1 Motivations

1.1 Notation. We write $\text{Spc}$ for the $\infty$-category of spaces or $\infty$-groupoids, $\text{Cat}_{\infty}$ for the $\infty$-category of $\infty$-categories, $\text{Sp}$ for the $\infty$-category of spectra, and $\text{Sp}_{K(n)}$ for the $\infty$-category of $K(n)$-local spectra (where $K(n)$ is a Morava $K$-theory).

Recall that the goal of [1] is to show that for any $\pi$-finite space $X$, prime number $p$, and Morava $K$-theory $K(n)$ (at $p$), the functors

$$\text{colim}_X, \text{lim}_X : \text{Fun}(X, \text{Sp}_{K(n)}) \to \text{Sp}_{K(n)}$$

are (canonically) equivalent. If we let $f: X \to \ast$ denote the unique morphism, then:

- The diagonal functor $\text{Sp}_{K(n)} \to \text{Fun}(X, \text{Sp}_{K(n)})$ is identified with the functor $f^* : \text{Fun}(\ast, \text{Sp}_{K(n)}) \to \text{Fun}(X, \text{Sp}_{K(n)})$ given by precomposition with $f$.

- The functor $\text{colim}_X$ is identified with the left adjoint $f_!$ of $f^*$, given by left Kan extension along $f$.

- The functor $\text{lim}_X$ is identified with the right adjoint $f^*$ of $f_*$, given by right Kan extension along $f$.

Rephrasing our problem, we're interested in studying the assignment

$$X \mapsto \text{Fun}(\ast, \text{Sp}_{K(n)})$$

and determining when there is a natural equivalence $f_! \Rightarrow f_*$. 

(1) To encode all the relevant functoriality in $X$, we should adopt the relative point of view and determine which morphisms $f : X \to Y$ in $\text{Spc}$ have the property that $f_! \Rightarrow f_*$. 

For setting up the general theory, it is not really relevant that we are working with local systems of $K(n)$-local spectra; we might as well consider any functor that assigns a morphism $f : X \to Y$ in $Spc$ a chain of three adjunctions $f_i \dashv f^* \dashv f_*$. 

The perspective we've taken is to assume that to any morphism $f : X \to Y$ in $Spc$, we have three adjoints $f_i \dashv f^* \dashv f_*$, and to try to construct an equivalence $Nm_f : f_i \Rightarrow f_*$. We can equivalently not assume the existence of the extreme right adjoint $f_*$ (or, alternatively, the extreme left adjoint $f_i$), and try to exhibit $f_i$ as a right adjoint to $f^*$. That is, we might as well restrict our attention to functors $Spc \to \text{Cat}_{\text{ladj}}^{\infty}$, where $\text{Cat}_{\text{ladj}}^{\infty}$ is the $\infty$-category of $\infty$-categories and left adjoint functors (see Notation 2.1).

The fact that we're considering functors to $\text{Cat}_{\text{ladj}}^{\infty}$ with source the $\infty$-category of spaces isn't particularly relevant for setting up the general theory; we can replace the $\infty$-category of spaces with an essentially arbitrary index $\infty$-category $\mathcal{X}$.

There is one non-obvious fact particular to our situation that is relevant to the general theory (Proposition 1.5). First we recall the colimit formula for left Kan extensions.

1.2 Recollection (comma $\infty$-categories). The comma $\infty$-category $X \downarrow_Y Z$ associated to two functors $f : X \to Y$ and $g : Z \to Y$ is the universal $\infty$-category fitting into a lax-commutative diagram

$$
\begin{array}{ccc}
X \downarrow_Y Z & \longrightarrow & X \\
\downarrow & \searrow & \downarrow f \\
Z & \longrightarrow & Y.
\end{array}
$$

Explicitly, $X \downarrow_Y Z$ can be computed as the iterated pullback

\begin{equation}
X \downarrow_Y Z = X \times_{\text{Fun}([0],Y)} \text{Fun}(\Delta^1,Y) \times_{\text{Fun}([1],Y)} Z.
\end{equation}

Notice that since all morphisms are invertible in an $\infty$-groupoid, if $Y$ is an $\infty$-groupoid, then the comma $\infty$-category $X \downarrow_Y Z$ is simply given by the pullback

$$
X \downarrow_Y Z = X \times_Y Z.
$$

(This can also be seen from the formal description (1.3) by noting that an $\infty$-category $Y$ is an $\infty$-groupoid if and only if evaluation at 0 or 1 defines an equivalence of $\infty$-categories $\text{Fun}(\Delta^1,Y) \Rightarrow Y$.)

1.4 Recollection (colimit formula for left Kan extensions). Let $C$ be an $\infty$-category with colimits and let $L_X : X \to C$ and $f : X \to Y$ be functors. Then for each object $y \in Y$, the value of the left Kan extension $\text{Lan}_f L_X$ of $L_X$ along $f$ at $y$ is given by the colimit

$$
(\text{Lan}_f L_X)(y) = \text{colim} \left( X \downarrow_Y \{y\} \longrightarrow X \xrightarrow{L_X} C \right),
$$

where $X \downarrow_Y \{y\}$ is the comma $\infty$-category.
1.5 Proposition ([1, Proposition 4.3.3]). Let $C$ be an $\infty$-category with colimits and

\[
\begin{array}{c}
X' \xrightarrow{f} Y' \\
\downarrow \quad \downarrow g \\
X \xrightarrow{f} Y
\end{array}
\]

a pullback square in $\text{Spc}$. Then the associated square

\[
\begin{array}{ccc}
\text{Fun}(X',C) & \xrightarrow{f^*} & \text{Fun}(Y',C) \\
\downarrow \quad \downarrow g^* \\
\text{Fun}(X,C) & \xrightarrow{f_!} & \text{Fun}(Y,C)
\end{array}
\]

commutes.

Proof. Note that by the universal property of the left Kan extension, for any local system $L_X : X \to C$ we have a natural transformation

$$\theta : \text{Lan}_f (L_X \circ \bar{g}) \to (\text{Lan}_f L_X) \circ g,$$

i.e., a natural transformation $f_! \circ \bar{g}^* (L_X) \to g^* f_!(L_X)$. We want to show that for each $y' \in Y'$, the morphism

$$\theta(y') : \text{Lan}_f (L_X \circ \bar{g})(y') \to (\text{Lan}_f L_X)(g(y'))$$

is an equivalence.

