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ON THE TOTAL CURVATURE OF KNOTS

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Introduction

The *total curvature* $\int_C |\mathfrak{x}''(s)| \, ds$ of a closed curve C of class C'', a quantity which measures the total turning of the tangent vector, was studied by W. Fenchel, who proved, in 1929, that, in three dimensional space, $\int_C |\mathfrak{x}''(s)| \, ds \geqq 2\pi$, equality holding only for plane convex curves. K. Borsuk, in 1947, extended this result to n dimensional space, and, in the same paper, conjectured that the total curvature of a knot in three dimensional space must exceed 4π. A proof of this conjecture is presented below.[1]

In proving this proposition, use will be made of a definition, suggested by R. H. Fox, of total curvature which is applicable to *any* closed curve. This general definition is validated by showing that the generalized total curvature $\kappa(C)$ is equal to $\int_C |\mathfrak{x}''(s)| \, ds$ for any closed curve C of class C''. Furthermore, the theorem of Fenchel and Borsuk is true for any closed curve, if the new definition of total curvature is used.

Closely related to the concept of total curvature is a new invarient $\mu(\mathfrak{C})$, the *crookedness* of the isotopy type $\mathfrak{C}$ of closed curves. This is either a positive integer or ∞, according as the type $\mathfrak{C}$ is or is not represented by a polygon. In terms of the concept of crookedness it is possible to provide an alternative formulation of the generalized total curvature as a Lebesgue integral over an $(n - 1)$ dimensional sphere. The crookedness $\mu(\mathfrak{C})$ of a type $\mathfrak{C}$ of simple closed curves is connected with the total curvatures of its representative curves C by the fundamental relation $2\pi\mu(\mathfrak{C}) = \text{g.l.b. } \kappa(C)$. Generally speaking this lower bound is not attained.

In the course of the paper several interesting incidental results are obtained: if the total curvature of a simple closed curve is finite, then there is an inscribed polygon equivalent to it by isotopy, and also if the curve is knotted there must be a plane which intersects it in at least six points.

I am indebted to R. H. Fox for substantial assistance in the preparation of this paper.

1. The Total Curvature of a Closed Polygon

By a *closed polygon* P in Euclidean n-space H^n, $n \geqq 1$, will be meant a finite sequence of points $\mathfrak{a}_0, \mathfrak{a}_1, \cdots, \mathfrak{a}_{m-1}, \mathfrak{a}_m = \mathfrak{a}_0$, of which is required only that

[1] Since the completion of this paper, there has appeared an independant proof, by I. Fáry, that the relation $\int_C |\mathfrak{x}''(s)| \, ds \geqq 4\pi$ holds for all knots [6].

$\mathfrak{a}_i \neq \mathfrak{a}_{i+1}$, and the line segments $\mathfrak{a}_i\mathfrak{a}_{i+1}$ for $i = 0, 1, \cdots, m - 1$. For convenience let $\mathfrak{a}_i$, where i is any integer, signify $\mathfrak{a}_{(i)}$, where (i) is the least positive residue mod m. The terms point and vector will be used synonymously, with every vector referred to a common origin, so that $\mathfrak{a}_{i+1} - \mathfrak{a}_i$ means the vector equal in length and parallel to the line segment $\mathfrak{a}_i\mathfrak{a}_{i+1}$. Denote by α_i the angle between the vectors $\mathfrak{a}_{i+1} - \mathfrak{a}_i$ and $\mathfrak{a}_i - \mathfrak{a}_{i-1}$ satisfying $0 \leqq \alpha_i \leqq \pi$. By *the total curvature* $\kappa(P)$ *of a closed polygon* P is meant the angle sum $\sum_{i=1}^{m} \alpha_i$.

1.1 LEMMA.[2] *The adjunction of a new vertex to a closed polygon cannot decrease its total curvature. The curvature may remain constant if either the new vertex* $\mathfrak{a}_j$ *and the two adjacent vertices* $\mathfrak{a}_{j-1}$ *and* $\mathfrak{a}_{j+1}$ *are collinear or* $\mathfrak{a}_{j-2}$, $\mathfrak{a}_{j-1}$, $\mathfrak{a}_j$, $\mathfrak{a}_{j+1}$, *and* $\mathfrak{a}_{j+2}$ *are coplanar. Otherwise it must definitely increase.*

Let P' be the closed polygon with vertices $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{j-1}, \mathfrak{a}_{j+1}, \cdots, \mathfrak{a}_m$. Let P be the closed polygon with vertices $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_{j-1}, \mathfrak{a}_j, \mathfrak{a}_{j+1}, \cdots, \mathfrak{a}_m$, obtained from P' by adjoining the vertex $\mathfrak{a}_j$. Denote by α_i' for $i = 1, 2, \cdots,$

