \( \pi \).

4. The Smooth Base Change Theorem

4.1. Definition. Let \( X \) be a scheme. The **characteristic** of \( X \) is the set

\[ \text{char}(X) := |\text{im}(X \to \text{Spec}(\mathbb{Z}))| = \{ \text{char}(\kappa(x)) \mid x \in X \}, \]

where \(|-|\) denotes the underlying space of a scheme.

4.2. Definition. Let \( X \) be a scheme and \( F \in \text{Shv}^{\text{ét}}_c(X) \). We say that \( F \) has **torsion prime to char(\( X \))** if for all primes \( p \in \text{char}(X) \), the multiplication by \( p \) map \( F \to F \) is a monomorphism.

4.3. Corollary (smooth base change [4, Ch. vi Thm. 4.1]). Let

\[
\begin{array}{ccc}
Y' & \xrightarrow{i} & Y \\
\downarrow{j} & & \downarrow{\pi} \\
X' & \xrightarrow{f} & X
\end{array}
\]

be a pullback square of schemes. If \( \pi \) is quasicompact, \( f \) is smooth, and \( F \in \text{Shv}^{\text{ét}}_c(Y) \) has torsion prime to \( \text{char}(X) \), then the base change morphism

\[
(f^{-1} \circ \mathcal{R}^i \pi_*)(F) \to (\mathcal{R}^i \pi_* \circ f^{-1}):(F)
\]

is an isomorphism for all \( i \geq 0 \).

4.4. Corollary ([4, Ch. vi Cor. 4.2]). Let \( \pi : Y \to X \) be a smooth proper morphism and \( F \in \text{Shv}^{\text{ét}}_c(Y) \) a constructible locally constant sheaf with torsion prime to \( \text{char}(X) \). Then for all \( i \geq 0 \), the sheaf \( \mathcal{R}^i \pi_*(F) \) is constructible and locally constant.

If, in addition, \( X \) is connected, then the groups \( H^i_\mathbb{Q}(Y, F|_{Y_s}) \) are isomorphic for all geometric points \( \bar{x} \to X \).
4.5. Corollary ([4], Ch. vi Cor. 4.3]). Let \( k \subset K \) be separably closed fields and \( X \) a \( k \)-scheme. Write \( f : X_k \to X \) for the basechange projection. If \( \mathcal{F} \in \text{Shv}_{ct}(X) \) is torsion with torsion prime to \( \text{char}(X) \), then the natural map
\[
H^i_{et}(X; \mathcal{F}) \to H^i_{et}(X_k; f^{-1}\mathcal{F})
\]
is an isomorphism for all \( i \geq 0 \).

4.6. Corollary ([4], Ch. vi Cor. 4.5]). Let \( \pi : X \to S \) be a morphism of finite type schemes that are locally of finite type over a field \( k \), and let \( \mathcal{F} \in \text{Cnst}_{(X)} \). If resolution of singularities holds for \( X \) (e.g., if \( \text{char}(k) = 0 \) or \( \dim(X) \leq 2 \)) and the torsion of \( \mathcal{F} \) is prime to \( \text{char}(X) \), then the sheaves \( R^i\pi_* (\mathcal{F}) \) are constructible for all \( i \geq 0 \).

5. Purity

5.1. Definition. Let \( S \) be a scheme. A smooth \( S \)-pair \((Z, X)\) is a closed immersion \( i : Z \hookrightarrow X \) of \( S \)-schemes.

A smooth \( S \)-pair \((Z, X)\) has codimension \( c \) if for all \( s \in S \), the fiber \( Z_s \) has pure codimension \( c \) in \( X_s \).

5.2. Theorem (cohomological purity [4], Ch. vi Thm. 5.1]). Let \( S \) be a scheme and \( i : Z \hookrightarrow X \) a smooth \( S \)-pair of codimension \( c \). Let \( \mathcal{F} \) be a locally constant torsion sheaf of torsion prime to \( \text{char}(X) \). Then the following equivalent statements hold.

(5.2.a) We have that \( R^{2c}i^!(\mathcal{F}) \) is locally isomorphic to \( i^{-1}\mathcal{F}(-c) \) (as a sheaf on \( Z \)), and for \( q \neq 2c \) we have that \( R^q i^!(\mathcal{F}) = 0 \).

(5.2.b) The unit \( \mathcal{F} \to j_*j^{-1}(\mathcal{F}) \) is an isomorphism, \( R^{2c-1}j_* (j^{-1}\mathcal{F}) \) is locally isomorphic to \( i_*i^{-1}\mathcal{F} \), and \( R^q j_* (j^{-1}\mathcal{F}) = 0 \) for \( q \neq 0, 2c - 1 \).