Since $C$ has all colimits, by the pointwise formula for left Kan extensions (Recollection 1.4), for all $y' \in Y'$ we have

$$\text{Lan}_f (L_X \circ \bar{g})(y') \cong \text{colim} \left( X' \downarrow_{Y'} \{y'\} \to X' \xrightarrow{\bar{g}} X \xrightarrow{L_X} C \right).$$

Since $Y'$ is an $\infty$-groupoid,

$$X' \downarrow_{Y'} \{y'\} \cong X' \times_{Y'} \{y'\},$$

(Recollection 1.2). Since the square (1.6) is a pullback square and $Y$ is an $\infty$-groupoid, we see that

$$X' \times_{Y'} \{y'\} = X \times_Y \{g(y')\} = X \downarrow_Y \{g(y')\},$$

(Recollection 1.2). Hence

$$\text{Lan}_f (L_X \circ \bar{g})(y') \cong \text{colim} \left( X \downarrow_Y \{g(y')\} \to X \xrightarrow{L_X} C \right) = \text{Lan}_f (L_X)(g(y')).$$

Moreover, this equivalence is induced by $\theta$. \qed
1.7 Remark. For an ∞-category $C$ with colimits, there is a version of Proposition 1.5 involving the $(−)_!$, right adjoints that says that given the pullback square (1.6) in $\text{Spc}$ there’s a natural equivalence
$$f^*g_* \cong \bar{g}_*\bar{f}^*$$
of functors $\text{Fun}(Y', C) \to \text{Fun}(X, C)$. The proof is the same as Proposition 1.5; one just uses the limit formula for right Kan extensions.

The identification $g^*f_! = \tilde{f}_!\tilde{g}^*$ in Proposition 1.5 is a Beck–Chevalley condition, which we examine in the next section.

2 Beck–Chevalley morphisms

2.1 Notation. We write $\text{Cat}_{\text{adj}}^{\infty} \subset \text{Cat}_{\infty}$ for the subcategory with objects any ∞-category and morphisms functors $C \to D$ which are left adjoints. We usually write $f_! : C \to D$ for a left adjoint and $f^* : D \to C$ for its corresponding right adjoint. In this case, we write
$$\eta_f : \text{id}_C \to f^*f_! \quad \text{and} \quad \varepsilon_f : f_!f^* \to \text{id}_D$$for the unit and counit of the adjunction $f_! \dashv f^*$, respectively.

2.2 Definition. Consider a commutative square $\sigma$
\[
\begin{array}{ccc}
C' & \xrightarrow{f} & D' \\
\downarrow{g} & & \downarrow{g} \\
C & \xrightarrow{f} & D
\end{array}
\]
in $\text{Cat}_{\text{adj}}^{\infty}$. The Beck–Chevalley morphism associated to the square $\sigma$ is the composite natural transformation
$$BC(\sigma) : \tilde{f}_!\tilde{g}^* \xrightarrow{\tilde{f}_!\tilde{g}^*\eta_f} \tilde{f}_!\tilde{g}^*f^*f_! \xrightarrow{-} \tilde{f}_!g^*\tilde{f}^*f_! \xrightarrow{\varepsilon_fg^*f_!} g^*f_!,$$where the middle equivalence comes from the identification of right adjoints
$$\tilde{g}^*f^* \cong g^*\tilde{f}^* .$$The Beck–Chevalley morphism is depicted diagrammatically as
\[
\begin{array}{ccc}
C' & \xrightarrow{f} & D' \\
\downarrow{g^*} & & \downarrow{g^*} \\
C & \xrightarrow{f_!} & D
\end{array}
\]
\[
\begin{array}{c}
\text{BC(}\sigma\text{)}
\end{array}
\]

We say that the square (2.3) satisfies the Beck–Chevalley condition if the Beck–Chevalley morphism $BC(\sigma) : \tilde{f}_!\tilde{g}^* \to g^*f_!$ is an equivalence.
2.4 Remark. As in Remark 1.7, there is a Beck–Chevalley morphism for the \((—)_*\) right adjoints: it is a natural transformation

\[
(f^* g)_* \rightarrow \tilde{g}_* \tilde{f}^* .
\]

This is the basechange morphism that one often sees in algebraic geometry, for example, in the smooth and proper basechange theorems for étale cohomology (see [3, Chapter vi Corollary 2.3 & Theorem 4.1]).

An alternative approach to ambidexterity is to dispose of the \((—)_!\) adjoints and instead work with the \((—)_*\) adjoints and the other Beck–Chevalley morphism (2.5).

2.6 Example. By giving a more careful proof of Proposition 1.5, we can conclude that for any \(\infty\)-category \(C\) with colimits and pullback square

\[
\begin{array}{ccc}
X' & f & \rightarrow & Y' \\
\tilde{g} & j & \downarrow & \tilde{g} \\
X & f & \rightarrow & Y
\end{array}
\]

in \(\text{Spc}\), the induced square

\[
\begin{array}{ccc}
\text{Fun}(X', C) & f^* & \rightarrow & \text{Fun}(Y', C) \\
\tilde{g}_* & \downarrow & \tilde{g}^* & \downarrow \\
\text{Fun}(X, C) & f^* & \rightarrow & \text{Fun}(Y, C)
\end{array}
\]

satisfies the Beck–Chevalley condition. (See also [2].)

2.7 Observation. Please observe that given commutative squares

\[
\begin{array}{ccc}
C' & \rightarrow & D' & \rightarrow & E' \\
\downarrow & & \downarrow & & \downarrow \\
C & \rightarrow & D & \rightarrow & E
\end{array}
\]

\(\sigma\) and \(\tau\) (from left to right) in \(\text{Cat}^{\text{adj}}_{\infty}\), the Beck–Chevalley morphism of the outer rectangle is equivalent to natural transformation given by the horizontal composite of the Beck–Chevalley morphisms

\[
\begin{array}{ccc}
C' & \rightarrow & D' & \rightarrow & E' \\
\downarrow & & \downarrow & & \downarrow \\
C & \rightarrow & D & \rightarrow & E
\end{array}
\]
Similarly, given commutative squares
\[
\begin{array}{ccc}
C'' & \rightarrow & D'' \\
\downarrow & & \downarrow \\
C' & \rightarrow & D' \\
\downarrow & & \downarrow \\
C & \rightarrow & D ,
\end{array}
\]
\[
\sigma' \quad \text{and} \quad \sigma \quad \text{(from top to bottom)} \quad \text{in} \quad \text{Cat}_{\infty}^{ladj}, \quad \text{the Beck–Chevalley morphism of the outer rectangle is equivalent to natural transformation given by the vertical composite of the Beck–Chevalley morphisms}
\[
\begin{array}{ccc}
C'' & \rightarrow & D'' \\
\downarrow & & \downarrow \\
C' & \rightarrow & D' \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
\]

Given our generalizations and reformulations of our goal from §1, we are interested in considering functors \( X \rightarrow \text{Cat}_{\infty}^{ladj} \) sending pullback squares to squares satisfying the Beck–Chevalley condition, and exhibiting the left adjoint \( f_i \) of \( f'' \) as a right adjoint of \( f_* \) for a certain class of morphisms \( f : X \rightarrow Y \) in \( X \).