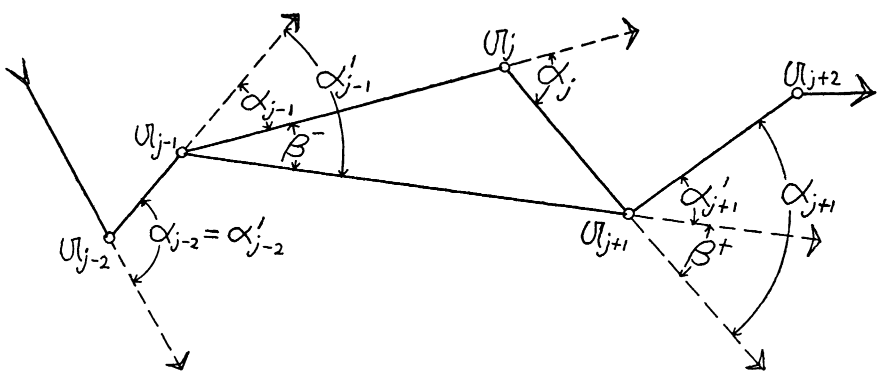

FIG. 1

$j - 1, j + 1, \cdots, m$ the respective exterior angles of P', and by α_i for $i = 1, 2, \cdots, j - 1, j, j + 1, \cdots, m$ the respective exterior angles of P. Denote by β^- the angle between $\mathfrak{a}_j - \mathfrak{a}_{j-1}$ and $\mathfrak{a}_{j+1} - \mathfrak{a}_{j-1}$, and by β^+ the angle between $\mathfrak{a}_{j+1} - \mathfrak{a}_{j-1}$ and $\mathfrak{a}_{j+1} - \mathfrak{a}_j$. (See Fig. 1.)

By the triangle inequality for spherical triangles $\alpha_{j-1} + \beta^- \geqq \alpha_{j-1}'$, where the equality can hold only if the three angles lie in a plane, that is, only if $\mathfrak{a}_{j-2}$, $\mathfrak{a}_{j-1}$, $\mathfrak{a}_j$, and $\mathfrak{a}_{j+1}$ are coplanar. Similarly $\alpha_{j+1} + \beta^+ \geqq \alpha_{j+1}'$, where the equality can hold only if $\mathfrak{a}_{j-1}$, $\mathfrak{a}_j$, $\mathfrak{a}_{j+1}$, and $\mathfrak{a}_{j+2}$ are coplanar. From the triangle with vertices $\mathfrak{a}_{j-1}$, $\mathfrak{a}_j$, and $\mathfrak{a}_{j+1}$ we have $\beta^- + \beta^+ = \alpha_j$. Therefore

$$\begin{aligned}\kappa(P) - \kappa(P') &= (\alpha_{j-1} - \alpha_{j-1}') + \alpha_j + (\alpha_{j+1} - \alpha_{j+1}') \\ &\geqq -\beta^- \qquad\quad + \alpha_j \qquad\quad -\beta^+ = 0.\end{aligned}$$

Hence $\kappa(P) \geqq \kappa(P')$ and the equality can hold only if either $\mathfrak{a}_{j-2}$, $\mathfrak{a}_{j-1}$, $\mathfrak{a}_j$, $\mathfrak{a}_{j+1}$, and $\mathfrak{a}_{j+2}$ are coplanar or $\mathfrak{a}_{j-1}$, $\mathfrak{a}_j$, and $\mathfrak{a}_{j+1}$ are collinear.

[2] This proof is essentially the same as a proof given by Borsuk [1. pp. 254–256].

1.2 Corollary. *If the vertices* $\mathfrak{a}_{j-2}$, $\mathfrak{a}_{j-1}$, $\mathfrak{a}_j$, *and* $\mathfrak{a}_{j+1}$ *of a closed polygon* P *are not coplanar, and the vertex* $\mathfrak{a}_j$ *is replaced by a vertex* $\mathfrak{a}'_j$ *which lies on the line segment* $\mathfrak{a}_j\mathfrak{a}_{j+1}$, *then* $\kappa(P)$ *is decreased.*

2. The Total Curvature of a Curve

By a *closed curve* C in Euclidean n-space H^n will be meant a continuous vector function $\mathfrak{x}(t) = (\mathfrak{x}_1(t), \cdots, \mathfrak{x}_n(t))$ of period l which is not constant in any t-interval. In particular any polygon can be described in this manner; it will be convenient to regard a polygon as a closed curve, ignoring the distinction between different parameterizations. A closed curve $\mathfrak{x}(t)$ is *simple* if $\mathfrak{x}(t_1) = \mathfrak{x}(t_2)$ only when $(t_1 - t_2)/l$ is an integer.

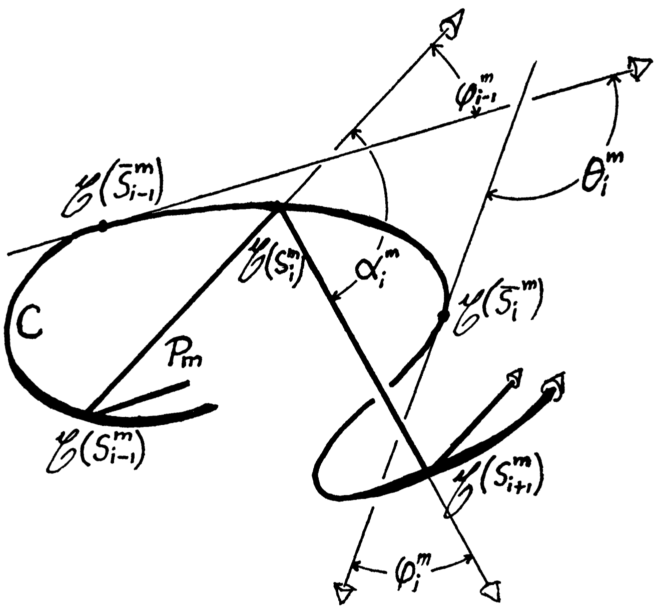

Fig. 2

A closed polygon P with vertices $\mathfrak{a}_1, \cdots, \mathfrak{a}_m$ is said to be *inscribed* in a closed curve $\mathfrak{x}(t)$ if there is a set of parameter values t_i such that $t_i < t_{i+1}$, $t_{i+m} = t_i + l$, and $\mathfrak{a}_i = \mathfrak{x}(t_i)$ for all integral values of i.