5.3. Observation. Assume the hypotheses of Theorem 5.2 that \( n \in \mathbb{Z} \) is prime to \( \text{char}(X) \), and the multiplication by \( n \) map \( \mathcal{F} \to \mathcal{F} \) is zero. If \( V \to X \) is a finite étale map and \( s \in \Gamma(V; \mathcal{F}) \), the map \( \mathbb{Z}/n \to \mathcal{F}|_V \) defined by \( s \) defines a map
\[
H^2_{Z\times V}(\mathbb{Z}/n) \to H^2_{Z\times V}(V; \mathcal{F}|_V).
\]

By varying \( V \) we get a morphism of étale sheaves on \( X \)
\[
\mathcal{F} \to \text{Hom}_X(i_*, R^{2c-i}i'(\mathbb{Z}/n), i_*, R^{2c-i}(\mathcal{F})) .
\]

Since \( R^{2c}i'(\mathbb{Z}/n) \) is locally isomorphic to \( \mathbb{Z}/n \) by Theorem 5.2 we see that \( R^{2c}i'(\mathbb{Z}/n) \) is locally free of finite type. Hence the proof of Theorem 5.2 shows that the morphism (5.4) is an isomorphism. Thus we may write the morphism (5.4) as an isomorphism
\[
i^{-1}(\mathcal{F}) \otimes R^{2c-i}(\mathbb{Z}/n) \simeq R^{2c-i}(\mathcal{F}) .
\]

5.5. Notation. Let \( S \) be a scheme, \( i : Z \hookrightarrow X \) a smooth \( S \)-pair of codimension \( c \), and \( n \) an integer prime to \( \text{char}(X) \). We (abusively) write \( T_{Z/X} := R^{2c-i}(\mathbb{Z}/n) \).

5.6. Corollary (Gysin sequence [4], Ch. vi Cor. 5.3]). Let \( S \) be a scheme, \( i : Z \hookrightarrow X \) a smooth \( S \)-pair of codimension \( c \), \( n \) an integer prime to \( \text{char}(X) \), and \( \mathcal{F} \in \text{LC}_{ct}(X) \) be killed by \( n \). Write \( f : X \to S \) for the structure morphism, \( j : U := X \setminus Z \hookrightarrow X \) for the open complement of \( Z \), and write \( i' = fi \) and \( j' = fj \). Then for \( 0 \leq r \leq 2c - 2 \) the comparison map
\[
R^r j_* (\mathcal{F}) \to R^r j'_*(\mathcal{F}|_U)
\]
is an isomorphism and there is a long exact sequence

\[
\begin{align*}
0 & \to R^{2c-1}f_* (F) \to R^{2c-1}f_* (F|_U) \to i'_*(i^{-1}(F) \otimes T_{Z/X}) \to R^{2c}f_* (F) \\
& \to R^{2c-2}f_* (F|_U) \to R^{2c-2}i'_*(i^{-1}(F) \otimes T_{Z/X}) \to \cdots 
\end{align*}
\]

The maps labeled \(i_*\) are called Gysin maps.

5.7. Example ([4, Ch. vi Ex. 5.6]). Let \(k\) be a separably closed field and \(n \in \mathbb{Z}\) prime to \(\text{char}(k)\). Then

\[
\begin{align*}
H^r_\text{ét}(\mathbb{P}^m_k; \mathbb{Z}/n) & \cong \begin{cases} 
\mathbb{Z}/n(-r/2), & r \text{ even } 0 \leq r \leq 2m \\
0, & \text{otherwise}
\end{cases} \\
\end{align*}
\]

With a few more results it is possible to show that if \(X\) is a smooth projective \(k\)-variety, for a generic curve \(i: C \to X\), the map

\[i^* : H^1_\text{ét}(X; (\mathbb{Z}/n)(1)) \to H^1_\text{ét}(C; (\mathbb{Z}/n)(1))\]

is an injection. Using Poincaré duality one can show that this result extends to smooth complete intersections.

5.8. Theorem (Deligne [4, Ch. vi Rem. 5.7]). Let \(S\) be a quasicompact scheme. Then for any morphism \(\pi: Y \to X\) of finite type \(S\)-schemes and \(F \in \text{Cnstr}(Y; \mathbb{Z}/n)\), there exists a dense open subscheme \(U_\pi \subset S\) such that:

\begin{enumerate}
\item[(5.8.a)] Over \(U_\pi\), the sheaves \(R^r\pi_* (F)\) are constructible for all but finitely many \(r \geq 0\).
\item[(5.8.b)] The formation of the sheaves \(R^r\pi_* (F)\) is compatible with all base changes along morphisms \(T \to S\) with image in \(U_\pi\).
\end{enumerate}

Moreover, if \(S\) is regular and \(0\) or \(1\)-dimensional, the sheaves \(R^r\pi_* (F)\) are constructible (over \(S\)) for all \(r \geq 0\).

6. Finiteness Theorems

6.1. Corollary ([4, Ch. vi Cor. 2.8]). If \(X\) is a proper \(k\)-scheme and \(F \in \text{Cnstr}(X)\), then \(H^i_\text{ét}(X; F)\) is finite for all \(i \geq 0\).

6.2. Corollary ([4, Ch. vi Cor. 5.5]). Let \(X\) be a smooth \(k\)-variety and \(F\) a finite locally constant étale sheaf on \(X\) with torsion prime to \(\text{char}(k)\). Then \(H^i_\text{ét}(X; F)\) is finite for all \(i \geq 0\).

7. Fundamental Classes

7.1. Notation. Throughout this section \(k\) denotes a separably closed field, \(n\) a integer prime to \(\text{char}(k)\), and \(\Lambda = \mathbb{Z}/n\).