2.8 Definition. Let \( X \) be an \( \infty \)-category with pullbacks. A functor \( C : X \rightarrow \text{Cat}_{\infty}^{ladj} \) is a **Beck–Chevalley functor** if for every pullback square
\[
\begin{array}{ccc}
X' & \stackrel{f}{\rightarrow} & Y' \\
\downarrow & \phi & \downarrow \\
X & \stackrel{f}{\rightarrow} & Y
\end{array}
\]
in \( X \), the induced square
\[
\begin{array}{ccc}
C_{X'} & \stackrel{i}{\rightarrow} & C_{Y'} \\
\downarrow & \phi & \downarrow \\
C_X & \stackrel{f_i}{\rightarrow} & C_Y
\end{array}
\]
in \( \text{Cat}_{\infty}^{ladj} \) satisfies the Beck–Chevalley condition.

2.9 Example. For any \( \infty \)-category \( C \) with colimits, the functor
\[
\text{Fun}(\cdot, C) : \text{Spc} \rightarrow \text{Cat}_{\infty}^{ladj}
\]
where the functoriality is in the \((\cdot)\) adjoints, is a Beck–Chevalley functor.
2.10 Remark. Beck–Chevalley functors $\mathbf{X} \to \text{Cat}_{\infty}^{\text{ladj}}$ are classified by functors $q : C \to \mathbf{X}$ that are both cartesian and cocartesian fibrations and for each pullback square in $\mathbf{X}$ the associated square in $\text{Cat}_{\infty}^{\text{ladj}}$ satisfies the Beck–Chevalley condition. Such fibrations are called Beck–Chevalley fibrations [1, Definition 4.1.3].

3 The definition of ambidexterity & basic properties

In this section we define ambidexterity of morphisms in an $\infty$-category $\mathbf{X}$ with pullbacks with respect to a Beck–Chevalley functor $C : \mathbf{X} \to \text{Cat}_{\infty}^{\text{ladj}}$. This definition is a rather intricate inductive definition; to warm up we recall how to define truncatedness in an $\infty$-category with pullbacks as well as the universal property of the counit of an adjunction

3.1 Recollection ([HTT, Lemma 5.5.6.15]). Let $\mathbf{X}$ be an $\infty$-category with finite limits (e.g., $\mathbf{X} = \text{Spc}$). For each integer $n \geq -2$ we define the class of $n$-truncated morphisms of $\mathbf{X}$ inductively as follows. A morphism $f : X \to Y$ in $\mathbf{X}$ is $(-2)$-truncated if $f$ is an equivalence. Now suppose that $n$-truncated morphisms in $\mathbf{X}$ have been defined for some integer $n \geq -2$. Then we say that a morphism $f : X \to Y$ in $\mathbf{X}$ is $(n+1)$-truncated if the diagonal morphism $\delta_f : X \to X \times_Y X$ is $n$-truncated.

Note that if $\mathbf{X} = \text{Spc}$ is the $\infty$-category of spaces, then a morphism $f$ is $n$-truncated in the sense just defined if and only if the fibers of $f$ are $n$-truncated spaces, so the notion just defined agrees with the classical notion of truncatedness.

3.2 Recollection (universal property of the (co)unit [HTT, Definition 5.2.2.7 & Proposition 5.2.2.8]). Given functors between $\infty$-categories $f : C \rightleftarrows D : g$, the functor $f$ is left adjoint to $g$ if and only if there exists a natural transformation $\eta : \text{id}_C \to gf$ such that for every pair of objects $c \in C$ and $d \in D$, the composite

$$\text{Map}_D(f(c), d) \xrightarrow{\eta(d)^*} \text{Map}_C(gf(c), g(d)) \xrightarrow{\eta(c)} \text{Map}_C(c, g(d))$$

is an equivalence in $\text{Spc}$. Dually, $f$ is left adjoint to $g$ if and only if there exists a natural transformation $\epsilon : fg \to \text{id}_D$ such that for every pair of objects $c \in C$ and $d \in D$, the composite

$$\text{Map}_C(c, g(d)) \xrightarrow{\epsilon(d)} \text{Map}_D(f(c), fg(d)) \xrightarrow{\epsilon(c)} \text{Map}_C(f(c), d)$$

is an equivalence in $\text{Spc}$.

Our approach to exhibiting a left adjoint $f_1 \dashv f^*$ as a right adjoint to $f^*$ is to define a counit transformation that exhibits $f_1$ as right adjoint to $f^*$.

3.3 Construction (definition of ambidexterity [1, Construction 4.1.8]). Let $\mathbf{X}$ be an $\infty$-category with pullbacks and $C : \mathbf{X} \to \text{Cat}_{\infty}^{\text{ladj}}$ a Beck–Chevalley functor. We define the following data for each integer $n \geq -2$:

(a.\ $n$) A class of morphisms in $\mathbf{X}$ which we call $n$-ambidextrous morphisms (with respect to the Beck–Chevalley functor $C$).
For each \( n \)-ambidextrous morphism \( f : X \to Y \) in \( X \), a natural transformation
\[
\mu_f^{(n)} : \text{id}_{\mathcal{C}_Y} \to f_! f^* \quad \text{(well-defined up to equivalence)}
\]
that exhibits \( f_! \) as a right adjoint to \( f^* \).

A morphism \( f : X \to Y \) in \( X \) is \((-2)\)-ambidextrous if \( f \) is an equivalence. In this case, we let \( \mu_{f}^{(-2)} = \varepsilon_f^{-1} \) be an inverse to the counit \( \varepsilon_f : f_! f^* \Rightarrow \text{id}_{\mathcal{C}_Y} \) of the adjunction \( f_! \dashv f^* \).