2.1 Lemma. *For any closed polygon* P, $\kappa(P) = \text{l.u.b. } \{\kappa(P')\}$ *where* P' *ranges over all polygons inscribed in* P.

If P is a polygon having two or more vertices coincident, it can be represented as the limit of a sequence of polygons with all vertices distinct; hence it only remains to prove the lemma for P with all vertices distinct.

If P'_0 is a representative inscribed polygon whose vertices include all but m of the vertices of P, we may adjoin the remaining vertices of P one by one to P'_0, producing a sequence of polygons $P'_0, P'_1, \cdots, P'_m$. By 1.1, $\kappa(P'_0) \leqq \kappa(P'_1) \leqq, \cdots, \leqq \kappa(P'_m)$; but $\kappa(P'_m) = \kappa(P)$. Therefore $\kappa(P) = \text{l.u.b. } \{\kappa(P')\}$.

For each closed curve C define *the total curvature* by $\kappa(C) = \text{l.u.b.}\ \{\kappa(P)\}$ where P ranges over all polygons inscribed in C. If C is itself a polygon, the preceding lemma shows that this definition is consistent with the definition of Section 1.

2.2 THEOREM. *If C is a closed curve of class C'' parameterized by arclength s, then* $\kappa(C) = \int_C \mid \mathfrak{x}''(s) \mid ds$.

If $\mathfrak{a}_1^m = \mathfrak{x}(s_1^m), \cdots, \mathfrak{a}_m^m = \mathfrak{x}(s_m^m)$ are the vertices of a polygon P_m inscribed in C, such that $\lim_{m\to\infty} \max_i \{(s_{i+1}^m - s_i^m)\} = 0$, it will first be shown that $\lim_{m\to\infty}\kappa(P_m) = \int_C \mid \mathfrak{x}''(s) \mid ds$ (Compare Fig. 2).

Define $\bar{s}_i^m = \frac{1}{2}(s_i^m + s_{i+1}^m)$ for every i and m. Denote by θ_i^m the angle between $\mathfrak{x}'(\bar{s}_{i-1}^m)$ and $\mathfrak{x}'(\bar{s}_i^m)$. The vector $\mathfrak{x}'(s)$ describes a curve L of length $\int_C \mid \mathfrak{x}''(s) \mid ds$ on the unit sphere S^{n-1}. The vectors $\mathfrak{x}'(\bar{s}_i^m)$ form the vertices of a spherical polygon of length $\sum_{i=1}^m \theta_i^m$ which is inscribed in L. Therefore $\lim_{m\to\infty} \sum_{i=1}^m \theta_i^m = \int_C \mid \mathfrak{x}''(s) \mid ds$.

Since $\mathfrak{x}''(s)$ is uniformly continuous, for each $\epsilon > 0$ there is a $\delta > 0$ such that $\mid \mathfrak{x}''(u) - \mathfrak{x}''(v) \mid < \epsilon$ for all $\mid u - v \mid < \delta$. From the identity

$$\mathfrak{x}(s_{i+1}^m) - \mathfrak{x}(s_i^m) = (s_{i+1}^m - s_i^m)\mathfrak{x}'(\bar{s}_i^m) + \int_{\bar{s}_i^m}^{s_{i+1}^m} \int_{\bar{s}_i^m}^{v} [\mathfrak{x}''(u) - \mathfrak{x}''(\bar{s}_i^m)]\, du\, dv + \int_{s_i^m}^{\bar{s}_i^m} \int_{v}^{\bar{s}_i^m} [\mathfrak{x}''(\bar{s}_i^m) - \mathfrak{x}''(u)]\, du\, dv$$

we have

$$\left| \frac{\mathfrak{x}(s_{i+1}^m) - \mathfrak{x}(s_i^m)}{s_{i+1}^m - s_i^m} - \mathfrak{x}'(\bar{s}_i^m) \right| < (s_{i+1}^m - s_i^m)\,\frac{\epsilon}{4} \text{ whenever } \max_i \{(s_{i+1}^m - s_i^m)\} < \delta.$$

If φ_i^m is the angle between $\mathfrak{x}(s_{i+1}^m) - \mathfrak{x}(s_i^m)$ and $\mathfrak{x}'(\bar{s}_i^m)$, then $\sin \varphi_i^m < (s_{i+1}^m - s_i^m)\,\epsilon/4$, since the end point of the vector $(\mathfrak{x}(s_{i+1}^m) - \mathfrak{x}(s_i^m))/(s_{i+1}^m - s_i^m)$ lies within a sphere of radius $(s_{i+1}^m - s_i^m)\,\epsilon/4$ about the end point of the unit vector $\mathfrak{x}'(\bar{s}_i^m)$. Hence for sufficiently small ϵ we have $\varphi_i^m < 2 \sin \varphi_i^m < (s_{i+1}^m - s_i^m)\,\epsilon/2$. The angle between $\mathfrak{x}(s_{i+1}^m) - \mathfrak{x}(s_i^m)$ and $\mathfrak{x}(s_i^m) - \mathfrak{x}(s_{i-1}^m)$ is α_i^m, while that between $\mathfrak{x}'(\bar{s}_i^m)$ and $\mathfrak{x}'(\bar{s}_{i-1}^m)$ is θ_i^m. Therefore $\alpha_i^m \leqq \theta_i^m + \varphi_i^m + \varphi_{i-1}^m$ and $\theta_i^m \leqq \alpha_i^m + \varphi_i^m + \varphi_{i-1}^m$, so that $\mid \alpha_i^m - \theta_i^m \mid < (s_{i+1}^m - s_{i-1}^m)\,\epsilon/2$, and $\mid \sum_{i=1}^m \alpha_i^m - \sum_{i=1}^m \theta_i^m \mid < l\epsilon$, where l is the length of C. Therefore

$$\lim_{m\to\infty} \kappa(P_m) = \lim_{m\to\infty} \sum_{i=1}^m \alpha_i^m = \lim_{m\to\infty} \sum_{i=1}^m \theta_i^m = \int_C \mid \mathfrak{x}''(s) \mid ds.$$

