Conductors and minimal discriminants of hyperelliptic curves

Padmavathi Srinivasan

MIT

AMS Summer Institute in Algebraic Geometry, Salt Lake City
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Outline

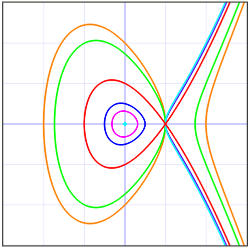
What are conductors and minimal discriminants?

Degenerating family of hyperelliptic curves → Measures of degeneracy

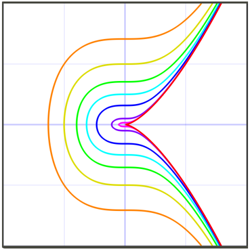
Measures of degeneracy → Artin conductor

Measures of degeneracy → Minimal discriminant

How are these related?



$y^2 = (x-1)(x^2-t)$



$y^2 = x^3 - t$

Outline

Notation

R: complete discrete valuation ring

K: fraction field of R

k: residue field of R, algebraically closed, char $\neq 2$

ν: discrete valuation $K \to \mathbb{Z} \cup \{\infty\}$

t: a uniformizer of R, *i.e.*, $\nu(t) = 1$.

Examples: $\mathbb{C}[[t]], \mathbb{Z}_p^{\mathrm{unr}}$

Notation

$f(x)$: monic, squarefree, even degree ≥ 4 polynomial in $R[x]$

X: smooth projective model of the plane curve $y^2 = f(x)$ over K

g: genus of X

If $f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_0$ factors as $(x - \alpha_1)\ldots(x - \alpha_d)$ in $\overline{K}[x]$, then

$$\mathrm{disc}(f) := \prod_{i<j}(\alpha_i - \alpha_j)^2$$

$$\in K[a_0, \ldots, a_{d-1}]$$

Minimal discriminant of X

$$f(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_0 = (x - \alpha_1)\ldots(x - \alpha_d)$$

The naive discriminant of $y^2 = f(x)$ is defined to be

$$\Delta_f := \nu(\mathrm{disc}(f)) = \nu\Big(\prod_{i<j}(\alpha_i - \alpha_j)^2\Big).$$

The minimal discriminant $\Delta_{\min}$ of X is given by

$$\Delta_{\min} = \Delta_{\min}(X) := \min\left(\Delta_f \;\middle|\; \begin{array}{l} f(x) \in R[x] \text{ such that } y^2 = f(x) \\ \text{is birational to } X \text{ over } K \end{array}\right)$$

$$\Delta_{\min} = 0 \iff X \text{ has good reduction}.$$

X : a hyperelliptic curve over K

$\mathscr{X}$: a proper, flat, regular R-scheme with $\mathscr{X}_K \simeq X$

$\mathscr{X}_{\overline{K}}$: geometric generic fiber of $\mathscr{X}$

$\mathscr{X}_k$: special fiber of $\mathscr{X}$

Fix a prime $\ell \neq \mathrm{char}\, k$. For any curve C over an algebraically closed field of char $\neq \ell$, let

$$\chi(C) := \sum_{i=0}^{2} (-1)^i \dim H^i_{\acute{e}t}(C, \mathbb{Q}_\ell)$$

δ: Swan conductor for the representation $H^1(\mathscr{X}_{\overline{K}}, \mathbb{Q}_\ell)$ (integer, ≥ 0, measure of wild ramification).

X : a hyperelliptic curve over K

$\mathscr{X}$: a proper, flat, regular R-scheme with $\mathscr{X}_K \simeq X$

$\mathscr{X}_{\overline{K}}$: geometric generic fiber of $\mathscr{X}$

$\mathscr{X}_k$: special fiber of $\mathscr{X}$

χ : ℓ-adic Euler Poincaré characteristic

δ : Swan conductor for the representation $H^1(\mathscr{X}_{\overline{K}}, \mathbb{Q}_\ell)$

$$-\mathrm{Art}(\mathscr{X}/R) = \chi(\mathscr{X}_k) - \chi(\mathscr{X}_{\overline{K}}) + \delta \underset{\text{(Saito)}}{=} \text{Deligne discriminant.}$$

Properties of the Artin Conductor

- $\mathrm{Art}(\mathscr{X}/R)$ is independent of ℓ.
- $-\mathrm{Art}(\mathscr{X}/R) \geq 0$.
  $-\mathrm{Art}(\mathscr{X}/R) = 0 \iff \mathscr{X}$ is smooth or $g = 1$ and $(\mathscr{X}_k)_{\mathrm{red}}$ is smooth.
- Let n be the number of components of $\mathscr{X}_k$ and let ϵ be the tame conductor. Then,

$$\begin{aligned} -\mathrm{Art}(\mathscr{X}/R) &= (n-1) + \epsilon + \delta \\ &\geq n - 1. \end{aligned}$$

- When $\mathscr{X}$ is regular and semi-stable,

$$-\mathrm{Art}(\mathscr{X}/R) = \#\text{ singular points of } \mathscr{X}_k.$$

Outline

Earlier results (Ogg, Saito, Liu)

Let $\mathscr{X}$ be the minimal proper regular model of X.

- If $g = 1$, then $-\mathrm{Art}(\mathscr{X}/R) = \Delta_{\min}$ [Ogg-Saito formula]. This also holds when char $k = 2$.
- If $g = 2$, Liu showed that $-\mathrm{Art}(\mathscr{X}/R) \leq \Delta_{\min}$. He showed that equality does not always hold.

Question: Does $-\mathrm{Art}(\mathscr{X}/R) \leq \Delta_{\min}$ hold for hyperelliptic curves of arbitrary genus?

Theorem ( )

Let X be a hyperelliptic curve over K and let $\mathscr{X}$ be the minimal proper regular model of X. Assume that

(a) *the Weierstrass points of X are K-rational, and,*

(b) *that the residue characteristic of K is not 2.*

Then,

$$-\operatorname{Art}(\mathscr{X}/R) \leq \Delta_{\min}.$$

Explicit construction of a regular model $\mathscr{X}'$

The first step in the proof is the explicit construction of a regular model $\mathscr{X}'$ for X (not necessarily minimal).

Lemma

Let $\mathrm{Bl}\ \mathbb{P}^1_R$ be an arithmetic surface birational to $\mathbb{P}^1_R$.
Let f be an element of the function field of $\mathbb{P}^1_R$.
Assume that the odd multiplicity components of the divisor of f on $\mathrm{Bl}\ \mathbb{P}^1_R$ are disjoint.
Then, the normalization of $\mathrm{Bl}\ \mathbb{P}^1_R$ in $K(x, \sqrt{f(x)})$ is a proper regular model for the hyperelliptic curve given by $y^2 = f(x)$.

1. Explicitly construct a regular model $\mathscr{X}'$ (not necessarily minimal).
2. $\mathscr{X}'$ is strict simple normal crossings, $\mathscr{X}'_k = \sum m_i \Gamma_i$ and $m_i \in \{1, 2\}$.

$$-\operatorname{Art}(\mathscr{X}'/R) = \sum_i \left\{ (1 - m_i)\chi(\Gamma_i) + \sum_{j \neq i} (m_j - 1)\Gamma_i.\Gamma_j \right\} + \sum_{i<j} \Gamma_i.\Gamma_j.$$

3. Similarly decompose $\Delta_{\min}$ into local terms.
4. Prove a local comparison inequality.
5. Add local inequalities.

$$-\operatorname{Art}(\mathscr{X}/R) \leq -\operatorname{Art}(\mathscr{X}'/R) \leq \Delta_{\min}.$$

Thank you!