7.2. Conventions. In this section all schemes are smooth \(k\)-varieties and all sheaves are \(\Lambda\)-modules.

7.3. Observation. If \(i: Z \to X\) is a smooth \(k\)-pair of codimension \(c\), then the spectral sequence

\[H^p_\text{ét}(Z; i^{-1}R^qf_!(F)) \Rightarrow H^p_\text{ét}(X; F)\]

gives canonical isomorphisms

\[H^p_\text{ét}(Z; i^{-1}R^{2c}i_!(F)) \cong H^{2c+p}_\text{ét}(X; F),\]

for any locally free sheaf \(F\) of finite rank. In particular,

\[\Gamma(Z; i^{-1}R^{2c}i_!(F)) \cong H^{2c}_\text{ét}(X; F).\]
7.4. **Observation.** If \( i : Z \hookrightarrow X \) is a smooth divisor of codimension 1 with open complement \( j : U = X \setminus Z \hookrightarrow X \), we have an isomorphism of exact sequences

\[
\begin{array}{ccc}
H^0_{\text{ét}}(U; \mathbb{G}_m) & \xrightarrow{\partial} & H^1_Z(X; \mathbb{G}_m) \\
\downarrow & & \downarrow \\
\Gamma(U; \mathcal{O}_X^*) & \xrightarrow{\text{ord}_Z} & \mathbb{Z}
\end{array}
\]

Moreover, the Kummer sequence gives an exact sequence

\[
H^1(Z; \mathbb{G}_m) \xrightarrow{\partial} H^1_{\text{ét}}(X; \mathbb{G}_m) \rightarrow H^1_{\text{ét}}(U; \mathbb{G}_m)
\]

7.5. **Definition.** If \( Z \hookrightarrow X \) is a smooth divisor of codimension 1, then define the fundamental class \( s_{Z/X} \in H^2_Z(X; \Lambda(c)) \) by setting \( s_{Z/X} = \partial(1) \), where \( \partial \) is the boundary morphism in the sequence (7.4).

7.6. **Observations.**

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7.8. **Theorem ([4 Ch. vi Thm. 6.1]).** There is a unique function \( (Z, X) \mapsto s_{Z/X} \) sending a smooth \( k \)-pair of codimension \( c \) to a fundamental class \( s_{Z/X} \in H^2_Z(X; \Lambda(c)) \) satisfying the following:

7.9. **Corollary ([4 Ch. vi Cor. 6.4]).** Let \( (Z, X) \) be a smooth \( k \)-pair of codimension \( c \). Then \( T_{Z/X} \) is canonically isomorphic to \( \Lambda(-c) \).

7.10. **Proposition ([4 Ch. vi Prop. 6.5]).**
For all \( x \in H^{r+2c}_{\text{ét}}(X; \Lambda) \) and \( z \in H^{s}_{\text{ét}}(X; \Lambda) \) we have \( i_{*}(i^{*}(x) \sim z) = x \sim i_{*}(z) \).

In particular, \( i_{*}i^{*}(x) = x \sim i_{*}(1_Z) \).

(7.10.b) Gysin maps compose: if \( i_1: Z \hookrightarrow Y \) and \( i_2: Y \hookrightarrow X \) are both smooth \( k \)-pairs, then \((i_2i_1)_{*} = i_2_{*}i_1_{*} \).

8. The Weak Lefschetz Theorem

8.1. Theorem ([4, Ch. vi Thm. 7.1]). Let \( X \) be an \( m \)-dimensional projective variety over a separably closed field and \( i: Z \hookrightarrow X \) the inclusion of a hyperplane section.

8.1.a) If \( X \) and \( Z \) are smooth, \( n \in \mathbb{Z} \) is prime to \( \text{char}(k) \), and \( F \in \text{LC}_{\text{ét}}(X; \mathbb{Z}/n) \), then the Gysin map \( i_{*}: H^{r}_{\text{ét}}(Z; i^{-1}F) \to H^{r+2}_{\text{ét}}(X; F(1)) \) is an isomorphism for \( r \geq m \) and a surjection for \( r = m - 1 \).

8.1.b) If \( F \in \text{Shv}_{\text{ét}}(X) \) is torsion, then the map \( H^{r}_{Z}(X; F) \to H^{r}_{\text{ét}}(X; F) \) is an isomorphism for \( r \geq m + 2 \) and a surjection for \( r = m + 1 \).

8.2. Theorem ([4, Ch. vi Thm. 7.2]). If \( X \) is a scheme affine and of finite type over a separably closed field, then \( \text{cd}(X) = \dim(X) \).

9. The Künneth Formula

9.1. Convention. Throughout this section \( \Lambda \) is a finite ring and unless otherwise stated all sheaves are \( \Lambda \)-modules.

9.2. Theorem (Künneth formula [4, Ch. vi Thm. 8.5]). Consider a pullback square of schemes

\[
\begin{array}{ccc}
X \times_S Y & \xrightarrow{f} & Y \\
\downarrow g & & \downarrow g \\
X & \xrightarrow{f} & S,
\end{array}
\]

and set \( h := g \tilde{f} = f \tilde{g} \). Assume the following:

9.2.a) \( S \) is quasicompact,

9.2.b) \( f \) and \( g \) are compactifiable,

9.2.c) \( F \in \text{Shv}_{\text{ét}}(X; \Lambda) \), \( G \in \text{Shv}_{\text{ét}}(Y; \Lambda) \), and \( F \) is flat as a \( \Lambda \)-module.