In order to show that $\int_C \mid \mathfrak{x}''(s) \mid ds = \text{l.u.b.}\ \{\kappa(P)\}$ for P inscribed in C, it only remains to show that $\kappa(P) \leqq \int_C \mid \mathfrak{x}''(s) \mid ds$.

Given any polygon P_k inscribed in C, we may form a sequence of polygons P_m for $m = k, k + 1, \cdots$ by adjoining vertices to P_k so that

$$\lim_{m\to\infty} \max_i \{(s_{i+1} - s_i)\} = 0.$$

By 1.1 $\kappa(P_k) \leqq \kappa(P_{k+1}) \leqq \cdots$, but $\lim_{m\to\infty} \kappa(P_m) = \int_C |\mathfrak{x}''(s)| \, ds$, and therefore $\kappa(P_k) \leqq \int_C |\mathfrak{x}''(s)| \, ds$.

3. The Crookedness of a Closed Curve

For each closed curve C and each unit vector $\mathfrak{b}$, define $\mu(C, \mathfrak{b})$ to be the number of maxima of the function $\mathfrak{b} \cdot \mathfrak{x}(t)$ (i.e. the number of parameter values t_0 for which $\mathfrak{b} \cdot \mathfrak{x}(t_0) \geqq \mathfrak{b} \cdot \mathfrak{x}(t)$ for t within some neighborhood of t_0) in a fundamental period. For each closed curve C define $\mu(C) = \min_{\mathfrak{b}} \{\mu(C, \mathfrak{b})\}$. We may call $\mu(C)$ the *crookedness* of C.

For every vector $\mathfrak{a}_{i+1} - \mathfrak{a}_i$ in the space H^n define $\mathfrak{b}_i = (\mathfrak{a}_{i+1} - \mathfrak{a}_i)/ | \mathfrak{a}_{i+1} - \mathfrak{a}_i |$. According to the convention introduced earlier, $\mathfrak{b}_i$ also denotes a point on the unit sphere S^{n-1}, the *spherical image* of $\mathfrak{a}_{i+1} - \mathfrak{a}_i$. Given a polygon P with vertices $\mathfrak{a}_1, \mathfrak{a}_2, \cdots, \mathfrak{a}_m$, a spherical polygon Q is formed on S^{n-1} by joining each $\mathfrak{b}_{i-1}$ to $\mathfrak{b}_i$ by a great circle arc of length α_i. This spherical polygon Q is called a *spherical image* of P, and is unique unless for some j the vector $\mathfrak{b}_j = -\mathfrak{b}_{j+1}$. Note that it may happen that $\mathfrak{b}_j = \mathfrak{b}_{j+1}$.

3.1 Theorem.[3] *For any closed curve C in H^n, $n \geqq 2$, the Lebesgue integral $\int_{S^{n-1}} \mu(C, \mathfrak{b}) dS$, where $\mathfrak{b}$ ranges over the unit sphere, exists and is equal to $(M_{n-1}\kappa(C))/2\pi$, where $M_{n-1} = (2\pi^{n/2})/\Gamma(n/2)$ is the measure of S^{n-1}.*

We will first consider the case in which the curve is a polygon P. For every point $\mathfrak{b}$ of S^{n-1}, let $S_{\mathfrak{b}}^{n-2}$ denote the great sphere of S^{n-1} which has a pole at $\mathfrak{b}$. An edge $\mathfrak{b}_{j-1}\mathfrak{b}_j$ of Q crosses $S_{\mathfrak{b}}^{n-2}$ if and only if $\mathfrak{b} \cdot (\mathfrak{a}_{j+1} - \mathfrak{a}_j)$ and $\mathfrak{b} \cdot (\mathfrak{a}_j - \mathfrak{a}_{j-1})$ have opposite sign, so that $\mathfrak{b} \cdot \mathfrak{a}_j$ is a maximum or minimum of $\mathfrak{b} \cdot \mathfrak{x}(t)$. Therefore, if $S_{\mathfrak{b}}^{n-2}$ contains no vertex of Q, (i.e. no edge of P is perpendiuclar to $\mathfrak{b}$) the number of intersections of Q with $S_{\mathfrak{b}}^{n-2}$ is $2\mu(P, \mathfrak{b})$. The set of points $\mathfrak{b}$ for which $S_{\mathfrak{b}}^{n-2}$ contains some vertex of Q is the union of the finite collection of great spheres $S_{\mathfrak{b}_i}^{n-2}$; in each component of the complement with respect to S^{n-1} of $\bigcup_i S_{\mathfrak{b}_i}^{n-2}$ the function $2\mu(P, \mathfrak{b})$ is constant. The integral $\int_{S^{n-1}} 2\mu(P, \mathfrak{b}) dS$, where dS is the element of surface on S^{n-1}, is therefore defined. The set of points $\mathfrak{b}$ for which $S_{\mathfrak{b}}^{n-2}$ meets a given segment $\mathfrak{b}_{i-1}\mathfrak{b}_i$ of length $0 \leqq \alpha_i \leqq \pi$ is a "double lune" bounded by the great spheres $S_{\mathfrak{b}_{i-1}}^{n-2}$ and $S_{\mathfrak{b}_i}^{n-2}$. Thus the contribution of $\mathfrak{b}_{i-1}\mathfrak{b}_i$ to $2\mu(P, \mathfrak{b})$ is 1 if $\mathfrak{b}$ is an interior point of this lune and 0 if $\mathfrak{b}$ is an exterior point. The measure of this lune is $(\alpha_i M_{n-1})/\pi$ where M_{n-1} is the measure of the entire sphere. Consequently