Assume that \((a_n)\) and \((b_n)\) have been defined for some integer \( n \geq -2 \). Let \( f : X \to Y \) be a morphism in \( X \), and write \( \delta_f : X \to X \times_Y X \) for the diagonal map, which fits into a commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\delta_f} & X \\
\downarrow & & \downarrow f \\
X \times_Y X & \xrightarrow{\text{pr}_1} & X \\
\downarrow \text{pr}_2 & & \downarrow f \\
X & \xrightarrow{f} & Y.
\end{array}
\]

Write \( \sigma \) for the pullback square in the diagram (3.4), so that the Beck–Chevalley morphism \( \text{BC}(\sigma) : \text{pr}_{1,1} \text{pr}_2^* \to f^* f_! \) is an equivalence. We say that \( f \) is weakly \((n+1)\)-ambidextrous if the diagonal \( \delta_f \) is \( n \)-ambidextrous. If \( f \) is weakly \((n+1)\)-ambidextrous, we define a natural transformation
\[
\nu_f^{(n+1)} : f^* f_! \to \text{id}_{\mathcal{C}_X}
\]
as the composite natural transformation
\[
f^* f_! \xrightarrow{\text{BC}(\sigma)^{-1}} \text{pr}_{1,1} \text{pr}_2^* \xrightarrow{\text{pr}_{1,1} \delta_f \delta_f^*} \text{pr}_{1,1} \delta_f \delta_f^* \text{pr}_2^* = \text{id}_{\mathcal{C}_X} \circ \text{id}_{\mathcal{C}_X} = \text{id}_{\mathcal{C}_X}.
\]

We say that a morphism \( f : X \to Y \) is \((n+1)\)-ambidextrous if the following condition is satisfied: for every pullback diagram
\[
\begin{array}{ccc}
X' & \xrightarrow{f} & Y' \\
\downarrow g & & \downarrow g \\
X & \xrightarrow{f} & Y
\end{array}
\]
in \( X \), the map \( f \) is weakly \((n+1)\)-ambidextrous and \( \nu_f^{(n+1)} : f^* f_! \to \text{id}_{\mathcal{C}_Y} \) is the counit for an adjunction \( f_! \dashv f^* \).

If \( f \) is \((n+1)\)-ambidextrous, we let \( \mu_f^{(n+1)} : \text{id}_{\mathcal{C}_Y} \to f_! f^* \) denote a compatible unit for the adjunction
\[
f^* : \mathcal{C}_Y \rightleftarrows \mathcal{C}_X : f_!
\]
determined by \( \nu_f^{(n+1)} \).
3.5 Definition ([1, Definition 4.1.1]). Let \( X \) be an \( \infty \)-category with pullbacks and let \( C : X \to \text{Cat}_{\infty}^{\text{adj}} \) be a Beck–Chevalley functor. We say that a morphism \( f : X \to Y \) in \( X \) is (weakly) ambidextrous (with respect to the Beck–Chevalley functor \( C \)) if \( f \) is (weakly) \( n \)-ambidextrous for some integer \( n \geq -2 \).

The following basic properties are easily deduced from the definitions.

3.6 Proposition ([1, Proposition 4.1.10]). Let \( X \) be an \( \infty \)-category with pullbacks, \( C : X \to \text{Cat}_{\infty}^{\text{adj}} \) a Beck–Chevalley functor, and \( f : X \to Y \) a morphism in \( X \). Then:

3.6.1. If \( f \) is weakly \( n \)-ambidextrous for some integer \( n \geq -2 \), then \( f \) is \( n \)-truncated.

3.6.2. For each integer \( n \geq -2 \), the class of \( n \)-ambidextrous morphisms are stable under pullback.

3.6.3. For each integer \( n \geq -1 \), the class of weakly \( n \)-ambidextrous morphisms are stable under pullback.

3.6.4. Let \( -1 \leq m \leq n \) be integers. If \( f \) is weakly \( m \)-ambidextrous, then \( f \) is weakly \( n \)-ambidextrous. Moreover, the natural transformations \( \nu_{f}^{(m)} : f^{*}f_{!} \to \id_{C_{X}} \) agree up to homotopy.

3.6.5. Let \( -2 \leq m \leq n \) be integers. If \( f \) is \( m \)-ambidextrous, then \( f \) is \( n \)-ambidextrous. Moreover, the natural transformations \( \mu_{f}^{(m)} : \id_{C_{Y}} \to f_{!}f^{*} \) agree up to homotopy.

3.6.6. Let \( -1 \leq m \leq n \) be integers. If \( f \) is (weakly) \( n \)-ambidextrous, then \( f \) is (weakly) \( m \)-ambidextrous if and only if \( f \) is \( m \)-truncated.

3.7 Notation. Let \( X \) be an \( \infty \)-category with pullbacks, \( C : X \to \text{Cat}_{\infty}^{\text{adj}} \) a Beck–Chevalley functor, and \( f : X \to Y \) a morphism in \( X \) that is weakly ambidextrous. Then we simply write \( \nu_{f} \) for \( \nu_{f}^{(n)} \) for some integer \( n \geq -2 \) such that \( f \) is weakly \( n \)-ambidextrous (so that \( \nu_{f} \) is well-defined up to homotopy). If \( f \) is ambidextrous, we write \( \mu_{f} \) for a compatible unit of \( \nu_{f} \).

3.8 Reformulation (the norm map [1, Remark 4.1.12]). Let \( X \) be an \( \infty \)-category with pullbacks, \( C : X \to \text{Cat}_{\infty}^{\text{adj}} \) a Beck–Chevalley functor, and \( f : X \to Y \) a morphism in \( X \). If \( f^{*} : C_{Y} \to C_{X} \) admits a right adjoint, we denote the right adjoint to \( f^{*} \) by \( f_{*} : C_{X} \to C_{Y} \). We then have an equivalence

\[
\Map_{\text{Fun}(C_{X}, C_{X})}(f^{*}f_{!} \to \id_{C_{X}}) = \Map_{\text{Fun}(C_{X}, C_{Y})}(f_{*}f^{*} \to \id_{C_{X}}).
\]

If \( f \) is weakly ambidextrous, we let \( \Nm_{f} : f_{!} \to f_{*} \) denote the image of \( \nu_{f} : f^{*}f_{!} \to \id_{C_{X}} \) under the equivalence (3.9). We call \( \Nm_{f} \) the norm map associated to \( f \).

We can reformulate the definition of ambidexterity as follows. A weakly ambidextrous morphism \( f : X \to Y \) is ambidextrous if and only if for every pullback square

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

is ambidextrous.
in $X$, the following conditions are satisfied:

(3.8.1) The morphism $\tilde{f}$ is weakly ambidextrous.

(3.8.2) The functor $\tilde{f}^*: C_{Y'} \to C_{X'}$ admits a right adjoint $\tilde{f}_!$.

(3.8.3) The norm map $Nm_{\tilde{f}}: \tilde{f}_! \to \tilde{f}_*$ is an equivalence.

### 4 Naturality properties of the norm

The goal of this section is to establish two naturality properties of the construction $f \mapsto \mu_f$ (or, equivalently, $f \mapsto Nm_f$).

