MOTIVIC HOMOTOPY THEORY AND POWER OPERATIONS IN MOTIVIC COHOMOLOGY

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SUBMITTED IN PARTIAL FULFILLMENT OF THE HONORS REQUIREMENTS FOR THE DEGREE OF BACHELOR OF ARTS TO THE

DEPARTMENT OF MATHEMATICS
HARVARD UNIVERSITY

CAMBRIDGE, MASSACHUSETTS
APRIL 2006
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Acknowledgments

First and foremost, I would like to thank my adviser, Mike Hopkins, without whom this thesis would not have been possible. I have learned a great deal from our discussions and he has been an inspiration throughout. I have started learning this theory as part of an intense summer seminar on the subject. I would like to thank the other participants, Aaron Silberstein, Thanos Papaioannou and David Roe, for making the learning process much more enjoyable. I am grateful to Chuck Weibel and Charles Rezk for valuable discussions. Finally, I would like to thank the Quincy B Team for providing much needed levity during the process.

Cambridge, Massachusetts

April 3, 2006

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Introduction

One of the very fascinating developments in mathematics came from Weil's realization in 1952 that arithmetic properties of varieties should be closely related to their topology. In formulating a program for the proof of his conjectures regarding the zeta functions of varieties over finite fields, Weil envisioned a cohomology theory that would shed light on the structure of the zeta functions. In particular, the idea was that a Lefshetz fixed point formula in this cohomology theory applied to the Frobenius map would count the number of \( \mathbb{F}_p \) points of a variety.

A great deal of effort was put into developing an appropriate cohomology theory for varieties over the next several decades. In the process, Grothendieck defined the notion of a scheme. It was in the language of schemes and sheaf cohomology that Grothendieck and Serre developed a suitable cohomology theory for schemes. A significant problem in this endeavor was the lack of a good topology for schemes. The natural topology, the Zariski topology, is much too coarse to allow such a cohomology.

A crucial insight came when Grothendieck realized that the reason one needed a topology in the first place was to have a notion of sheaves. In categorical terms, one can think of a sheaf as a functor from the category of the open sets of a topological space (where the morphisms are inclusions) to some abelian category which behaves well under coverings. In particular, to define a sheaf all one needs is an appropriate category and a notion of coverings associated to a scheme. A Grothendieck topology formalizes this idea. Taking open sets to be étale maps and coverings to be étale coverings, one gets the étale topology of a scheme. In this topology, Grothendieck considered the cohomology with coefficients in the constant sheaf \( \mathbb{Q}_l \), the \( l \)-adic cohomology, for \( l \neq p \). This cohomology theory was suitable for obtaining a Lefshetz trace formula and redefining the terms in the zeta function as the characteristic polynomials of the action of Frobenius on \( l \)-adic cohomology.

The full Weil conjectures, however, still eluded mathematicians. What remained was essentially to show that the characteristic polynomials of the Frobenius did not depend on the choice of \( l \). Grothendieck envisioned a universal category of such cohomology theories, the category of motives, as well as a motivic cohomology theory which would unite the \( l \)-adic cohomologies as well as related cohomology theories. However, the development of such a theory relies on the Standard Conjectures of Grothendieck which are still unresolved. Deligne ultimately proved the Weil conjectures using a different approach. Nevertheless, the construction of motives remained
an important problem in algebraic geometry and much work has been done assuming a good category of motives.

In the meanwhile, the study of generalized cohomology theories became an important part of algebraic topology, which led to the development of spectra. The category of spectra is roughly the topological analogue of the category of motives, in that the Brown representability theorem guarantees that any generalized cohomology theory is representable in the category of spectra.

Motivated by the developments in algebraic topology, Voevodsky sought to import homotopy-theoretic methods into algebraic geometry by developing a homotopy theory of schemes. Voevodsky constructed a triangulated category, which assuming the existence of the category of motives would be its derived category. Roughly speaking, this category is the abelianization of the homotopy category of schemes (see [Wei04] for a more precise relationship).

Furthermore, Voevodsky was able to construct a motivic cohomology theory, which is known to have many of the desired properties. In terms of motivic homotopy theory, motivic cohomology is just the analogue of singular cohomology. Using his theory, Voevodsky was able to prove the Milnor conjecture, relating the Milnor $K$-theory of a field with its Galois cohomology.

In this thesis, we describe the development of motivic homotopy theory. We provide the construction of motivic cohomology in purely homotopy theoretic terms, which makes many of its key properties much more transparent. Our goal is to provide a coherent narrative of the theory focusing on the homotopical and cohomological calculations. In this way, rather than presenting all the technical details, we present enough of the machinery to be able to make useful computations.

Having presented enough of the theory to be able to reasonably work with it, we move on to the problem of determining the mod 2 motivic Steenrod algebra, that is, the algebra of stable motivic cohomology operations. Our approach differs from that of [Voe03] in several ways. Firstly, we work solely within the framework of motivic homotopy theory and do not appeal to étale cohomology or Milnor $K$-theory. Furthermore, we import the ideas from homotopy theory relating cohomology theories and formal groups laws to construct an algebra closely related to the dual of the motivic Steenrod algebra.

Since the theory brings together relatively disparate areas of mathematics, it relies on quite a bit of background. We have tried not to assume too much machinery from algebraic topology and geometry. From the algebraic geometry side, we have tried not to assume anything outside the scope of [Har77]. We also assume some familiarity with algebraic topology; [Ada74] should provide enough background. While we have tried to make the thesis readable without prior knowledge of simplicial sets and model categories, familiarity with these is definitely helpful; [GJ99] is an excellent reference for both.
Chapter 1

The Category of Spaces

In this chapter, we describe the construction of an appropriate category in which one can reasonably talk about the homotopy of schemes. First, we discuss a straightforward, naive approach following [Mor04] and outline some of its shortcomings. This provides us with some insight on what to expect from the actual category of spaces. We then get into the construction of this category, which is a model category of simplicial Nisnevich sheaves. We recall the basics of Grothendieck topologies and model categories and describe the model category structure on the category of Nisnevich sheaves following [MV99]. This will form the framework within which we work throughout the thesis.