Then there is a canonical quasi-isomorphism

\[
R_{\tilde{g}}f_{*}(F) \otimes_{\Lambda} R_{\tilde{g}}g_{*}(G) \cong R_{h*}(F \boxtimes A G).
\]

If, moreover, \( R_{\tilde{g}}f_{*}(F) \) is flat for all \( r \geq 0 \), then for all \( m \geq 0 \) we have a canonical isomorphism

\[
\bigoplus_{r+s=m} R_{\tilde{g}}f_{*}(F) \otimes_{\Lambda} R_{\tilde{g}}g_{*}(G) \cong R_{h*}(F \boxtimes A G).
\]

9.3. Lemma (proper base change [4, Ch. vi Lem. 8.9]). Consider a pullback square of schemes

\[
\begin{array}{ccc}
X \times_S Y & \xrightarrow{f} & Y \\
\downarrow g & & \downarrow g \\
X & \xrightarrow{f} & S,
\end{array}
\]

where \( g \) is proper, and let \( \mathcal{G} \in \text{Shv}_{\text{ét}}(Y; \Lambda) \). Then there is a canonical quasi-isomorphism

\[
f^{*}R_{\tilde{g}}_{*}(\mathcal{G}) \cong R_{\tilde{g}}_{*}(f^{*}G).
\]
9.4. Corollary ([4, Ch. vi Cor. 8.13]). Let $X$ and $Y$ be schemes over a separably closed field $k$, $\mathcal{F} \in \text{Shv}_{\text{ét}}(X; \Lambda)$, $\mathcal{G} \in \text{Shv}_{\text{ét}}(Y; \Lambda)$. If $X$ and $Y$ are proper over $k$, $\mathcal{F}$ is flat, and $H^r_{\text{ét}}(X; \mathcal{F})$ is flat for all $r \geq 0$, then for all $m \geq 0$ we have a canonical isomorphism

$$\bigoplus_{r+s=m} H^r_{\text{ét}}(X; \mathcal{F}) \otimes_{\Lambda} H^s_{\text{ét}}(Y; \mathcal{G}) \xrightarrow{\text{induced by the cup product pairings}} H^m_{\text{ét}}(X \times Y; \mathcal{F} \boxtimes \mathcal{G})$$

9.5. Theorem ($\ell$-adic Künneth formula [4, Ch. vi Cor. 8.21]). Let $k$ be a separably closed field and $\ell$ a prime different from $\text{char}(k)$. Let $X$ and $Y$ be compactifiable $k$-schemes and $A$ the integral closure of $\mathbb{Z}_\ell$ in a finite extension of $\mathbb{Q}_\ell$. If $F \in \text{Cnstr}(X; A)$, $G \in \text{Cnstr}(Y; A)$, and $F$ and $G$ are flat $A$-modules, then:

- (9.5.a) We have a canonical isomorphism $H^\ast_{\text{c}}(X; F) \otimes_A H^\ast_{\text{c}}(Y; G) \xrightarrow{\text{induced}} H^\ast_{\text{c}}(X \times Y; F \boxtimes G)$.

- (9.5.b) For all $m \geq 0$ there is a short exact sequence

$$0 \longrightarrow \bigoplus_{r+s=m} H^r_{\text{c}}(X; F) \otimes_A H^s_{\text{c}}(Y; G) \longrightarrow H^m_{\text{c}}(X \times Y; F \boxtimes G) \longrightarrow \bigoplus_{r+s=m} \text{Tor}^1_A(H^r_{\text{c}}(X; F), H^s_{\text{c}}(Y; G)) \longrightarrow 0.$$

10. The Cycle Class Map

10.1. Convention. Fix an algebraically closed field $k$. All schemes in this section are smooth $k$-varieties.

10.2. Notation. Fix an integer $n$ prime to $\text{char}(k)$, and write $\Lambda = \mathbb{Z}/n$. For a smooth $k$-variety $X$, write $H^r_{\text{c}}(X; \Lambda) = \bigoplus_{r \geq 0} H^2_{\text{c}}(X; \Lambda(r))$.

10.3. Remark. As a graded group, the degree $r$ piece of $H^\ast_{\text{c}}(X; \Lambda)$ is $H^2_{\text{c}}(X; \Lambda(r))$. Moreover, the cup product makes $H^\ast_{\text{c}}(X; \Lambda)$ a graded-commutative ring.

10.4. Recollection. Let $X$ be a smooth $k$-variety. An elementary $r$-cycle on $X$ is a closed integral subscheme $Z \subset X$ of codimension $r$. The group of algebraic $r$-cycles $C^r(X)$ is the free abelian group on the set of elementary $r$-cycles. Write $C^\ast(X) = \bigoplus_{r \geq 0} C^r(X)$.

Elements of $C^\ast(X)$ are called algebraic cycles on $X$.

10.5. Definition. Let $X$ be a smooth $k$-variety. An elementary $r$-cycle $Z \subset X$ and an elementary $r'$-cycle $Z' \subset X$ intersect properly if each irreducible component of $Z \cap Z'$ has codimension $r + r'$. In this case $Z \cdot Z'$ is defined and belongs to $C^{r+r'}(X)$.

Algebraic cycles $Z, Z' \in C^\ast(X)$ intersect properly if every elementary cycle of $Z$ intersects every elementary cycle of $Z'$ properly, in which case $Z \cdot Z'$ is defined in the obvious manner.