#### 4.1 Proposition ([1, Proposition 4.2.1])

Let $X$ be an $\infty$-category with pullbacks, $C: X \to \text{Cat}_{\omega}^{\text{adj}}$ a Beck–Chevalley functor, and let $\tau$ be a pullback diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & Y' \\
\downarrow \phi & & \downarrow g \\
X & \xrightarrow{f} & Y
\end{array}
\]

in $X$. Then:

(4.1.1) If $f$ is weakly ambidextrous, then $\tilde{f}$ is weakly ambidextrous and the diagram

\[
\begin{array}{cccc}
\tilde{f}^* & \xrightarrow{\tilde{f}^* \tilde{g}^*} & f^* f_! & \xrightarrow{\sim} & g^* f_! \\
\vert & & \downarrow \phi^* & & \downarrow \gamma^* \\
\tilde{g}^* & & & & \tilde{g}^*
\end{array}
\]

commutes up to homotopy.

(4.1.2) If $f$ is ambidextrous, then $\tilde{f}$ is ambidextrous and the diagram

\[
\begin{array}{cccc}
g^* & \xrightarrow{\mu_f g^*} & g^* f_! & \xrightarrow{\sim} & f_! g^* f^* \\
\vert & & \downarrow g^* \mu_f & & \downarrow f_! \tilde{g}^* f^* \\
\tilde{f}_! \tilde{g}^* & \xrightarrow{\sim} & \tilde{f}_! g^* f^* & \xrightarrow{BC(\tau)_{f^*}} & g^* f_! f^*
\end{array}
\]

commutes up to homotopy.

#### 4.2 Reformulation (Proposition 4.1 in terms of norms [1, Remark 4.2.3])

In the situation of Proposition 4.1, assume that $f$ and $\tilde{f}$ are weakly ambidextrous and that the functors $f^*$ and $\tilde{f}^*$ admit right adjoints $f_!$ and $\tilde{f}_!$. Then we can reformulate assertion (4.1.1) as follows: the morphism

\[
\begin{array}{cccc}
\tilde{f}^* \tilde{g}^* & \xrightarrow{BC(\tau)} & f^* f_! & \xrightarrow{\sim} & g^* f_! \\
\vert & & \downarrow \phi^* & & \downarrow g^* \mu_f \\
\tilde{f}_! \tilde{g}^* & \xrightarrow{\sim} & \tilde{f}_! g^* f^* & \xrightarrow{N\text{m}_f} & g^* f_! f^*
\end{array}
\]

is homotopic to $N\text{m}_f \tilde{g}^*$, where the last morphism is the Beck–Chevalley morphism involving the $(-)_*$ adjoints (Remark 2.4).
Proof of Proposition 4.1. The statement that \( f \) is (weakly) ambidextrous implies that \( \tilde{f} \) is (weakly) ambidextrous is immediate from the definitions.

Recall that if \( f \) is weakly ambidextrous, then \( f \) is \( n \)-truncated for some integer \( n \geq -2 \) (3.6.1). We prove both (4.1.1) and (4.1.2) simultaneously by induction on the truncatedness of \( f \), which we denote by \( n \). If \( n = -2 \), then \( f \) is an equivalence, hence \( \tilde{f} \) is an equivalence, and (4.1.1) and (4.1.2) are obvious.

So assume that \( n \geq -1 \) and that \( f \) (and hence \( \tilde{f} \)) is weakly ambidextrous. Thus we have a pullback square \( \rho \)

\[
\begin{array}{ccc}
X' & \xrightarrow{\delta_f} & X' \times_Y X' \\
\downarrow{\tilde{\partial}} & & \downarrow{\pi} \\
X & \xrightarrow{\delta_f} & X \times_Y X,
\end{array}
\]

where the morphism \( \pi \) is induced by \( g \) and \( \tilde{g} \), and \( \delta_f \) and \( \delta_f \) are \((n - 1)\)-truncated and ambidextrous by the assumption that \( f \) is weakly ambidextrous. Let \( \sigma \) denote the pullback square

\[
\begin{array}{ccc}
X \times_Y X & \xrightarrow{\Pr_1} & X \\
\downarrow{\Pr_2} & & \downarrow{f} \\
X & \xrightarrow{f} & Y,
\end{array}
\]

let \( \sigma \) denote the pullback square

\[
\begin{array}{ccc}
X' \times_Y X' & \xrightarrow{\Pr_1} & X' \\
\downarrow{\Pr_1} & & \downarrow{f} \\
X' & \xrightarrow{f} & Y',
\end{array}
\]

and let \( \xi \) denote the pullback square

\[
\begin{array}{ccc}
X' \times_Y X' & \xrightarrow{\Pr_1} & X' \\
\downarrow{\pi} & & \downarrow{\tilde{g}} \\
X \times_Y X & \xrightarrow{\Pr_1} & X.
\end{array}
\]
Now consider the diagram

\[
\begin{array}{c}
\begin{array}{c}
\tilde{f}^* \tilde{g}^* \xrightarrow{\text{BC}(\rho)^{-1}} \tilde{f}_! \tilde{g}_! \quad \begin{array}{c}
\tilde{f}_! \tilde{g}_! \delta_1^* \delta_2^* \pi_1 \quad \text{pr}^*_2 \quad \sim \quad \tilde{g}^*
\end{array}
\end{array}
\end{array}
\]

where the long composites at the top and bottom of the diagram are the definitions of \( v f \tilde{g}^* \) and \( \tilde{g}^* v f \), respectively, and morphisms labeled with ‘\( \sim \)’ and no other decorations are given by identification of adjoints. Our goal is to show that the outer rectangle of (4.4) commutes up to homotopy; we do this by showing that each sub-diagram commutes up to homotopy.

The diagrams in the middle column of (4.4) commute up to homotopy by the inductive hypothesis (for (4.1.2)). The upper-right diagram in (3.4) commutes because we're just identifying adjoints. The lower-right diagram in (3.4) commutes because Beck–Chevalley morphisms compose horizontally (Observation 2.7) and we have a commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
X' \xrightarrow{\delta_1} X' \times_Y X' \xrightarrow{\text{pr}_1} X'
\end{array}
\end{array}
\]

where the long composites on the top and bottom are the identity on \( X' \) and \( X \), respectively. To see that the left third rectangle in (3.4) commutes up to homotopy, it suffices to show that the diagram

\[
\begin{array}{c}
\begin{array}{c}
\tilde{f}^* \tilde{g}^* \quad \text{pr}_1 \quad \sim \quad \tilde{g}^*
\end{array}
\end{array}
\]
commutes up to homotopy. This again follows from the fact that Beck–Chevalley morphisms compose vertically (Observation 2.7) and the fact that the outer rectangles of the two diagrams

\[
\begin{array}{ccc}
X' \times_Y X' & \xrightarrow{pr_1} & X' \\
\downarrow \pi & & \downarrow \check{g} \\
X \times_Y X & \xrightarrow{pr_1} & X \\
\downarrow pr_2 & & \downarrow f \\
X & \xrightarrow{f} & Y
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
X' \times_Y X' & \xrightarrow{pr_1} & X' \\
\downarrow \pi & & \downarrow \check{g} \\
X' & \xrightarrow{f} & Y' \\
\downarrow \check{g} & & \downarrow g \\
X & \xrightarrow{f} & Y
\end{array}
\]

are the same.