1.1 A Naive Approach

A naive approach to trying to create a homotopy theory of schemes would be to exactly follow the basic definitions from topology, with the affine line, $\mathbb{A}^1$ playing the role of the unit interval. In particular, suppose that we have two morphisms of schemes

$$f, g : X \to Y$$

We can then try to define them to be homotopic if there exists a morphism

$$h : X \times \mathbb{A}^1 \to Y$$

such that $h|_{X \times \{0\}} = f$ and $h|_{X \times \{1\}} = g$. This then defines an equivalence relation on morphisms. We can then let $[X, Y]$ denote the set of equivalence classes of morphisms between $X$ and $Y$.

There are several problems with this approach, however. For instance, there is no way of constructing the cone $C(f)$ of a morphism $f : X \to Y$ in such a way that one gets long exact sequences of homotopy sets. Taking the projective space $\mathbb{P}^1$, we see that because it has a covering with two open affine spaces whose intersection is $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$ that $\mathbb{P}^1$ should be the suspension of $\mathbb{G}_m$. However, the set $[\mathbb{P}^1, X]$
generally has no reasonable group structure. For instance, $[\mathbb{P}^1, \mathbb{P}^1]$ is only a monoid (since there are no morphisms corresponding to sheaves $\mathcal{O}(n)$ for $n < 0$).

The above discussion suggests that the problem with constructing a homotopy theory of schemes is that there aren’t enough objects and morphisms in the category of schemes. Thus we want to embed the category of schemes into a larger category on which one can do homotopy theory. There are a number of technical difficulties that are involved in doing this. Firstly, we have to make precise what we mean by having a homotopy category. In order to do this, one makes use of Quillen’s model categories [Qui67], though the definition we will use here follows [MV99] and is a little stronger. We will enlarge our category to the category of simplicial Nisnevich sheaves and we’ll see that it has a natural model category structure.

1.2 Nisnevich Sheaves

In his effort to define étale cohomology, Grothendieck generalized the notion of a topology by realizing that in order to define sheaves, all one needs is the notion of a covering.

A covering of a scheme $Y$ is a family $\{g_i : U_i \to Y\}_{i \in I}$ of morphisms such that $Y = \bigcup g_i(U_i)$.

Definition 1.2.1. A site on a scheme $X$ is a full subcategory $\mathcal{C}/X$ of schemes over $X$ which is closed under fiber products, together with a class $\mathcal{C}$ of coverings of all objects of $\mathcal{C}/X$, such that

1. All isomorphisms are in $\mathcal{C}$;
2. If $\{U_\alpha \xrightarrow{p_\alpha} U\}$ is a covering, and for all $\alpha$, $\{U_{\alpha,\beta} \xrightarrow{q_{\alpha,\beta}} U_\alpha\}$ is a covering then $\{U_{\alpha,\beta} \xrightarrow{p_\alpha q_{\alpha,\beta}} U\}$ is also a covering.
3. If $\{U_\alpha \xrightarrow{p_\alpha} U\}$ is a covering and $V \to U$ is a morphism, then $\{V \times_U U_\alpha \xrightarrow{q_\alpha} V\}$, where $q_\alpha$ is the projection onto the first factor, is also a covering of $V$.

The following are several important examples of sites over a scheme $X$:

- The (small) Zariski site, $X_{\text{Zar}}$, which is the category of open immersions over $X$ with the coverings given by all morphisms in this category.
- The (small) étale site, $X_{\text{ét}}$, which is the category of étale morphisms over $X$ in which the coverings consist of all morphisms.
- The (big) flat site, $X_{\text{fl}}$, which is the category of schemes of locally finite type over $X$, and the covering morphisms are flat.
For our purposes, it turns out that the right topology to consider is the category of smooth schemes of finite type over a field \( k \), in which the coverings are the Nisnevich coverings (though the basic results are still true if one replaces \( \text{Spec} k \) by a more general scheme). We denote this site by \( S_{\text{Nis}} \).

**Definition 1.2.2.** A Nisnevich covering of \( U \) is an étale covering \( \{ U_\alpha \xrightarrow{f_\alpha} U \} \) such that for every \( x \in U \) there exists \( \alpha \) and \( y \in U_\alpha \) such that

\[
f_\alpha(y) = x \quad \text{and} \quad \kappa(x) = \kappa(y).
\]

As one would expect, a presheaf on a site \( T \) is just a functor \( P : T^{\text{op}} \rightarrow \mathcal{C} \) to some category \( \mathcal{C} \) (since, for a topological space, \( T \) would be the category of open sets with morphisms given by inclusions). In particular, if there is a morphism \( U' \rightarrow U \) in \( T \), we get the restriction map

\[
\text{res}_{U',U} : P(U) \rightarrow P(U')
\]

The Grothendieck topology allows one to define sheaves on a site. In particular, a presheaf \( P \) on a site \( T \) is a sheaf if for every covering \( \{ U_i \rightarrow U \} \), \( P(U) \) is the coequalizer in the following sequence

\[
P(U) \rightarrow \prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i \times_U U_j)
\]

There is another characterization of Nisnevich sheaves, which will be very useful to us.