$$\int_{S^{n-1}} 2\mu(P, \mathfrak{b}) \, dS = \frac{M_{n-1}}{\pi} \sum_{i=1}^{m} \alpha_i = \frac{M_{n-1}}{\pi} \kappa(P).$$

[3] This theorem is related to Crofton's formula. Cf. [3. p. 81].

If C is an arbitrary closed curve $\mathfrak{x}(t)$, let P_m be a set of inscribed polygons $\mathfrak{x}_m(t)$ with vertices $\mathfrak{a}_1^m = \mathfrak{x}(t_1^m), \cdots, \mathfrak{a}_m^m = \mathfrak{x}(t_m^m)$ such that each P_m contains all the vertices of P_{m-1} and satisfying $\lim_{m\to\infty} \kappa(P_m) = \kappa(C)$ and $\lim_{m\to\infty} \max_i \{(t_{i+1}^m - t_i^m)\} = 0$. The values of $\mathfrak{b}$ for which $\mathfrak{b}\cdot\mathfrak{x}(t)$ or any $\mathfrak{b}\cdot\mathfrak{x}_m(t)$ has an interval of constancy form a set of measure zero, and therefore have no effect on the integral. Such values will be ignored for the remainder of the proof.

We first show that $\mu(C, \mathfrak{b}) = \lim_{m\to\infty} \mu(P_m, \mathfrak{b})$. It is certainly true that $\mu(C, \mathfrak{b}) \geqq \mu(P_m, \mathfrak{b}) \geqq \mu(P_{m-1}, \mathfrak{b})$. If $\mu(C, \mathfrak{b}) < \infty$, it is possible to select a neighborhood of each of the $\mu(C, \mathfrak{b})$ maxima of $\mathfrak{b}\cdot\mathfrak{x}(t)$ and of each of its minima sufficiently small so that a polygon with a vertex in each of these neighborhoods must have at least $\mu(C, \mathfrak{b})$ maxima; which is certainly true of P_m for m sufficiently large. If $\mu(C, \mathfrak{b}) = \infty$, the set of values of t for which $\mathfrak{b}\cdot\mathfrak{x}(t)$ is a maximum must contain a denumerable subset $\{t_{2i}\}$ such that either $t_0 < t_2 < \cdots < \lim_{i\to\infty} t_{2i} < t_0 + l$ or $t_0 > t_2 > \cdots > \lim_{i\to\infty} t_{2i} > t_0 - l$. In either case we may select a series of intermediate values t_{2i+1} such that each $\mathfrak{b}\cdot\mathfrak{x}(t_{2i}) > \mathfrak{b}\cdot\mathfrak{x}(t_{2i+1})$ and $\mathfrak{b}\cdot\mathfrak{x}(t_{2i}) > \mathfrak{b}\cdot\mathfrak{x}(t_{2i-1})$. Given any $2j < \infty$ we may select neighborhoods of the $\mathfrak{x}(t_i)$, for $i < 2j$, so small that any polygon with at least one vertex in each neighborhood has at least $j - 1$ maxima; which is true of P_m for m sufficiently large. Therefore $\mu(P_m, \mathfrak{b})$ increases without finite bound as $m \to \infty$. Each of the integrals $\int_{S^{n-1}} \mu(P_m, \mathfrak{b})\, dS$ exists; and the nondecreasing sequence of positive functions $\mu(P_m, \mathfrak{b})$ approaches $\mu(C, \mathfrak{b})$. Therefore[4] the integral $\int_{S^{n-1}} \mu(C, \mathfrak{b})\, dS$ exists and equals $\lim_{m\to\infty} \int_{S^{n-1}} \mu(P_m, \mathfrak{b})\, dS = \lim_{m\to\infty} \frac{(M_{n-1})}{2\pi} \kappa(P_m) = \frac{(M_{n-1})}{2\pi} \kappa(C)$.

3.2 Corollary. $\kappa(C) \geqq 2\pi\mu(C)$.

Since
$$\frac{M_{n-1}}{2\pi} \kappa(C) = \int_{S^{n-1}} \mu(C, \mathfrak{b})\, dS \geqq \int_{S^{n-1}} \mu(C)\, ds = M_{n-1}\mu(C).$$

By a *convex curve* will be meant a closed plane curve described by $\mathfrak{x}(t)$ such that any line contains $\mathfrak{x}(t)$ either for not more than two values of t within a fundamental period or for all values of t within some interval.