Now we prove the inductive step for (4.1.2). Assume that \(f\) is ambidextrous so that the natural transformations \(\nu f : f^* f_1 \to \text{id}_{C_X}\) and \(\nu \check{f} : \check{f}^* \check{f}_1 \to \text{id}_{C_{Y'}}\) are counits of adjunctions \(f^* \dashv f_1\) and \(\check{f}^* \dashv \check{f}_1\). Then by the universal property of the unit \(\mu f\), the composite map

\[
\text{Map}_{\text{Fun}(C_X, C_Y)}(g^* f^* f_1, f^* f_1) \xrightarrow{\mu g^* f_1} \text{Map}_{\text{Fun}(C_X, C_{Y'})}(g^* f_1, f^* f_1)
\]

is an equivalence. Moreover, by the triangle identity, the natural transformation \(g^* \mu f : g^* \to g^* f_1 f^*\) corresponds to the identity \(g^* f_1 \to g^* f_1\) under the equivalence \(\alpha\). Thus proving that the diagram appearing in (4.1.2) commutes up to homotopy is equivalent to showing that the composite

\[
g^* f_1 \xrightarrow{\mu g^* f_1} \check{f}_1 \check{f}^* g^* f_1 = \check{f}_1 \check{g}^* f^* f_1 \xrightarrow{\check{g}^* f_1 \nu f} g^* f_1 f^* f_1 \xrightarrow{g^* f_1 \nu f} g^* f_1
\]
is homotopic to the identity. To prove this, consider the diagram

\[
\begin{array}{ccccccccc}
g^* f_1 & \mu_f & \longrightarrow & \tilde{f}_1 f^* g^* f_1 & \longrightarrow & f_1 g^* f_1 f_1 & \longrightarrow & g^* f_1 \\
\downarrow_{BC(\tau)} & & & \downarrow_{BC(\tau)} & & \downarrow_{\nu_f} & & \downarrow_{BC(\tau)} \\
\tilde{f}_1 \tilde{g}^* & \longrightarrow & \tilde{f}_1 \tilde{f}_1 \tilde{g}^* & \longrightarrow & \tilde{f}_1 \tilde{g}^*. 
\end{array}
\]

The left-hand square and right-hand triangle obviously commute, and the middle square commutes by the inductive step for (4.1.1). To show that the top composite is homotopic to the identity, it suffices to show that the bottom composite is homotopic to the identity: this is true by the triangle identity since \(\mu_f\) and \(\nu_f\) are a compatible unit and counit.

**4.5 Corollary** (adjoint formulation of Proposition 4.1 [1, Corollary 4.2.6]). Let \(X\) be an \(\infty\)-category with pullbacks, \(C : X \to \text{Cat}_{\text{ladj}}\) a Beck–Chevalley functor, and let \(\tau\) be a pullback diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{f} & Y' \\
\downarrow & & \downarrow \\
X & \xrightarrow{f} & Y
\end{array}
\]

in \(X\). Then:

1. **(4.5.1)** If \(f\) is weakly ambidextrous, then \(\tilde{f}\) is weakly ambidextrous and the diagram

\[
\begin{array}{ccc}
\tilde{g}_1 f^* \tilde{f}_1 & \xrightarrow{BC(\tau) \tilde{f}_1} & \tilde{f}_1 g_1 \\
\downarrow_{\sim} & & \downarrow_{\sim} \\
\tilde{g}_1 f^* \tilde{f}_1 & \longrightarrow & \tilde{f}_1 g_1 
\end{array}
\]

commutes up to homotopy.

2. **(4.5.2)** If \(f\) is ambidextrous, then \(\tilde{f}\) is ambidextrous and the diagram

\[
\begin{array}{ccc}
g_1 & \longrightarrow & \tilde{g}_1 f^* \tilde{f}_1 \\
\downarrow & & \downarrow_{\mu_f} \\
g_1 f_1 f_1 & \longrightarrow & \tilde{g}_1 \tilde{f}_1 \tilde{g}_1 f_1
\end{array}
\]

commutes up to homotopy.

The proof of the following proposition is similar to the proof of Proposition 4.1, though a little more involved.

**4.6 Proposition** ([1, Proposition 4.2.2]). Let \(X\) be an \(\infty\)-category with pullbacks, \(C : X \to \text{Cat}_{\text{ladj}}\) a Beck–Chevalley functor, and suppose we are given morphisms \(f : X \to Y\) and \(g : Y \to Z\) in \(X\). Then:
(4.1.1) If $f$ and $g$ are weakly ambidextrous, then $gf$ is weakly ambidextrous and $\nu_{gf}$ is homotopic to the composition

$$
(gf)^* (gf)_! = f^* g^* g_! f_! \xrightarrow{f^* \nu_{gf}} f^* f_! \xrightarrow{\nu_f} \text{id}_{C_X}.$$

commutes up to homotopy.

(4.1.2) If $f$ and $g$ are ambidextrous, then $gf$ is ambidextrous and $\mu_{gf}$ is homotopic to the composition

$$
\text{id}_{C_Z} \xrightarrow{\mu_g} g_! g^* \xrightarrow{g_! \mu_g} g_! f_! f^* \xrightarrow{\mu_{gf}} (gf)_!(gf)^*.$$

4.7 Reformulation (Proposition 4.6 in terms of norms [1, Remark 4.2.4]). In the situation of Proposition 4.6, assume that $f$ and $g$ are weakly ambidextrous and that the functors $f^*$ and $g^*$ admit right adjoints $f_!$ and $g_*$. Then $(gf)^*$ is left adjoint to $(gf)_*$ := $g_* f_!$ and we can reformulate assertion (4.6.1) as follows: the norm $Nm_{gf} : (gf)_! \to (gf)^*$ is given by the composite

$$
(gf)_! \xrightarrow{Nm_{gf}} g_! f_! \xrightarrow{g_* f_!} g_* g_! (gf)_* = (gf)^*.
$$

5 Ambidexterity for local systems

So far we have considered ambidexterity for arbitrary Beck–Chevalley functors $X \to \text{Cat}_{\text{adj}}^{\infty}$. We now specify to the case of $C$-valued local systems, i.e., $X = \text{Spc}$ and we consider the Beck–Chevalley functor $\text{Fun}(\_ , C) : \text{Spc} \to \text{Cat}_{\text{adj}}^{\infty}$, where $C$ is an $\infty$-category with colimits (recall Proposition 1.5).