**Definition 1.2.3.** An elementary distinguished square is a cartesian square

\[
\begin{array}{ccc}
U \times_X V & \longrightarrow & V \\
\downarrow & & \downarrow p \\
U & \longrightarrow & X \\
& i & \\
\end{array}
\]

in which \( i \) is an open embedding, \( p \) is an étale morphism and

\[
p : p^{-1}(X - U)_{\text{red}} \xrightarrow{\cong} (X - U)_{\text{red}}
\]

**Theorem 1.2.4.** \([\text{MV99}]\) A presheaf is a sheaf in the Nisnevich topology if and only if it takes elementary distinguished squares into cartesian squares of sets.

A fundamental property of Nisnevich sheaves, which for instance follows from the analogous fact about étale sheaves \([\text{Mil80, II.1.7}]\) is that every representable presheaf is a Nisnevich sheaf. Furthermore, unlike Zariski sheaves, Nisnevich sheaves have the following local characterization.
Lemma 1.2.5. [MV99] Suppose that $U,V$ and $X$ form an elementary distinguished square. Then the sheaf represented by $X$ is isomorphic to the pushout of $U$ and $V$ along $U \cap V = U \times_X V$ in the category of Nisnevich sheaves, i.e. the represent functor (which sends the scheme to the functor it represents)

$$\text{Rep} : \text{Sch}/X \to \text{Shv}(\text{Sch}/S)_{\text{Nis}}$$

takes elementary distinguished squares to cocartesian ones. In particular, $V/(U \times_X V) \to X/U$ is an isomorphism of Nisnevich sheaves.

Once we have the notion of a sheaf, we can define sheaf cohomology in the usual way. For details, the reader is referred to [Mil80, Ch. III]. For the Nisnevich topology, we have for all sheaves $M$ of abelian groups and for all schemes of dimension $d$,

$$H^n_{\text{Nis}}(X; M) = 0 \quad \text{for} \ n > d.$$ 

In this way, the Nisnevich topology behaves similarly to the Zariski topology with respect to cohomology.

1.3 Model Categories and Simplicial Sheaves

Model categories are a way of formalizing the structure that one has in topological spaces that allows one to pass to the homotopy category. In particular, it defines such a structure for an arbitrary category.

Definition 1.3.1. A model category is a category $\mathcal{C}$ with three distinguished classes of morphisms: weak equivalences, cofibrations and fibrations satisfying the following properties:

1. $\mathcal{C}$ has all small limits and colimits

2. If $f$ and $g$ are two composable morphisms and two of $f, g, f \circ g$ are weak equivalences then so is the third.

3. If $f$ is a retract of $g$, i.e. there is a commutative diagram of the form

$$\begin{array}{ccc}
A & \rightarrow & C \\
\downarrow f & & \downarrow g \\
B & \rightarrow & D
\end{array} \quad \begin{array}{ccc}
&& \downarrow f \\
\rightarrow & & \\
\rightarrow & & \\
B & \rightarrow & D
\end{array}$$

and $g$ is a weak equivalence, fibration or cofibration, then so is $f$.

4. Suppose that there is a commutative solid arrow diagram of the form

$$\begin{array}{ccc}
U & \rightarrow & X \\
\downarrow i & & \downarrow p \\
V & \rightarrow & Y
\end{array}$$

and $p$ is a fibration, then so is $i$.
where $i$ is a cofibration and $p$ is a fibration. Then if either $i$ or $p$ is a weak equivalence, the dotted arrow exists (though is not necessarily unique) making the diagram commute.

5. Any morphism $f : X \to Y$ can be functorially factored as

(a) $f = p \circ i$ where $p$ is a fibration and $i$ is a trivial cofibration; and

(b) $f = q \circ j$ where $q$ is a trivial fibration and $j$ is a cofibration.

Given a model category $C$, we can define an associated homotopy category $\text{Ho}(C)$ which has the same objects as $C$ but the morphisms are weak equivalence classes of morphisms between objects. We have the obvious map $\gamma : C \to \text{Ho}(C)$ which sends objects to themselves and morphisms to their weak equivalence class.

We can characterize $\text{Ho}(C)$ as the universal category in the following sense. Suppose that $F : C \to D$ is a functor which sends weak equivalences to isomorphisms. Then there exists a unique functor $F_* : \text{Ho}(C) \to D$ such that $F_* \circ \gamma = F$. In general, we can always formally invert a class of morphisms $\Sigma$ of a category $C$ to get a functor $\gamma : C \to C[\Sigma]$ which is universal in this way. In general, one must be willing to pass to a higher set theoretic universe to do this. In the case of passing to the homotopy category of a model category, however, one can avoid this by standard homotopical algebra constructions [GJ99, II.1].

It is a fundamental theorem in topology that the category of compactly generated Hausdorff spaces is a model category, and, in fact, it forms a prototypical example of a model category.

Another important model category in topology is the category of simplicial sets, $\Delta^{\text{op}} \text{Set}$, which has an equivalent homotopy category to that of compactly generated Hausdorff spaces. Recall that a simplicial set $X$ is a contravariant functor

$$X : \Delta^{\text{op}} \to \text{Set}$$

where $\Delta$ is the simplicial category, whose objects are finite sets $[n] = \{1, \ldots, n\}$ and the morphisms are (weakly) order preserving functions between them. For a decent introduction to simplicial sets and their model category structure, the reader is referred to [May67] and [GJ99].