10.6. Observation. For a morphism $\pi : Y \to X$ of smooth $k$-varieties we can sometimes define a morphism $\pi^* : C^\ast(X) \to C^\ast(Y)$ by setting

$$\pi^*(Z) := \Gamma_{\pi} \cdot (Y \times Z)$$

for an elementary cycle $Z \subset X$, where $\Gamma_{\pi} \subset Y \times X$ is the graph of $\pi$, and extending linearly to algebraic cycles. The map $\pi^*$ makes sense as long as all of the intersection products $\Gamma_{\pi} \cdot (Y \times Z)$ are well-defined.
10.7. **Remark.** According to [37], Prop. 1.3, $\pi^*$ is well-defined if $\pi$ is a flat map of algebraic $k$-schemes (though we only consider smooth $k$-schemes), or in the case that both the source and target are smooth (which is the case that we are interested in).

10.8. **Observation.** If $\pi : Y \to X$ is a proper morphism of smooth $k$-varieties, there is a morphism $\pi_* : C^*(Y) \to C^*(X)$ defined on elementary cycles $Z \subset Y$ by

$$\pi_*(Z) = \begin{cases} 0, & \text{dim}(\pi(Z)) < \text{dim}(Z) \\ \deg(\pi|_Z)\pi(Z), & \text{otherwise} \end{cases}$$

and extended linearly to $C^*(Y)$.

10.9. **Lemma** (projection formula for cycles). Let $\pi : Y \to X$ be a proper morphism of smooth $k$-varieties, $Z \in C^*(Y)$, and $Z' \in C^*(X)$. Then

$$\pi^*(Z') \cdot Z = Z' \cdot \pi_* (Z),$$

if the intersection products $\pi^*(Z') \cdot Z$ and $Z' \cdot \pi_*(Z)$ are defined.

10.10. **Definition.** Let $X$ be a smooth $k$-variety. The cycle class map $\text{cl}_X : C^*(X) \to H^*_\text{ét}(X; \mathcal{O})$ is the homomorphism of graded groups defined as follows.

(a) If $i : Z \hookrightarrow X$ is a smooth elementary $r$-cycle, then define $\text{cl}_X(Z) = i_*(1_Z)$, where

$$i_* : H^0_{\text{ét}}(Z; \mathcal{O}) \to H^{2r}_{\text{ét}}(X; \mathcal{O}(r))$$

is the Gysin map and $1_Z \in H^0_{\text{ét}}(Z; \mathcal{O})$ is the identity. Equivalently, $\text{cl}_X(Z)$ is the image of the fundamental class $s_{Z/X}$ under the map $H^{2r}_{\text{ét}}(X; \mathcal{O}(r)) \to H^{2r}_{\text{ét}}(X; \mathcal{O}(r))$.

(b) This definition extends to singular elementary cycles by virtue of the following lemma, and extends to each graded piece $C^r(X)$ by linearity.

10.11. **Lemma** ([37, Ch. vi Lem. 9.1]). Let $X$ be a smooth $k$-variety. For any reduced closed subscheme $Z \subset X$ of codimension $r$, we have $H^s_{\text{ét}}(X; \mathcal{O}) = 0$ for $s < 2r$.

10.12. **Proposition** ([37, Ch. vi Prop. 9.3]). Let $i : Z \hookrightarrow X$ be a closed immersion of smooth $k$-varieties and $c = \text{codim}_X(Z)$. Then for all $r \geq 0$, the square

$$\begin{array}{c}
C^r(Z) \\
\downarrow \text{cl}_Z \\
H^{2r}_{\text{ét}}(Z; \mathcal{O}(r)) \\
i_* \\
H^{2(r+c)}_{\text{ét}}(X; \mathcal{O}(r+c))
\end{array}$$

commutes.

10.13. **Proposition** ([37, Ch. vi Prop. 9.4]). Let $X$ and $Y$ be smooth $k$-varieties. The square

$$\begin{array}{ccc}
C^*(X) \times C^*(Y) & \to & C^*(X \times Y) \\
\downarrow \text{cl}_X \times \text{cl}_Y \\
H^*_{\text{ét}}(X; \mathcal{O}) \times H^*_{\text{ét}}(Y; \mathcal{O}) & \to & H^*_{\text{ét}}(X \times Y; \mathcal{O}) \\
\downarrow \text{pr}_1^* \times \text{pr}_2^* \\
H^*_{\text{ét}}(X \times Y; \mathcal{O}) & \to & H^*_{\text{ét}}(X \times Y; \mathcal{O})
\end{array}$$

commutes.
10.14. **Proposition** ([4], Ch. vi Prop. 9.4]). Let $X$ be a smooth $k$-variety and $Z$ and $Z'$ algebraic cycles on $X$. If $Z$ and $Z'$ intersect transversally, then
\[
\text{cl}_X(Z \cdot Z') = \text{cl}_X(Z) \sim \text{cl}_X(Z').
\]

10.15. **Remark** ([4], Ch. vi Rem. 9.6]). The map $\text{cl}_X^1 : C^1(X) \to H^2_{\text{ét}}(X; \mathcal{L}(1))$ is the composite of the canonical maps $C^1(X) \to \text{Pic}(X)$ given by $Z \mapsto O_X(Z)$ and $\text{Pic}(X) \to H^2_{\text{ét}}(X; \mathcal{L}(1))$ coming from the identifications $\text{Pic}(X) \cong H^1_{\text{ét}}(X; \mathcal{G}_m)$ and $\mathcal{L}(1) \cong \mathcal{L}$ along with the Kummer sequence.