5.1 Definition ([1, Definition 4.3.4]). Let $C$ be an $\infty$-category with colimits.

- A space $X$ is weakly $C$-ambidextrous if the unique morphism $f : X \to *$ is weakly $C$-ambidextrous (with respect to the Beck–Chevalley functor $\text{Fun}(\_ , C) : \text{Spc} \to \text{Cat}_{\text{adj}}^{\infty}$).
- A space $X$ is $C$-ambidextrous if $X$ is weakly $C$-ambidextrous and the natural transformation $\nu_f : f^* f_! \to \text{id}_{\text{Fun}(X,C)}$ is the counit of an adunction (so that $f^* \dashv f_!$).

5.2 Proposition ([1, Proposition 4.3.5]). Let $C$ be an $\infty$-category with colimits and $f : X \to Y$ and $f : X \to Y$ a morphism in $\text{Spc}$. Then:

- The morphism $f$ is ambidextrous if and only if $f$ is $n$-truncated for some integer $n \geq -2$ and each fiber $X_y$ of $f$ is $C$-ambidextrous.
- The morphism $f$ is weakly ambidextrous if and only if $f$ is $n$-truncated for some integer $n \geq -2$ and each fiber $X_y$ of $f$ is weakly $C$-ambidextrous.

5.3 Corollary ([1, Corollary 4.3.6]). Let $C$ be an $\infty$-category with colimits and $f : X \to Y$ a morphism between truncated spaces. If $Y$ is $C$-ambidextrous and each fiber $X_y$ of $f$ is $C$-ambidextrous, then $X$ is $C$-ambidextrous.
The proof of Proposition 5.2 uses the following fact to reduce to proving the claim in a special case.

5.4 Lemma ([1, Lemma 4.3.8]). Let C be an ∞-category with colimits and X a space. Then Fun(X, C) is generated under colimits by objects of the form x c, where x : * → X is a point of X and c ∈ C = Fun(*,C).

Lemma 5.4 is also used in the proof of the following proposition.

5.5 Proposition ([1, Proposition 4.3.9]). Let C be an ∞-category with colimits and X a truncated space. Let f : X → * denote the unique morphism. Then X is C-ambidextrous if and only if the following conditions are satisfied:

(5.5.1) X is weakly C-ambidextrous (i.e., Map_X(x, x') is C-ambidextrous for all x, x' ∈ X).

(5.5.2) The pullback functor f* admits a right adjoint f_!

(5.5.3) The functor f* preserves colimits.

Proof. If X is C-ambidextrous, then (5.5.1) is obvious and (5.5.2) and (5.5.3) follow from the fact that the left adjoint f_! of f* is also right adjoint to f*.

Now assume that (5.5.1)–(5.5.3) are satisfied. Using (5.5.2), the counit ν_f : f*f_! → id_{Fun(X,C)} corresponds to the norm map Nm_X : f_! → f_* (Reformulation 3.8), and our goal is to show that Nm_f is an equivalence. That is, we want to show that for every local system L : X → C, the natural transformation

Nm_f(L) : f_!L → f_*L

is an equivalence in C. By assumption (5.5.3), the collection of objects L ∈ Fun(X, C) for which Nm_f(L) is an equivalence is closed under colimits, so by Lemma 5.4 we are reduced to the case where L = x_c for x : * → X a point and c ∈ C. By assumption (5.5.1), for every point x ∈ X the morphism x : * → X is C-ambidextrous. By Reformulation 4.7, the composite

c = (fx)c = f_!L \xrightarrow{Nm_f(L)} f_*L \xrightarrow{f_*Nm_c(L)} f_*x_c = (fx)_c = c

is homotopic to the identity. Since x : * → X is X-ambidextrous, the norm Nm_x is an equivalence, hence Nm_f is an equivalence as well.

5.6 Remark. Note that Proposition 5.5 gives a characterization of (weakly) C-ambidextrous morphisms that doesn’t explicitly mention the natural transformations ν_f or μ_f.

6 Semiadditivity & ambidexterity of Eilenberg–MacLane spaces

6.1 Definition ([1, Definition 4.4.1]). Let n ≥ −2 be an integer. A space X is a finite n-type if X is π-finite and n-truncated.
6.2 Definition ([1, Definition 4.4.2]). Let $C$ be an $\infty$-category with colimits and $n \geq -2$ be an integer. We say that $C$ is $n$-semiadditive if every finite $n$-type is $C$-ambidextrous.

6.3 Examples.

(6.3.1) Since a space $X$ is a finite $(-2)$-type if and only if $X$ is contractible, every $\infty$-category with colimits is $(-2)$-semiadditive.

(6.3.2) Since the only $(-1)$-types are contractible and empty spaces, an $\infty$-category with colimits $C$ is $(-1)$-semiadditive if and only if $C$ is pointed.

(6.3.3) Since a space $X$ is a finite $0$-type if and only if $X$ is equivalent to a finite set, an $\infty$-category with colimits is $0$-semiadditive if and only if $C$ is semiadditive in the usual sense, i.e., the finite products in $C$ are also finite coproducts.

(6.3.4) Any stable $\infty$-category with colimits is $0$-semiadditive.

6.4 Notation ([1, Notation 4.4.15]). Let $C$ be a $0$-semiadditive $\infty$-category with colimits and $n \geq 0$ be an integer. Write $[n] : \text{id}_C \to \text{id}_C$ for the natural transformation determined by the composite

$$c \xrightarrow{\Delta} c^{\times n} \xrightarrow{\psi} c$$

of the diagonal and codiagonal for each object $c \in C$.

6.5 Proposition ([1, Proposition 4.4.16]). Let $C$ be a $0$-semiadditive $\infty$-category with limits and assume that there exists a prime number $p$ with the following property:

(6.5.1) For every integer $n \geq 1$ which is relatively prime to $p$, the natural transformation $[n] : \text{id}_C \to \text{id}_C$ is an equivalence

Then $C$ is $1$-semiadditive if and only if the Eilenberg–MacLane space $K(\mathbb{Z}/p, 1)$ is $C$-ambidextrous.

Proof. The fact that the $1$-semiadditivity of $C$ implies that $K(\mathbb{Z}/p, 1)$ is $C$-ambidextrous is immediate from the definition of $1$-semiadditivity.

For the other direction, suppose that $K(\mathbb{Z}/p, 1)$ is $C$-ambidextrous. Let $X$ be a finite $1$-type; our goal is to show that $X$ is $C$-ambidextrous. By applying Corollary 5.3 to the map $X \to \pi_0(X)$, we can reduce to the case where $X$ is connected, so that $X \simeq BG$ for some finite group $G$.