The first axiom for a model category requires that the category contain all small limits and colimits. Thus, to construct a model category including schemes, the first thing we can do is embed the category of schemes into the category of presheaves, its universal completion and cocompletion. The problem with presheaves, however, is that if $X$ is a scheme which is covered by two open subsets $U$ and $V$, the pushout $Z$ of the diagram

$$\begin{array}{ccc}
U \times_X V & \longrightarrow & V \\
\downarrow & & \downarrow \\
U & \longrightarrow & Z
\end{array}$$
in the category of presheaves is not represented by the scheme $X$. To resolve this problem, we pass from presheaves to sheaves. While the Zariski topology suffices for this problem, it turns out that the right topology to use for these sheaves is the Nisnevich topology. We will thus consider the category of sheaves $Shv(S_{Nis})$.

Now, in order to simplify the construction of the model category structure, we will pass from sheaves to simplicial sheaves $\Delta^{op}Shv(S_{Nis})$. A simplicial sheaf is just a functor
\[
\mathcal{F} : \Delta^{op} \to Shv(S_{Nis}).
\]
Thus, a simplicial sheaf $\mathcal{F}$ is given by a collection of sheaves $\mathcal{F}_n$ along with appropriate face and degeneracy maps. As we will see later, we can then induce a model category structure on the category of sheaves which gives an equivalent homotopy category.

We can now use the model category structure on simplicial sets to define the model category structure, [MV99].

**Definition 1.3.2.** Let $f : X \to Y$ be a morphism of simplicial sheaves.

1. $f$ is a weak equivalence if for any $U$, the morphism of simplicial sets $X(U) \to Y(U)$ is a weak equivalence.
2. $f$ is a cofibration if it is a monomorphism.

From these, the class of fibrations is induced from the axioms of a model category. Let $W$ (resp. $C$, $F$) denote the class of weak equivalences (resp. cofibrations, fibrations).

**Theorem 1.3.3.** [Jar87] The triple $(W, C, F)$ is a model category structure on $\Delta^{op}Shv(S_{Nis})$.

Let $\mathcal{H}_s$ be the homotopy category corresponding to this model category.

Now that we have a model category to work with, we can impose the condition that $\mathbb{A}^1$ be contractible. To do this, we use the technique of Bousfield localization [GJ99] to make the map $\mathbb{A}^1 \to S$ a weak equivalence.

**Definition 1.3.4.** A simplicial sheaf $Z \in \Delta^{op}Shv(S_{Nis})$ is $\mathbb{A}^1$ local if the projection $Y \times \mathbb{A}^1 \to Y$ induces a bijection
\[
\text{Hom}_{\mathcal{H}_s(S)}(Y, Z) \to \text{Hom}_{\mathcal{H}_s(S)}(Y \times \mathbb{A}^1, Z)
\]
for all $Y \in \Delta^{op}Shv(S_{Nis})$.

A morphism $Y \to Z$ in $\Delta^{op}Shv(S_{Nis})$ is an $\mathbb{A}^1$ weak equivalence if, for any $\mathbb{A}^1$-local object $W$ the induced map
\[
\text{Hom}_{\mathcal{H}_s(S)}(Z, W) \to \text{Hom}_{\mathcal{H}}(Y, W)
\]

is a bijection.
Theorem 1.3.5. The class of monomorphisms and $A_1$ weak equivalences defines a model category structure on $\Delta^{op}\text{Shv}(S_{Nis})$.

The following lemma from [MV99] is very useful when working with $\text{Spc}$, which follows from the fact that the model category structure on $\text{Spc}$, is proper, i.e. the class of weak equivalences is closed under base change by fibrations and cobar change by cofibrations.

Lemma 1.3.6. [GJ99, II.9.8] Consider the cocartesian squares

\[
\begin{array}{ccc}
X_i \xrightarrow{a_i} Y_i \\
\downarrow b_i \downarrow d \\
X'_i \xrightarrow{c_i} Y'_i
\end{array}
\]

for $i = 1, 2$ such that $a_i$ are monomorphisms, and let $f_X, f_Y, f_{X'}, f_{Y'}$ be a morphism from the first square to the second such that $f_X, f_Y$ and $f_{X'}$ are $A_1$ weak equivalences. Then $f_{Y'}$ is an $A_1$ weak equivalence.

We denote by $\text{Spc}$ the category of spaces $\Delta^{op}\text{Shv}(S_{Nis})$, and by $\mathcal{H}$ the associated homotopy category. As a slightly subtle point, we will largely be concerned with pointed spaces, i.e. for each $X \in \text{Spc}$ a choice of map $S \to X$. Let $\text{Spc}_\bullet$ be the resulting category. This category naturally inherits a model category structure from $\text{Spc}$ and we denote by $\mathcal{H}_\bullet$ the associated homotopy category. One has to be careful here because the homotopy type of a pointed space can depend on the choice of basepoint. Nevertheless, we will mostly deal with spaces in which all rational points are equivalent up to automorphism in which case all choices of basepoint are equivalent.

1.4 Geometric Realization

In this section, we will construct a geometric realization functor from $\text{Spc}$ to the $\text{Shv}(S_{Nis})$ which will enable us to define a model category structure on the latter category which has an equivalent homotopy category.