10.16. **Example** ([4], Ch. vi Ex. 9.7]). If $L_r \subset P^m_k$ is a complete intersection of codimension $r$, then the Gysin map $\mathcal{L} \to H^{2r}_{\text{ét}}(P^m_k; \mathcal{L}(r))$ is an isomorphism. Hence $H^{2r}_{\text{ét}}(P^m_k; \mathcal{L}(r))$ is generated by $\text{cl}_{2r}(P^m_k(L_1))$. 

11. **Chern Classes**

11.1. **Convention.** Let $k$ be an algebraically closed field. In this section all schemes are smooth quasiprojective $k$-varieties.

11.2. **Notation.** Let $n \in \mathbb{Z}$ be prime to $\text{char}(k)$. We write $\mathcal{L} = \mathbb{Z}/n$.

11.3. **Proposition** (projective bundle formula ([4], Ch. vi Prop. 10.1)). Let $X$ be a smooth quasiprojective $k$-variety and $E$ a rank $m$ vector bundle on $X$. Let $\xi \in H^2_{\text{ét}}(P(E); \mathcal{L}(1))$ be the image of $O_{P(E)}(1)$ under the map $\text{Pic}(P(E)) \to H^2_{\text{ét}}(P(E); \mathcal{L}(1))$ coming from the Kummer sequence. Let $\pi : P(E) \to X$ be the structure morphism. Then the map $H^*_{\text{ét}}(X; \mathcal{L}[1]/(t^m+1)) \to H^*_{\text{é}}(P(E); \mathcal{L})$ given by $t^r \mapsto \text{cl}_{P(E)}(L^r)$ is an isomorphism of graded rings.

11.4. **Corollary.** Let $X$ be a smooth quasiprojective $k$-variety and $E$ a rank $m$ vector bundle on $X$. There are unique elements $c_r(E) \in H^2_{\text{ét}}(X; \mathcal{L})$ such that
\[
\begin{align*}
c_0(E) &= 1, \\
c_r(E) &= 0, \quad \text{for } r > m, \\
\sum_{r=0}^m c_r(E)t^{m-r} &= 0.
\end{align*}
\]
The element $c_r(E)$ is called the $r^\text{th}$ Chern class of $E$.

11.5. **Definition.** Let $X$ be a smooth quasiprojective $k$-variety and $E$ a vector bundle on $X$. The total Chern class of $X$ is the element $c(E) = \sum_{r \geq 0} c_r(E) \in H^*(X; \mathcal{L})$. 
11.6. Definition. Let \( X \) be a smooth quasiprojective \( k \)-variety and \( \mathcal{E} \) a vector bundle on \( X \). The **Chern polynomial** of \( X \) is the element
\[
\text{ch}_t(\mathcal{E}) := \sum_{r \geq 0} c_r(\mathcal{E}) t^r \in \bigoplus_{r \geq 0} H^{2r}_{\text{ét}}(X; \Lambda(r)) t^r.
\]

11.7. Notation. Write \( \text{Var}_k \) for the category of smooth quasiprojective \( k \)-varieties.

11.8. Theorem (\cite[Ch. vii Thm. 10.3]{EGA}). The Chern classes of smooth quasiprojective \( k \)-varieties are uniquely characterized by the following properties.

\[(11.8.a)\] Functoriality: If \( \pi : Y \to X \) is a morphism in \( \text{Var}_k \) and \( \mathcal{E} \) is a vector bundle on \( X \), then for all \( r \geq 0 \) we have
\[
c_r(\pi^{-1}(\mathcal{E})) = \pi^*(c_r(\mathcal{E})).
\]

\[(11.8.b)\] Normalization: If \( \mathcal{E} \) is a vector bundle on \( X \), then
\[
\text{ch}_t(\mathcal{E}) = 1 + p_X(\mathcal{E}) t,
\]
where \( p_X : \text{Pic}(X) \to H^2_{\text{ét}}(X; \Lambda) \) is the natural morphism from the Kummer sequence.

\[(11.8.c)\] Additivity: If \( X \in \text{Var}_k \) and
\[
0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0.
\]
is a short exact sequence of vector bundles on \( X \), then
\[
\text{ch}_t(\mathcal{E}) = \text{ch}_t(\mathcal{E}') \text{ch}_t(\mathcal{E}'').
\]

11.9. Remark. By \((11.8.c)\) the total Chern class defines a homomorphism of abelian groups \( c : K_0(X) \to H^*_\text{ét}(X; \Lambda) \). Moreover, this homomorphism respects the gradings.

11.10. Corollary. Let \( X \in \text{Var}_k \). If \((\dim(X) - 1)! \) is invertible in \( \Lambda \), then the cycle class map \( \text{cl}_X \) defines a homomorphism \( \text{cl}_X : \text{CH}^\bullet(X) \to H^*_\text{ét}(X; \Lambda) \) which is natural in \( X \).

11.11. Remark. \([\text{Corollary 11.10}]\) is true with \( \Lambda \) replaced by \( \mathbb{Q}_\ell \), where \( \ell \neq \text{char}(k) \). In this case, the invertibility condition on \((\dim(X) - 1)! \) is vacuous.

