Zero-pointed Manifolds and Poincaré/Koszul Duality

(Andy)

I Zero-pointed Manifolds

Recall, $V$ a $\otimes$-presentable symmetric monoidal $\infty$-category and an $n$-disk algebra $A \in \text{Alg}_n(V)$, we defined the factorization homology

$$\int_{(-)} A : \text{Mfld}_n \to V.$$  

Note. This picture is unsatisfactory for both Poincaré and Koszul duality.

- Poincaré: There’s a natural isomorphism

  $$\mathbb{H}^c_c(M) \cong H_{n-i}(M)$$

  for $M$ an oriented $n$-manifold.

- The functoriality of $\mathbb{H}^c_c$ comes from $(\cdot)$, extension by zero. From
the point of homotopy theory, this comes from the Pontryagin-Thom collapse of an open embedding.
- This requires more than just the functionality in open embeddings.

> Koszul: Most naturally about $E_n$-algebras and $E_n$-coalgebras.
- Disk$_n$ captures $E_n$-algebras but not $E_n$-coalgebras.

Idea. Enlarge Mfld$_n$ to a category ZMfld$_n$ that encodes this functionality.

Def. A zero-pointed manifold $M_*$ is a pointed locally compact Hausdorff space $M$ such that:

1. $M := M_*$ is endowed with the structure of a smooth manifold.
2. There exists a compact $F$ with boundary $\partial F = \partial L \cup \partial R$ such that

$$M_* = (F \setminus \partial R) / \partial L$$

Examples. The following are zero-pointed manifolds:

> $M_4 = M \sqcup S^3$
> $M^+ = 1$-pt compactification of $M$. 
Def. A continuous map of zero-pointed manifolds $f : M^*_+ \to N^*_+$ is a zero-pointed embedding if $f^{-1}(N^*_+) \to N^*_+$ is an embedding.

Def.

> $\mathcal{ZMfld}_n :=$ zero-pointed manifolds and embeddings.

> $\text{Disk}_n^+, \subset \mathcal{ZMfld}_n$, the full subcategory spanned by finite wedges of $\mathbb{R}^n^+$.

> $\text{Disk}_n^+, \subset \mathcal{ZMfld}_n$, the full subcategory spanned by finite wedges of $(\mathbb{R}^n)^+$.

> $\mathcal{Mfld}_n^+, \subset \mathcal{ZMfld}_n$, the full subcategory spanned by finite wedges of $M^*_+$ for $M \in \mathcal{Mfld}_n$.

> $\mathcal{Mfld}_n^+, \subset \mathcal{ZMfld}_n$, the full subcategory spanned by finite wedges of $M^*_+$ for $M \in \mathcal{Mfld}_n$.

These are all symmetric monoidal under wedge.
Note. There is an involution

\[ \sim : \mathcal{M}_{\text{fld}}_n \to \mathcal{M}_{\text{fld}}_n^\circ \]

\[ M_* \mapsto (M_*^*)^\perp \mapsto * \]

\[ (M^+)^\sim \cong M^+ \quad \text{and} \quad (M^+)^\circ \cong M^+ . \]

In particular,

\[ (\text{Disk}_{n,+})^\circ \cong \text{Disk}_n^+ . \]

Claim. Let \( V \) be a symmetric monoidal \( \infty \)-category. There is an equivalence of \( \infty \)-categories

\[ \text{Fun}_{\otimes} (\text{Disk}_n, V) \cong \text{Alg}^\text{aug}_n(V) . \]

Hence

\[ \text{Fun}_{\otimes} (\text{Disk}^+, V) \cong \text{coAlg}^\text{aug}_n(V) . \]
Def. Let $M_*$ be a zero-pointed manifold. For $i \geq 0$, write

$$\text{Conf}_i(M_*):= \{ c: [i] \rightarrow M_* | c|_{c^{-1}(M)} \text{ is injective} \} \subseteq (M_*)^i.$$ 

$$\text{Conf}_i^c(M_*):= (M_*)^i / \{ c: [i] \rightarrow M | c|_{c^{-1}(M)} \text{ not injective} \}.$$

⚠ $\text{Conf}_i(M_*)$ and $\text{Conf}_i^c(M_*)$ are generally not zero-pointed manifolds because of problems with the basepoint. They are homotopy equiv. to zero-pointed manifolds with corners.
II Factorization (co)homology of zero-pointed manifolds