Let $\Delta[n]$ be the constant simplicial sheaf given by the $n$-simplex. Let $\Delta^\bullet$ be the cosimplicial object in the category of schemes, given by

\[
\Delta^n = \text{Spec} \left( k[t_0, \ldots, t_n] / \left( \sum t_i = 1 \right) \right).
\]

Now, we can define the geometric realization functor

\[
| \ |_{A_1} : \text{Spc} \to \text{Shv}(S_{Nis})
\]

as the functor which sends $\Delta[n]$ to the scheme $\Delta^n$. Now, we can extend this to the functor we want by taking a simplicial sheaf $X$ to

\[
|X| = \left( \bigsqcup X_n \times \Delta^n \right) / \sim
\]
where \( \sim \) gives the usual relation in terms of the face and degeneracy maps. In more categorical terms, the geometric realization functor is a coend (a kind of diagonal colimit, see [Mac71, IX.6])

\[
|X| = \int^{[n]} X_n \times \Delta^n
\]

There is a functor which is right adjoint to geometric realization, which is the “singular chains” functor, \( \text{Sing} : \text{Shv}(S_{Nis}) \to \text{Spc} \), given by, for \( Y \in \text{Shv}(S_{Nis}), Z \in S_{Nis} \).

\[
\text{Sing}(Y)_n(Z) = Y(Z \times \Delta^n)
\]

**Theorem 1.4.1.** [MV99] The adjoint functors

\[
\text{Sing} : \text{Shv}(S_{Nis}) \to \text{Spc}
\]

\[
| \cdot | : \text{Spc} \to \text{Shv}(S_{Nis})
\]

take \( \mathbb{A}^1 \) weak equivalences to \( \mathbb{A}^1 \) weak equivalences and the corresponding functors between homotopy categories are mutually inverse equivalences.

Thus, we get a model category structure on \( \text{Shv}(S_{Nis}) \) which gives the same homotopy category and is compatible with the inclusion \( \text{Shv}(S_{Nis}) \to \text{Spc} \). Therefore, we could just take spaces to be the category of Nisnevich sheaves. Nevertheless, we use simplicial sheaves because it makes the many of the topological constructions that we use somewhat more intuitive. Furthermore, for simplicial sheaves we have the following very convenient result.

**Lemma 1.4.2.** [MV99, 2.2.14] Let \( f : X \to Y \) be a morphism of simplicial sheaves such that for any \( n \geq 0 \) the corresponding morphism of sheaves of sets \( f_n : X_n \to Y_n \) is and \( \mathbb{A}^1 \) weak equivalence. Then \( f \) is a weak equivalence.
Chapter 2

Topological Constructions

In this chapter, we import a number of important constructions from topology into our category of spaces. Namely, we construct wedges, smash products (the coproduct and product respectively in the pointed category) and Thom spaces. We give a proof of the homotopy purity theorem following [MV99], which is an algebraic analogue of the tubular neighborhood theorem for smooth schemes. Finally, we discuss the construction of classifying spaces for algebraic groups.

2.1 Smash Product, Spheres and Thom Spaces

We can use the fact that our category of spaces is given by simplicial sheaves to import various structures from the category of simplicial sets into our setting. Recall that given two pointed simplicial sets, we have the following construction:

- The wedge $X \lor Y$ is the coequalizer of the diagram
  $\ast \rightrightarrows X \sqcup Y$
  in the category of simplicial sets, pointed in the canonical way.

- The smash product $X \wedge Y$ is the quotient
  \[(X \times Y)/(X \times \{\ast\} \lor \{\ast\} \times Y)\]
  which is pointed by the image of $(X \times \{\ast\} \lor \{\ast\} \times Y)$. The unit for the smash product is $(\ast)_+$, a point with an additional disjoint basepoint adjoined.

We can form the same definitions in the category $\textbf{Spc}$, and in fact, we have that for sheaves $X, Y \in \textbf{Spc}$

- The wedge $X \lor Y$ is the sheaf associated to the presheaf $U \mapsto X(U) \lor Y(U)$ and
• The smash product $X \wedge Y$ is the sheaf associated to the presheaf $U \mapsto X(U) \wedge Y(U)$.

As in the category of topological spaces, where spheres play a fundamental role, we have several distinguished objects in $\text{Spc}$:

• The simplicial circle $S^1_s$, which is the constant simplicial sheaf corresponding to the simplicial circle $\Delta^1/\partial\Delta^1$;

• The Tate circle, $S^1_t$ which is the sheaf represented by $\mathbb{A}^1 - \{0\}$; and

• $T$, the quotient sheaf $\mathbb{A}^1/(\mathbb{A}^1 - \{0\})$.

Now, we can define sphere in our category by wedging these in an appropriate way. It turns out that there are two distinct spheres in $\text{Spc}$:

$$S^n_s = (S^1_s)^n; S^n_t = (S^1_t)^n$$

We can now define the bigraded sphere to be

$$S^{p,q} = S_{s}^{p-q} \wedge S_{t}^{q}$$

The indexing here is to conform to the usual bigraded indexing in motivic cohomology. We shall see that in the stable category, this collection of spaces play a role analogous to that of the sphere spectrum in topology.

**Lemma 2.1.1.** In the homotopy category $\mathcal{H}(S)$, we have a canonical isomorphisms

$$S^1_s \wedge S^1_t \cong T \cong \mathbb{P}^1$$

**Proof.** Consider the following cartesian, cocartesian square given by the standard affine covering of $\mathbb{P}^1$.

$$\begin{array}{ccc}
\mathbb{A}^1 - \{0\} & \longrightarrow & \mathbb{A}^1 \\
\downarrow & & \downarrow \\
\mathbb{A}^1 & \longrightarrow & \mathbb{P}^1
\end{array}$$

Since it is an elementary distinguished square, it follows that $T \cong \mathbb{P}^1/\mathbb{A}^1$ but from Lemma 1.3.6, we have that this is weakly equivalent to $\mathbb{P}^1/0 = \mathbb{P}^1$. Now, since the square is also cartesian we have that $\mathbb{P}^1 = \Sigma_s(\mathbb{A}^1 - \{0\})$. \qed

We thus have three suspension functors on $\text{Spc}$ given by:

$$\Sigma_s(X) = S^1_s \wedge X; \quad \Sigma_t(X) = S^1_t \wedge X; \quad \Sigma_T(X) = T \wedge X$$

and by the lemma, we have that $\Sigma_T \cong \Sigma_s \circ \Sigma_t$ on the level of the homotopy category. We will also use the notation $\Sigma^{p,q}$ for the obvious composition of functors. Now, analogous to the usual topological definition of Thom spaces, we make the following definition.
Definition 2.1.2. Let $X$ be a smooth scheme over $S$ and $E$ a vector bundle over $X$. The Thom space of $E$ is the pointed sheaf

$$Th(E) = Th(E/X) = E/(E - i(X))$$

where $i : X \to E$ is the zero section of $E$.