12. Poincaré Duality

12.1. Notation. We write \( \Lambda = \mathbb{Z}/n \), and \( n \) is always taken to be prime to \( \text{char}(X) \) for any scheme \( X \) that we take cohomology of with coefficients in a \( \Lambda \)-module.

12.2. Notation. If \( X \) is a separated variety of dimension \( d \) and \( x \in X \) is a closed point, we abusively write \( \text{cl}_X(x) \) for the image of the fundamental class \( s_{x/X} \) under the natural map
\[
H^d_{\text{ét}}(X; \Lambda(d)) \to H^d_{\text{ét}}(X; \Lambda(d)).
\]

12.3. Theorem (Poincaré duality for smooth varieties \cite[Ch. vii Thm. 11.1]{EGA}). Let \( k \) be a separably closed field and \( X \) a dimension \( d \) smooth separated \( k \)-variety. Then:

\[(12.3.a)\] There is a unique **trace map** \( \eta_X : H^{2d}_c(X; \Lambda(d)) \to \Lambda \) such that \( \eta_X(\text{cl}_X(x)) = 1 \) for all closed points \( x \in X \). Moreover, \( \eta_X \) is an isomorphism.

\[(12.3.b)\] If \( F \in \text{Cnstr}(X; \Lambda) \), the natural pairings
\[
H^r_c(X; F) \times \text{Ext}^{2d-r}_X(F, \Lambda(d)) \to H^{2d}_c(X; \Lambda(d)) \xrightarrow{\eta_X} \Lambda
\]
are perfect for \( 0 \leq r \leq 2d \).

**Proof Idea.** To construct the trace map:

\[(12.3.a)\] If \( X \) is a dimension \( d \) separated variety over a separably closed field, then
\[
H^d_{c}(X; \Lambda(d)) \cong \Lambda.
\]
If \( \pi : Y \to X \) is a separated étale morphism, where \( X \) is a dimension \( d \) separated variety over a separably closed field, then for every closed point \( y \in Y \),

\[
\pi_* : H^d_c(Y; \Lambda(d)) \to H^d_c(X; \Lambda(d))
\]

sends \( \text{cl}_Y(y) \) to \( \text{cl}_X(\pi(y)) \).

To prove that the pairing is perfect:

1. If \( \pi : X' \to X \) is finite étale, the pairings are perfect for \( (X', F') \) if and only if they are perfect for \( (X, \pi_* F') \).
2. If \( U \subset X \) is an open subvariety, the pairings are perfect for \( (X, F) \) if and only if they are perfect for \( (U, F|_U) \).
3. If \( \pi : X \to S \) is a smooth projective morphism with 1-dimensional fibers, the pairings are perfect for locally constant constructible sheaves. To do this, we show that the pairings are perfect for sheaves with support in a smooth closed subvariety \( Z \subset X \).
4. The pairings are perfect for constant sheaves.
5. The pairings are perfect for locally constant sheaves.
6. By pulling back to an open subvariety on which a constructible sheaf is locally constant, we conclude the theorem.

**12.4. Corollary** ("classical" Poincaré duality [4, Ch. vi Cor. 11.2]). Let \( X \) be a dimension \( d \) separated variety over a separably closed field. Then for any locally constant constructible \( \Lambda \)-module, the cup product pairing

\[
H^r_c(X; F) \times H^{2d-r}_c(X; F(d)) \to H^d_c(X; \Lambda(d)) \overset{\eta_X}{\longrightarrow} \Lambda
\]

is perfect for \( 0 \leq r \leq 2d \).

**12.5. Corollary** ([4, Ch. vi Cor. 11.5]). Let \( X \) be a dimension \( d \) separated variety over a separably closed field. If \( X \) is not complete, then for any locally constant torsion sheaf \( F \) on \( X \) we have \( H^d_{et}(X; F) = 0 \).

**12.6. Theorem** ([4, Ch. vi Thm. 11.7]). Let \( X \) be a dimension \( d \) projective variety over a separably closed field. Then for all \( r \geq 0 \), the group \( N^r(X) \) of \( r \)-cycles modulo numerical equivalence is finitely generated.

**12.7. Proposition** (Poincaré duality for compactifiable morphisms [4, Ch. vi Prop. 11.8]). Let \( \pi : X \to S \) be a smooth compactifiable morphism of schemes with \( d \)-dimensional fibers. There exists a unique relative trace map \( \eta_{X/S} : R^d_c \pi_* (\Lambda(d)) \to \Lambda \) such that for any geometric point \( s \to S \) and any closed point \( x \in X_s \), the trace map

\[
\eta_{X_s/s} : H^d_c(X_s; \Lambda(d)) \to \Lambda
\]

sends \( \text{cl}_{X_s}(x) \) to \( 1 \). If, moreover, the fibers of \( \pi \) are connected, then \( \eta_{X/S} \) is an isomorphism.

Moreover, \( \eta_{X/S} \) is compatible with composition of compactifiable morphisms and base change along arbitrary morphisms.