Def. The factorization homology of a zero-pointed manifold $M^*_n \in \text{Mfld}_n$ with coefficients in $\text{AAlg}^{\text{aug}}_{\text{gn}}(V)$ is the colimit

$$\int_{M^*_n} \text{A} := \text{Colim } \left( \text{Disk}_{n^+}/M^*_n \longrightarrow \text{Disk}_{n^+} \stackrel{\text{A}}{\longrightarrow} V \right).$$

For an augmented $n$-disk coalgebra $C \in \text{CoAlg}^{\text{aug}}_{\text{gn}}(V)$

$$\int_{M^*_n} C := \text{lim } \left( \text{Disk}_{n^+}/M^*_n \stackrel{C}{\longrightarrow} \text{Disk}_{n^+} \longrightarrow V \right).$$

Fact. For $M \in \text{Mfld}_n$ and $\text{AAlg}^{\text{aug}}_{\text{gn}}(V)$

$$\int_{M^+} \text{A} \simeq \int_M \text{A}.$$

\text{forget the augmentation.}
Thm. These exist when $V$ has sifted (co)limits, and are symmetric monoidal as functors from zero-pointed manifolds and $\otimes$ commutes with sifted (co)limits.

If they exist, there is a natural map

$$\int_{M_+} A \longrightarrow \int_{M_+^*} \left( \left( \int_{(-)} A \right) \right)_{\text{Disc}^n} \cong \int_{(\mathbb{R}^n)_+} A$$

We call this the Poincaré/Koszul duality map.

Claim (Why $\int_{(\mathbb{R}^n)_+} A$ is the Koszul dual). $\int_{(\mathbb{R}^n)_+} A \cong \text{Bar}^n(A)$.

Example. For $M_+ = S^n_+$, the Poincaré/Koszul duality map is a map

$$HH_*(A) \longrightarrow HH^*(\text{DA})$$

Koszul Dual
Fact. Let $\bar{M}$ be a compact manifold with boundary and $M_* = \bar{M}/\partial \bar{M}$. Then

$$\int_{M_*} A = \int_{\bar{M}} A.$$  (factorization homology of manifold with boundary)

Here the right-hand side is the colimit over

$$\text{Disk}_n, /\sim \xrightarrow{\text{collapse}} (\text{Disk}_{n+1}) /\sim \xrightarrow{A} V.$$

finite coproducts of $(\mathbb{R}^n, \text{Disk}^n)$. 

Hence $\int_{(\mathbb{R}^n)} A = \int_{\text{Disk}^n} A$.

By excision

$$\int_{\text{Disk}^n} A = \left( \int_{\partial^+ \text{Disk}^n} A \right) \mathcal{O} \left( \int_{\text{Disk}^{n-1}} A \right) \mathcal{O} \left( \int_{\mathbb{R} \times \text{Disk}^{n-1}} A \right) \approx 1 \mathcal{O} 1 \mathcal{O} 1.$$ 

by induction

$$1 \approx \text{Bar}^{n-1}(A).$$
Linear Poincaré Duality

Here, \((V, \otimes) = (S, \otimes)\) where \(S\) is a stable presentable \(\infty\)-category.

In this setting,

\[
\text{Alg}_n(S) = \text{Mod}_{O(n)}(S),
\]

\[
\text{coAlg}_n(S) = \text{Mod}_{O(n)}(S).
\]

Thm. If \(E, F \in \text{Mod}_{O(n)}(S)\) are such that \((R^n)^\ast \otimes E \simeq F\) (equiv. \(E \simeq F(R^n)\))

then Poincaré/Koszul Duality holds and is equivalent to the statement that

\[
\text{Fr}_{M^+} \otimes E \simeq \text{Map}_{O(n)}(\text{Fr}_{M^+}, F).
\]

Here \(\text{Fr}_{M^+} = \text{ZEmb}(\mathbb{R}^n, M^+)\).

\(\int_{M^+} E \simeq \int_{M^+} F\)

Basically by definition by doing Kan extensions.
Example. This implies Atiyah duality: take \( F = S \) with the trivial action. So that \( E = \Omega^n S = S^{-n} \) with the nontrivial action:

\[ \Rightarrow (M^*)^{-\tau} = S M^* \]