Now, for any vector bundle $E$ over $X$, let $\mathbb{P}(E) \to X$ be the corresponding projective bundle over $X$.

Proposition 2.1.3. Let $E$ be a vector bundle over $X$ and $\mathbb{P}(E) \to \mathbb{P}(E \oplus \mathcal{O})$ be the embedding at infinity. Then the canonical morphism of pointed sheaves

$$\mathbb{P}(E \oplus \mathcal{O})/\mathbb{P}(E) \to Th(E)$$

is a weak equivalence.

Proof. The open covering $\mathbb{P}(E \oplus \mathcal{O}) = E \cup (\mathbb{P}(E \oplus \mathcal{O}) - X)$ (where the closed embedding of $X$ into $\mathbb{P}(E \oplus \mathcal{O})$ is the composition of the embedding of $E$ with the zero section) gives following cocartesian square

$$
\begin{array}{ccc}
E \cap (\mathbb{P}(E \oplus \mathcal{O}) - X) = E - X & \to & \mathbb{P}(E \oplus \mathcal{O}) - X \\
\downarrow & & \downarrow \\
E & \to & \mathbb{P}(E \oplus \mathcal{O})
\end{array}
$$

It follows that

$$Th(E) = E/(E - X) \cong \mathbb{P}(E \oplus \mathcal{O})/(\mathbb{P}(E \oplus \mathcal{O}) - X)$$

Since the embedding at infinity factors through $\mathbb{P}(E \oplus \mathcal{O}) - X$, we get the desired map

$$\mathbb{P}(E \oplus \mathcal{O})/\mathbb{P}(E) \to \mathbb{P}(E \oplus \mathcal{O})/(\mathbb{P}(E \oplus \mathcal{O}) - X) \cong Th(E)$$

and by lemma 1.3.6, it suffices to show that the embedding $\mathbb{P}(E) \to \mathbb{P}(E \oplus \mathcal{O}) - X$ is a weak equivalence. But by [Gro61, §8], this map is isomorphic to the zero section of the canonical line bundle over $\mathbb{P}(E)$, and hence an $\mathbb{A}^1$ weak equivalence. \qed

2.2 Homotopy Purity Theorem

We now come to one of the most fundamental theorems in the theory, the homotopy purity theorem, which is the analogue of the tubular neighborhood theorem for manifolds in this context.

Theorem 2.2.1. Let $i : Z \to X$ be a closed embedding of smooth schemes over $S$. Let $N_{X,Z} \to Z$ be the normal vector bundle of the embedding. Then there is a canonical isomorphism

$$Th(N_{X,Z}) \cong X/(X - i(Z))$$

in the homotopy category.
To prove this theorem, we use the technique of the deformation to the normal cone [Ful98, Ch. 5]. Deformation to the normal cone is the construction in intersection theory which gives a deformation from the given embedding of a subscheme $Z$ of $X$ to the zero section embedding of $Z$ to the normal cone $N = N_ZX = \text{Spec}(\bigoplus I^t/I^{t+1})$ where $I$ is the ideal sheaf defining $Z$. Explicitly, there is a closed embedding

$$Z \times \mathbb{A}^1 \hookrightarrow M^\circ$$

with $M^\circ = M^\circ_{X,Z}$ a scheme of dimension $\text{dim}(X) + 1$, such that over $t \neq 0$, the embedding of $Z \times \{t\}$ in $M^\circ_t$ is isomorphic to the given embedding of $Z$ in $X$ and the embedding of $W \times \{0\}$ in $M^\circ_0$ is isomorphic to the zero section of embedding of $Z$ in $N_ZX$.

We can construct $M^\circ$ and the closed embedding as follows. Let $M$ be the blowup of $X \times \mathbb{A}^1$ along $Z \times \{0\}$. Since the normal cone to $X \times \{0\}$ is $N \oplus \mathcal{O}$, the exceptional divisor of this blowup is $P(N \oplus \mathcal{O})$. Now, considering the sequence of embeddings

$$Z = Z \times \{0\} \hookrightarrow Z \times \mathbb{A}^1 \hookrightarrow X \times \mathbb{A}^1$$

we see that the blowup of $Z \times \mathbb{A}^1$ along $Z \times \{0\}$ is embedded as a closed subscheme of $M$; but since $Z \times \{0\}$ is a Cartier divisor on $Z \times \mathbb{A}^1$, this blowup is isomorphic to $Z \times \mathbb{A}^1$. In this way, we have a closed embedding

$$Z \times \mathbb{A}^1 \hookrightarrow M$$

which is the desired map outside of $Z \times \{0\}$. Now, at 0, the sequence of embeddings

$$Z \times \{0\} \hookrightarrow X \times \{0\} \hookrightarrow X \times \mathbb{A}^1$$

shows that the blowup $\tilde{X}$ of $X$ along $Z$ is embedded as a closed subscheme of $M$. The desired $M^\circ$ is then the complement of $\tilde{Y}$ in $M$. One can see explicitly that the preimage at 0 is $\mathbb{P}(N \oplus \mathcal{O})$ [Ful98].