**Proof Idea.** Work fiberwise, using Poincaré duality for varieties and the assumptions that the fibers are equidimensional. The assumption that \( \pi \) is compactifiable allows us to globalize the result on fibers.
13. The Zeta Function & the Weil Conjectures

13.1. Notation. In this section all schemes are $F_q$-varieties. Given an $F_q$-variety $X$, we write $\overline{X} := X \times_{\text{Spec}(F_q)} \text{Spec}(\overline{F}_q)$.

13.2. Definition. Let $X$ be an $F_q$-variety. The Zeta function of $X$ is the formal power series with $Q$-coefficients defined by

$$Z(X, t) := \exp \left( \sum_{n \geq 1} \frac{\#X(F_q^n)}{n} t^n \right).$$

13.3. Conjectures (Weil). Let $X$ be a smooth projective $F_q$-variety of dimension $d$.

13.3.a) Rationality of the Zeta function: we have that

$$Z(X, t) = P_1(X, t) \cdots P_{2d-1}(X, t) \frac{P_0(X, t) \cdots P_{2d}(X, t)}{P_0(X, t) \cdots P_{2d}(X, t)},$$

where each $P_i(X, t)$ is a polynomial with coefficients in a field of characteristic 0.

13.3.b) Integrality: $P_0(X, t) = 1 - t$, $P_{2d}(X, t) = 1 - q^d t$, each each $P_i(X, t)$ has roots whose inverses are algebraic integers.

13.3.c) Functional equation: $Z(X, q^{-d}t^{-1}) = \pm q^{d/2} t^\chi Z(X, t)$, where $\chi$ is the Euler characteristic of $X$.

13.3.d) Riemann Hypothesis: the inverses of the roots of $P_i(X, t)$ and all of their conjugates have complex absolute value $q^{r/2}$.

13.3.e) Specialization: If $X$ is the specialization of a smooth projective variety $\overline{X}$ over a number field, then

$$\deg(P_i(X, t)) = \beta_i(\overline{X}(\mathbb{C}), \text{num}) \cdot \beta_i(Y) \text{ denotes the } r^{th} \text{ Betti number of a space } Y.$$

13.4. Theorem (Lefschetz trace formula [4, Ch. vi Thm. 12.3]). Let $\phi: X \to X$ be an endomorphism of a smooth projective variety over an algebraically closed field $k$ such that the intersection $I_\phi \cdot \Delta_X$ is defined. Then for any prime $\ell \neq \text{char}(k)$ we have

$$(I_\phi \cdot \Delta_X) = \sum_{r=0}^{2 \dim(X)} (-1)^r \text{tr}(\phi^* | H^r_{\text{et}}(X; \mathbb{Q}_\ell))).$$

13.5. Theorem ([3, Ch. iv Thm. 1.2]). Let $X$ be a smooth projective $F_q$-variety. For all $r \geq 0$, and primes $\ell \neq \text{char}(F_q)$, the polynomials

$$P_{r, \ell}(X, t) := \det(\text{id} - t \text{Fr}^* | H^r_{\text{et}}(\overline{X}; \mathbb{Q}_\ell)).$$

in $\mathbb{Q}_\ell[t]$ have coefficients in $Z$. Moreover the polynomials $P_{r, \ell}(X, t)$ are independent of the prime $\ell$.

13.6. Notation. Let $X$ be a smooth projective $F_q$-variety. Write $P_r(X, t)$ for the common polynomial $P_{r, \ell}(X, t) \in \mathbb{Z}[t]$.

13.7. Theorem (Riemann hypothesis for varieties [3, Ch. iv Thm. 1.2]). Let $X$ be a smooth projective $F_q$-variety and $\ell \neq \text{char}(F_q)$ a prime. Then the eigenvalues of $\text{Fr}^*$ on $H^r_{\text{et}}(\overline{X}; \mathbb{Q}_\ell)$ and all of their conjugates have complex absolute value $q^{r/2}$, and the inverse roots of $P_i(X, t)$ are algebraic integers of absolute value $q^{r/2}$. 
13.8. **Theorem** (rationality of the Zeta function [4, Ch. vi Thm. 12.4]). Let $X$ be a smooth projective $\mathbb{F}_q$-variety. Then

$$Z(X, t) = \frac{P_1(X, t) \cdots P_{2d-1}(X, t)}{P_0(X, t) \cdots P_{2d}(X, t)}.$$  

13.9. **Theorem** (functional equation for the Zeta function [2, Ch. iv Thm. 1.2; 4, Ch. vi Thm. 12.6]). Let $X$ be a smooth projective $\mathbb{F}_q$-variety of dimension $d$. Then

$$Z(X, q^{-d}t^{-1}) = (-1)^{d-1}N q^{d/2}t^\chi Z(X, t),$$

where $\chi$ is the Euler characteristic of $X$ and $N$ is the multiplicity of the eigenvalue $q^{d/2}$ of $\text{Fr}^*$ on $H^d_{\text{ét}}(\overline{X}; \mathbb{Q}_\ell)$ (for any prime $\ell \neq \text{char}(\mathbb{F}_q)$).

13.10. **Theorem** ([4, Ch. vi 13.1]). Let $X$ be a separated $\mathbb{F}_q$-variety. Then for any prime $\ell \neq \text{char}(\mathbb{F}_q)$ we have

$$Z(X, t) = \prod_{r \geq 0} \det(\text{id} - t \text{Fr}^* | H^d_{\text{ét}}(\overline{X}; \mathbb{Q}_\ell))^{(-1)^r}.$$  

**References**