3.3 Lemma. *The necessary and sufficient condition that a closed polygon P in H^2 be convex is that for every $\mathfrak{b}$ either $\mu(P, \mathfrak{b}) = 1$ or $\mu(P, \mathfrak{b}) = \infty$.*

It is clear that this condition is necessary. Suppose that P is a closed plane polygon such that either $\mu(P, \mathfrak{b}) = 1$ or $\mu(P, \mathfrak{b}) = \infty$ for each $\mathfrak{b}$ in the plane. For any $\mathfrak{b}$ such that $\mu(P, \mathfrak{b}) = 1$, any line perpendicular to $\mathfrak{b}$ will intersect P in at most two points. If $\mathfrak{b}$ is a vector for which $\mu(P, \mathfrak{b}) = \infty$, and if H^1 is a line perpendicular to $\mathfrak{b}$ which intersects P a finite number, say r, of times, then it is always possible to rotate H^1 about one of its points of intersection with P in the proper direction so that r is not decreased. Hence there is a $\bar{\mathfrak{b}}$ and a $\bar{H}^1$ perpendicular to $\bar{\mathfrak{b}}$ such that $\mu(P, \bar{\mathfrak{b}}) < \infty$ and the number of intersections of $\bar{H}^1$ with P is $\bar{r} \geqq r$. But by the first case, $\bar{r} \leqq 2$, and therefore $r \leqq 2$.

[4] [4. Theorem 12.6 p. 28].

3.4 THEOREM. *For any closed curve C, $\kappa(C) \geqq 2\pi$. The equality holds if and only if C is convex.*

Since $\mu(C) \geqq 1$ for every curve, or since every curve has an inscribed polygon P for which $\kappa(P) = 2\pi$, we have $\kappa(C) \geqq 2\pi$. Since any curve which is not convex has an inscribed polygon which is not convex, and since a polygon inscribed in a convex curve must be convex, it only remains to prove the second portion of the theorem for polygons.

It is proved in plane geometry that the sum of the exterior angles of a convex polygon is 2π. If there were a non-planar polygon P for which $\kappa(P) = 2\pi$, we could select four consecutive non-coplanar vertices (neglecting vertices for which $\alpha_i = 0$). By 1.2 there would be a new polygon P' such that $\kappa(P') < \kappa(P) = 2\pi$, which is impossible. If there were a non-convex plane polygon P for which $\kappa(P) = 2\pi$, then 3.3 states that there would be a direction $\bar{\mathfrak{b}}$ for which $1 < \mu(P, \bar{\mathfrak{b}}) < \infty$, but there is a neighborhood of any such $\bar{\mathfrak{b}}$ within which $\mu(P, \mathfrak{b})$ is constant. This means that $\kappa(P) = \int_{S^1} \mu(P, \mathfrak{b})\, dS > \int_{S^1} dS = 2\pi$.

4. The Curvature and Crookedness of Isotopy Types of Curves

In H^n the closed curve described by $\mathfrak{x}(t)$ of period l, and the closed curve described by $\bar{\mathfrak{x}}(t)$ of period $\bar{l}$ are said to be *equivalent by isotopy* if there is some isotopy of H^n onto itself, which transforms $\mathfrak{x}(ul)$ into $\bar{\mathfrak{x}}(u\bar{l})$ for all u.

By a *curve type* $\mathfrak{C}$ in H^n is meant an equivalence class of closed curves under isotopy. A curve type is *simple* if the representative closed curves are simple.

A simple curve type $\mathfrak{C}$ and its members are said to be *unknotted* if $\mathfrak{C}$ is that type which contains all circles. If a simple curve type contains no circles then the type and its members are said to be *knotted.*

A curve type $\mathfrak{C}$ and its members are said to be *tame*[5] if $\mathfrak{C}$ contains a polygon. Otherwise they are said to be *wild.*[5] It is well known that in H^2 every simple closed curve is unknotted. In H^n for $n > 3$, every simple tame curve is unknotted.

For each curve type $\mathfrak{C}$, define $\kappa(\mathfrak{C}) = \text{g.l.b. } \kappa(C)$ and $\mu(\mathfrak{C}) = \min \mu(C)$, where C ranges over all members of $\mathfrak{C}$.

4.1 LEMMA. *For each $\mathfrak{c}$ and $\mathfrak{p}$ in H^{n-1} such that $|\mathfrak{c} - \mathfrak{p}| < r$, there is an isotopy, $f_u^{\mathfrak{cp}}(\mathfrak{x})$, $0 \leqq u \leqq 1$, of H^{n-1} onto itself which transforms $\mathfrak{c}$ into $\mathfrak{p}$ and leaves fixed all points of H^{n-1} outside the $(n-2)$-sphere of radius r and center $\mathfrak{c}$, such that $f_u^{\mathfrak{cp}}(\mathfrak{x})$ is a continuous function of u, $\mathfrak{x}$, $\mathfrak{c}$, and $\mathfrak{p}$.*

For example:

$$f_u^{\mathfrak{cp}}(\mathfrak{x}) = \begin{cases} \mathfrak{x} - u\left[1 - \dfrac{|\mathfrak{x} - \mathfrak{c}|}{r}\right](\mathfrak{p} - \mathfrak{c}) & \text{for } |\mathfrak{x} - \mathfrak{c}| \leqq r. \\ \mathfrak{x} & \text{for } |\mathfrak{x} - \mathfrak{c}| \geqq r. \end{cases}$$

4.2 THEOREM. *For any simple closed curve C, such that $\mu(C) < \infty$, there is a polygon P inscribed in C and equivalent to C by isotopy.*

[5] This definition was given by Fox and Artin [5].