Now, we get the following diagram

$$Z \times \mathbb{A}^1 \xrightarrow{f} M^\circ \xrightarrow{p} X \times \mathbb{A}^1$$

with

$$p^{-1}(Z \times \{0\}) = \mathbb{P}(N \oplus \mathcal{O}) \quad \text{and} \quad f(Z \times \{0\}) = \mathbb{P}(\mathcal{O}).$$

By Proposition 2.1.3, we get an $\mathbb{A}^1$ weak equivalence of pointed sheaves

$$Th(N) \cong p^{-1}(Z \times \{0\})/(p^{-1}(Z \times \{0\}) - f(Z \times \{0\}))$$

Now, since $p$ is an isomorphism outside $p^{-1}(X \times \{0\})$, we have a map

$$g : X = X \times \{1\} \to M^\circ$$
with
\[ g(X) \cap f(Z \times \mathbb{A}^1) = g(Z) \]
Now, since \( p^{-1}(Z \times \{0\}) \cap f(Z \times \mathbb{A}^1) = f(Z \times \{0\}) \), we get two morphisms:
\[ \tilde{g} : X/(X - Z) \rightarrow M^o/(M^o - f(Z \times \mathbb{A}^1)) \]
\[ \alpha : Th(N) \rightarrow M^o/(M^o - f(Z \times \mathbb{A}^1)) \]
The content of the homotopy purity theorem, then, is that these two maps are \( \mathbb{A}^1 \) weak equivalences. We will do this in several steps.

**Step 1.** By [Gro67, Cor. 17.12.2d], we have that there exists a finite Zariski covering \( U = \{U_i\} \) of \( X \) such that for any \( i \), there exists an étale morphism \( q_i : U_i \rightarrow \mathbb{A}^n \) such that \( q_i^{-1}(\mathbb{A}^{n-c} \times \{0, \ldots, 0\}) = Z \cap U_i \) for some \( c \). The same condition clearly holds for all intersections of the form \( U_{i_1} \cap \ldots \cap U_{i_k} \). Now, consider the simplicial sheaf \( \mathcal{X} \) given by
\[ \mathcal{X}_n = \left( \bigsqcup U_i \right)^{n+1}_X \]
We clearly have a map \( \mathcal{X} \rightarrow X \). Since the geometric realization of \( \mathcal{X} \) is clearly \( X \), this map is a weak equivalence.

We also have simplicial sheaves \( Z \) and \( \mathcal{M} \) with \( Z \) having terms \( (\bigsqcup (U_i \cap Z))^{n+1} \), and \( \mathcal{M}^o \) formed by applying the construction of \( \mathcal{M}^o \) termwise to the closed embedding \( Z \rightarrow \mathcal{X} \). We then get the following commutative diagram
\[
\begin{array}{ccc}
\mathcal{X}/(\mathcal{X} - Z) & \longrightarrow & \mathcal{M}^o/(\mathcal{M}^o - f(Z \times \mathbb{A}^1)) \\
\downarrow & & \downarrow \\
X/(X - Z) & \longrightarrow & M^o/(M^o - f(Z \times \mathbb{A}^1)) \leftarrow Th(N)
\end{array}
\]

in which the vertical arrows are weak equivalences by Lemma 1.3.6. Thus, by Lemma 1.4.2, we see that it suffices to show that \( \tilde{g} \) and \( \alpha \) are weak equivalences when restricted to the \( U_i \).

**Step 2.** By the above discussion, we can assume that the embedding \( Z \hookrightarrow X \) is such that there exists an étale morphism \( q : X \rightarrow \mathbb{A}^n \) such that \( q^{-1}(\mathbb{A}^{n-c} \times \{0, \ldots, 0\}) = Z \).

Now, consider \( W = X \times_{\mathbb{A}^n} (Z \times \mathbb{A}^c) \), where the structure map \( Z \times \mathbb{A}^c \) is given by \( q \times Id \). Over \( \mathbb{A}^{n-c} \times \{0, \ldots, 0\} \) the fiber of \( W \) is the closed subscheme \( Z \times_{\mathbb{A}^{n-c}} Z \).
Since \( Z \) is separated, the diagonal morphism \( \Delta : Z \rightarrow Z \times_{\mathbb{A}^{n-c}} Z \) is closed. But by [Gro67, Cor. 17.4.2], it is also open. Hence \( Z \times_{\mathbb{A}^{n-c}} Z \) is the disjoint union of the diagonal embedding and a closed subscheme \( Y \). Let \( U = W - Y \). We then have two étale projections
\[ pr_1 : U \rightarrow X \quad \text{and} \quad pr_2 : U \rightarrow Z \times \mathbb{A}^c \]
such that \( pr_1^{-1}(Z) \to Z \) and \( pr_2^{-1}(Z \times \{0\}) \to Z \times \{0\} \) are isomorphisms. It follows that in the following commutative diagram

\[
\begin{array}{ccc}
X/(X - Z) & \longrightarrow & M^o/(M^o - f(Z \times \A^1)) \\
\uparrow & & \uparrow \quad \downarrow \uparrow \\
U/(U - pr_1^{-1}(Z)) & \longrightarrow & M^o_u/(M^o_u - f(Z \times \A^1)) \\
\uparrow & & \uparrow \quad \downarrow \uparrow \\
(Z \times \A^c)/((Z \times \A^c) - (Z \times \{0\})) & \longrightarrow & M^o_{Z \times \A^c}/(M^o_{Z \times \A^c} - f(Z \times \A^1)) \\
\uparrow & & \downarrow \uparrow \\
& & \quad \downarrow \uparrow \\
\end{array}
\]

the vertical arrows are weak equivalences. Hence, it suffices to prove the theorem for \( X = Z \times \A^c \) and the zero embedding.