Now we reduce to the case where $G$ is a $P$-group. Let $P \subset G$ be a $p$-Sylow subgroup, and consider the maps $g : BP \to BG$ induced by the inclusion $P \hookrightarrow G$ and $f : BG \to \ast$. Then $g$ is equivalent to a covering space with finite fibers, hence is $C$-ambidextrous since $C$ is $0$-semiadditive. We want to show that $\text{Nm}_f : f_! \to f_\ast$ is an equivalence. Let $L \in \text{Fun}(BG, C)$ and let $\alpha$ denote the composite

$$\alpha : L \longrightarrow g_* g^* L = g_! g^* L \longrightarrow L,$$

where the middle equivalence comes from the fact that $g$ is ambidextrous. We claim that $\alpha$ is an equivalence. To prove this, it suffices to show that for every point $x \in BG$, the
morphism \( x^* \alpha : x^* L \to x^* L \) is an equivalence. Unwinding the definitions, we see that \( x^* \alpha \) is given by the morphism \( \#(G/P) : x^* L \to x^* L \). Since \( P \) is a \( p \)-Sylow subgroup of \( G \), the number \( \#(G/P) \) is relatively prime to \( p \), so that \( \#(G/P) \) is an equivalence by assumption (6.5.1). Since \( \alpha \) is an equivalence, we see that \( L \) is a retract of \( g_i g^* L \). Hence to see that \( \text{Nm}_f(L) \) is an equivalence, it suffices to prove that \( \text{Nm}_f(g_i g^* L) \) is an equivalence. We may therefore assume that \( L = g_i L' \) for some \( L' \in \text{Fun}(BP, C) \). Consider the composite

\[
\text{Nm}_{fg}(L') : (fg)_i L' \xrightarrow{\text{Nm}_f(g, L')} f_* g_i L' \xrightarrow{f_* g_i} (fg)_i L' ,
\]

where the second morphism is an equivalence since \( g \) is \( C \)-ambidextrous. By the 2-of-3 property we see that to prove that \( \text{Nm}_f(g_i g^* L) \) is an equivalence, so we can replace \( G \) by \( P \) and assume that \( G \) is a \( p \)-group.

With this reduction, we now proceed by induction on the cardinality of the \( p \)-group \( G \). If \( G \) is trivial, there is nothing to prove. For the induction step, we can choose a normal subgroup \( N \triangleleft G \) of order \( p \). It follows from the inductive hypothesis that \( B(G/N) \) is \( C \)-ambidextrous.

Consider the fiber sequence

\[
K(\mathbb{Z}/p, 1) \cong BN \to BG \to B(G/N) ,
\]

so an application of Corollary 5.3 and the assumption that \( K(\mathbb{Z}/p, 1) \) is \( C \)-ambidextrous show that \( BG \) is \( C \)-ambidextrous, completing the proof.

6.6 Proposition ([1, Proposition 4.4.17]). Let \( C \) be a 0-semiadditive co-category with limits and \( p \) be a prime number. If the natural transformation \( [p] : \text{id}_C \to \text{id}_C \) is an equivalence, then for every finite \( p \)-group \( G \), the Eilenberg–MacLane space \( BG \) is \( C \)-ambidextrous.

Proof. As in the proof of Proposition 6.5, we can reduce to the case where \( G = \mathbb{Z}/p \). Consider the maps \( g : * \to BG \) given by any point and \( f : BG \to * \), so that \( g \) is equivalent to a covering space with finite fibers (namely the projection \( EG \to BG \)). We need to show that \( \text{Nm}_f : f_* \to f_* \) is an equivalence. Let \( L \in \text{Fun}(BG, C) \) and let \( \alpha \) denote the composite

\[
\alpha : L \to g^* g^* L = g_i g^* L \to L ,
\]

where the middle equivalence comes from the fact that \( g \) is ambidextrous. As in the proof of Proposition 6.5, we see that for each \( x \in BG \), the map \( x^* \alpha : x^* L \to x^* L \) is homotopic to \( [p] : x^* L \to x^* L \), hence an equivalence by assumption. Thus \( L \) is a retract of \( g_i(g^* L) \). Thus it suffices to show that \( \text{Nm}_{fg} \) induces as equivalence \( f_i g_* (g^* L) \Rightarrow f_* g_i (g^* L) \). Set \( L' := g^* L \). Since \( fg = \text{id}_* \), the composite map

\[
\text{Nm}_{fg} : L' = (fg)_i L' \xrightarrow{\text{Nm}_f} f_* g_i L' \xrightarrow{\text{Nm}_g} (fg)_i L' = L'
\]

is an equivalence. By the 2-of-3 property, we are reduced to proving that \( \text{Nm}_{fg} \) induces an equivalence \( f_* g_i L' \Rightarrow f_* g_i L' \). This follows from the assumption that \( C \) is 0-semiadditive.
6.7 Corollary ([1, Corollary 4.4.18]). Let $C$ be a $0$-semiadditive $\infty$-category with limits. If for each integer $n \geq 1$ the natural transformation $[n] : \text{id}_C \rightarrow \text{id}_C$ is an equivalence, then $C$ is $1$-semiadditive.

6.8 Proposition ([1, Proposition 4.4.20]). Let $C$ be a stable $\infty$-category with limits and colimits and let $p$ be a prime number such that the natural transformation $[p] : \text{id}_C \rightarrow \text{id}_C$ is an equivalence. Then the Eilenberg–MacLane spaces $K(\mathbb{Z}/p, m)$ are $C$-ambidextrous for $m \geq 1$.

6.9 Corollary ([1, Corollary 4.4.21]). Let $C$ be a stable $\infty$-category with limits and colimits. Assume that for each object $c \in C$, the endomorphism ring $\text{Ext}^0_C(c, c)$ is a $\mathbb{Q}$-algebra. Then $C$ is $n$-semiadditive for every integer $n \geq -2$.

6.10 Example ([1, Example 4.4.22]). Let $R$ be an $E_1$-ring spectrum with the property that $\pi_0(R)$ is a $\mathbb{Q}$-vector space. Then the $\infty$-category of left $R$-module spectra is $n$-semiadditive for every integer $n \geq -2$.

6.11 Corollary ([1, Corollary 4.4.23]). Let $C$ be a stable $\infty$-category with limits and colimits and $p$ a prime number. Assume that for each object $c \in C$, the endomorphism ring $\text{Ext}^0_C(c, c)$ is a $\mathbb{Z}(p)$-module. Then $C$ is $n$-semiadditive if and only if the Eilenberg–MacLane spaces $K(\mathbb{Z}/p, m)$ are $C$-ambidextrous for $1 \leq m \leq n$.

References