If $\mathfrak{b}$ is a unit vector for which $\mu(C, \mathfrak{b}) < \infty$, there are a finite set of values $t_1 < t_2 < \cdots < t_{2\mu(C,\mathfrak{b})} < t_1 + l$, for which $\mathfrak{b} \cdot \mathfrak{x}(t)$ has a maximum or minimum. About each point $\mathfrak{x}(t_i)$ construct a cylinder $Z_i^{n-1}(0)$ with generators parallel to $\mathfrak{b}$ which intersects C in exactly two points $\mathfrak{x}(t_i^-)$ and $\mathfrak{x}(t_i^+)$ such that both lie on a base of the cylinder and such that $\mathfrak{x}(t_i^+)$ is the center of this base. It will first be shown that there is an isotopy of the closed n-cell bounded by $Z_i^{n-1}(0)$ onto itself which leaves $Z_i^{n-1}(0)$ fixed and transforms the curve segment $\mathfrak{x}(t)$ for $t_i^- \leqq t \leqq t_i^+$ onto the polygonal line $\mathfrak{x}(t_i^-)$, $\mathfrak{x}(t_i)$, $\mathfrak{x}(t_i^+)$.

Each hyperplane H^{n-1} perpendicular to $\mathfrak{b}$ which intersects $Z_i^{n-1}(0)$ intersects it in a sphere S^{n-2}. Perform the isotopy of each H^{n-1} onto itself which transforms the curve segment $\mathfrak{x}(t)$ for $t_i \leqq t \leqq t_i^+$ into the axis of the cylinder, and which leaves all points outside of the n-cell bounded by the cylinder fixed, as defined by 4.1. Select a continuous sequence of coaxial cylinders $Z_i^{n-1}(v)$, $0 \leqq v < \infty$, such that any $Z_i^{n-1}(\bar{v})$ is contained within all $Z_i^{n-1}(v)$ with $v < \bar{v}$, such that each cylinder intersects C only in the center of one base and in one other point of that base, and such that $Z_i^{n-1}(v)$ tends to the point $x(t_i)$ as $v \to \infty$. Rotate each $Z_i^{n-1}(v)$ about its axis so that each point $\mathfrak{x}(t)$ for $t_i^- \leqq t < t_i$ is transformed into the plane determined by $\mathfrak{x}(t_i^-)$ and the axis of the cylinders. Since we have transformed $\mathfrak{x}(t)$ for $t_i^- \leqq t \leqq t_i^+$ onto a plane curve within $Z_i^{n-1}(0)$, it is certainly possible to transform it onto the polygonal line $\mathfrak{x}(t_i^-)$, $\mathfrak{x}(t_i)$, $\mathfrak{x}(t_i^+)$, still within $Z_i^{n-1}(0)$, producing an equivalent curve $\bar{C}$ described by $\bar{\mathfrak{x}}(t)$. This curve is divided into $4\mu(C, b) = 4\mu(\bar{C}, b)$ distinct segments by the points $\mathfrak{x}(t_i^+)$ and $\mathfrak{x}(t_i^-)$. If $g > 0$ is the g.l.b. of the distances between distinct and nonconsecutive curve segments, then for each point of $\bar{C}$ which is not within any of the Z_i^{n-1} we may construct the sphere S^{n-2} which has its center at the point, lies in a hyperplane perpendicular to b, and has radius $g/3$. Since no two of these spheres can intersect, we have in effect, constructed a tube around each curve segment outside the cylinders, with no two tubes intersecting. It is now possible to inscribe a polygonal line, lying completely within the tube, in each segment of $\bar{C}$. Perform the isotopy of each H^{n-1} perpendicular to $\mathfrak{b}$ onto itself which transforms each of these curve segments onto the corresponding polygonal line and which leaves fixed all points outside of the S^{n-2}. We have thus transformed $\bar{C}$, and therefore C, into an inscribed polygon P, equivalent by isotopy. (Note for future reference that $\mu(C, b) = \mu(P, b)$.)

4.3 Corollary. *The necessary and sufficient condition that a simple curve type $\mathfrak{C}$ be tame is that $\mu(\mathfrak{C}) < \infty$.*

4.4 Corollary. *The total curvature of a tame knot cannot equal the curvature of its type.*

Assume that C is a tame knot of type $\mathfrak{C}$ with $\kappa(C) = \kappa(\mathfrak{C})$. Let P be a polygon of type $\mathfrak{C}$ inscribed in C. Then $\kappa(P) \leqq \kappa(C)$. Since P cannot lie in any plane, we may select four consecutive non-coplanar vertices (if we ignore vertices for which $\alpha_i = 0$). By 1.2 we may select a new polygon $\bar{P}$, still a member of $\mathfrak{C}$, and having $\kappa(\bar{P}) < \kappa(P) \leqq \kappa(C) = \kappa(\mathfrak{C})$; which is impossible.

4.5 Corollary. *The crookedness of any knot is greater than or equal to* 2.

If C is a curve with $\mu(C) = 1$, then, in the proof of 4.2 we can select the two cylinders with a common base. The first isotopy will then transform C into a plane quadrilateral, which is certainly unknotted.

4.6 COROLLARY. *The total curvature of any knot is greater than* 4π.

4.7. THEOREM. *If* $\mathfrak{C}$ *is a simple curve type, then* $\kappa(\mathfrak{C}) = 2\pi\mu(\mathfrak{C})$.

It has already been shown that $\kappa(C) \geqq 2\pi\mu(C)$ for any $C \epsilon \mathfrak{C}$ and therefore that $\kappa(\mathfrak{C}) \geqq 2\pi\mu(\mathfrak{C})$. If $\mu(\mathfrak{C}) = \infty$, this proves the proposition. If $\mu(\mathfrak{C}) < \infty$, we may select a curve C of $\mathfrak{C}$ and a direction $\mathfrak{b}$ such that $\mu(C, \mathfrak{b}) = \mu(\mathfrak{C})$. By 4.2 there is a polygon P which is a member of $\mathfrak{C}$ such that $\mu(P, \mathfrak{b}) = \mu(C, \mathfrak{b})$. For convenience we will select a new coordinate system so that $\mathfrak{b}$ is parallel to the x_1 axis. We may then define the isotopy $F_u(x_1, x_2, x_3, \cdots, x_n) = (x_1, ux_2, ux_3, \cdots, ux_n)$ for $0 < u \leqq 1$. This evidently transforms P into an

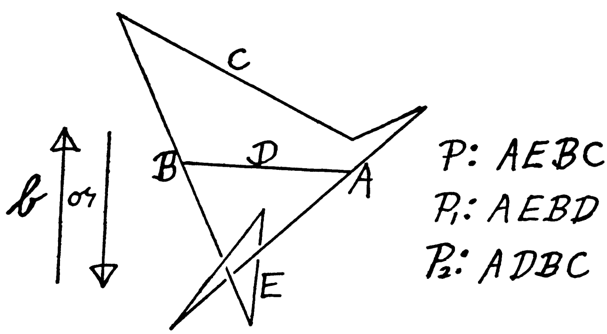

FIG. 3

equivalent polygon P_u, such that $\mu(P_u, \mathfrak{b}) = \mu(P, \mathfrak{b})$. If $\mathfrak{a}_i^u$, $1 \leqq i \leqq m$, is the set of vertices of P_u, we may divide it into four subsets:

(a) vertices $\mathfrak{a}_i^u$ such that $\mathfrak{b}\cdot\mathfrak{a}_{i-1}^u < \mathfrak{b}\cdot\mathfrak{a}_i^u < \mathfrak{b}\cdot\mathfrak{a}_{i+1}^u$,
(b) vertices $\mathfrak{a}_i^u$ such that $\mathfrak{b}\cdot\mathfrak{a}_{i-1}^u > \mathfrak{b}\cdot\mathfrak{a}_i^u > \mathfrak{b}\cdot\mathfrak{a}_{i+1}^u$,
(c) vertices $\mathfrak{a}_i^u$ such that $\mathfrak{b}\cdot\mathfrak{a}_{i-1}^u < \mathfrak{b}\cdot\mathfrak{a}_i^u > \mathfrak{b}\cdot\mathfrak{a}_{i+1}^u$,
(d) vertices $\mathfrak{a}_i^u$ such that $\mathfrak{b}\cdot\mathfrak{a}_{i-1}^u > \mathfrak{b}\cdot\mathfrak{a}_i^u < \mathfrak{b}\cdot\mathfrak{a}_{i+1}^u$.

(If an equality were to hold, we would have $\mu(P_u, \mathfrak{b}) = \infty$.) Evidently the number of vertices in (c) equals the number in (d) equals $\mu(\mathfrak{C})$. However for members of (c) and (d), $\lim_{u\to 0}(\alpha_i^u) = \pi$; whereas for (a) and (b), $\lim_{u\to 0}(\alpha_i^u) = 0$. Therefore $\lim_{u\to 0}\kappa(P_u) = 2\pi\mu(\mathfrak{C})$.

As another interesting consequence of Theorem 4.2, we have the following.

4.8 THEOREM. *Given a knot* C *in* H^3 *for which* $\mu(C) < \infty$, *there is a plane whose intersection with* C *consists of at least six components.*

Since every such C has an inscribed polygon which is knotted, and since a plane intersects a curve at least as many times as it intersects an inscribed polygon, it only remains to prove the theorem for knotted polygons. If there is a polygon which does not satisfy the theorem, there must be one having a minimum number of sides. If P is such a polygon, we have: $4\pi < \kappa(P) = \frac{1}{2}\int_{S^2}\mu(P, \mathfrak{b})\,dS$. Therefore there must be some unit vector $\mathfrak{b}$ such that $2 < \mu(P, \mathfrak{b})$ and also

$\mathfrak{b} \cdot \mathfrak{a}_j \neq \mathfrak{b} \cdot \mathfrak{a}_k$ for every pair of distinct vertices $\mathfrak{a}_j$ and $\mathfrak{a}_k$ of P. If we select a plane perpendicular to $\mathfrak{b}$ and move it parallel to itself in the direction of $\mathfrak{b}$ until it intersects P, it must first intersect P in a minimum (i.e. an $\mathfrak{a}_j$ such that $\mathfrak{b} \cdot \mathfrak{a}_j < \mathfrak{b} \cdot \mathfrak{a}_{j \pm 1}$). After this it will intersect P in two points, until it intersects another minimum, after which it will have four intersections. If it next reaches another minimum, the theorem is proved. If it next reaches a maximum, there will then be only two intersections. Join these two points by a line segment, so that two new polygons are formed by this segment and the sides of P. (See Fig. 3.) At least one of these new polygons, P_1, must be knotted. Since $\mu(P) \geqq 3$, each of the new polygons P_1 and P_2 must have at least five sides. Since P_2 has five or more sides, P_1 must have fewer sides than P; and therefore there must be some plane intersecting P_1 in six or more components. It is clear that this plane must intersect the original polygon P itself in six or more components.

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