**Step 3.** We can thus assume that \( X = Z \times \A^c \) and the embedding is the zero section. From the definition of the blowup, we see that \( M^o \) is isomorphic to the total space of the canonical line bundle over \( \P^n_Z \). Therefore, by Lemma 1.3.6, the morphism

\[
q' : M^o/(M^o - \{0, \ldots, 0\} \times \A^c_Z) \to \P^n_Z/(\P^n_Z - Z)
\]

is an \( \A^1 \) weak equivalence. Since \( q' \circ \alpha \) is the canonical isomorphism of sheaves, we have that \( \alpha \) is an \( \A^1 \) weak equivalence.

Now, composing the projection \( M^o \to \P^n_Z \) with the immersion \( g : \A^c_Z \to M^o \), we get the canonical embedding \( \A^c_Z \to \P^n_Z \) which takes \( \{0, \ldots, 0\} \) to the class of \( \{0, \ldots, 0, 1\} \). Thus the morphism

\[
\A^c_Z/(\A^c_Z - \{0, \ldots, 0\}) \to \P^n_Z/(\P^n_Z - Z)
\]

is an isomorphism and so \( \tilde{g} \) is also an \( \A^1 \) weak equivalence.

### 2.3 Classifying Spaces

Suppose \( G \) is an algebraic group in \( \text{Spc} \) and let \( E(G) \) be the simplicial sheaf of groups with \( n \)-th term given by \( E(G) = X^{n+1} \) with the faces and degeneracies induced by partial projections and diagonals respectively. The subgroup of vertices of \( E(G) \) is \( G \) and so \( G \) acts on \( E(G) \) on the left and right. Let \( B(G) \) be the quotient \( E(G)/G \), with the projection given by

\[
(g_0, g_1, \ldots, g_n) \mapsto (g_0g_1^{-1}, g_1g_2^{-1}, \ldots, g_{n-1}g_n^{-1})
\]

Thus we get a classifying space of \( B(G) \).

There is another, not necessarily equivalent, version of the classifying space of an algebraic group \( G \) called the geometric classifying space, \( B_{gm}G \).

To construct \( B_{gm}G \), fix a faithful representation \( \rho : G \to \text{GL}_n \). Let \( U_i \subset \A^n \) be the maximal open subscheme of \( \A^n = (\A^n)^1 \) on which \( G \) acts freely via the
diagonal representation $\rho^i$. The geometric quotient $V_i = U_i / G$ is smooth over $k$. We define $B_{gm}(G)$ to be the colimit $V_{\infty}$ of the spaces $V_i$ with the embeddings $V_i \hookrightarrow V_{i+1}$ induced by the standard inclusions $\mathbb{A}^{ni} \subset \mathbb{A}^{n(i+1)}$. Morel and Voevodsky [MV99, 4.2.9] show that up to $\mathbb{A}^1$ weak equivalence, this space does not depend on the choice of representation.

The distinction between the two classifying spaces is that $BG$ classifies Nisnevich principal $G$-bundles, while $B_{gm}G$ classifies étale principal $G$-bundles [MV99, 4.2]. In particular, we have the following relation between them.

**Lemma 2.3.1.** [MV99, 4.1.18, 4.2.7] There is a natural morphism $BG \rightarrow B_{gm}G$, which is an $\mathbb{A}^1$ weak equivalence if and only if “Hilbert’s Theorem 90” holds for $G$; that is, for every finite field extension $K$ of $k$, $H^1_{et}(K, G) = 0$.

It follows that for $\mathbb{G}_m$, $B_{gm}\mathbb{G}_m = BG_m$. 
Chapter 3

Stable Homotopy Category and Motivic Cohomology

In this chapter, we describe the stable motivic homotopy category following the usual construction in homotopy theory. We then use this category to give a definition of motivic cohomology in terms of spectra and show that it coincides with Voevodsky’s original definition in terms of motivic chain complexes. Further, we state the motivic version of the Thom isomorphism theorem and use it to get the Gysin sequence for motivic cohomology. Finally, we mention a version of the Künneth theorem for motivic cohomology which will be important in calculations.

3.1 The Stable Homotopy Category

In topology it is often a good idea to pass from the category of spaces to the category of spectra, which is the stable category. Brown’s famous representability theorem states that any cohomology theory on the category of topological spaces is representable in the category of spectra. Roughly speaking, one passes to the stable category by inverting the suspension functor. The goal of this section is to introduce an analogue in the case of motivic homotopy.

The right notion of suspension, it turns out, is with respect to $T = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$. Recall that we have $T = S^1_s \wedge S^1_t = \mathbb{P}^1$. We then have the following definition.

Definition 3.1.1. A $T$-spectrum $E$ is a sequence of pointed spaces $E_i$ along with bounding maps $T \wedge E_i \to E_{i+1}$. A morphism $E \to F$ of $T$-spectra is a sequence of morphisms $E_i \to F_i$ which commute with the bounding maps.

The category of $T$-spectra has an evident model category structure with the notion of stable $\mathbb{A}^1$-weak equivalence. We can obtain the stable homotopy category $\text{SHot}^T$ by localizing the category of $T$-spectra by inverting $\mathbb{A}^1$ weak equivalences. In the same way that one does this in topology, one can define the smash product on spectra $E \wedge F$ by considering $E_n \wedge F_m$ and using an appropriate function $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$ [Ada74].
As in topology, we have the $T$-suspension functor

$$\Sigma^\infty_T : \text{Spc}_* \to T\text{-Spectra}$$

given by $\Sigma^\infty_T(X)_i = T^\wedge i \wedge X$ with the obvious bounding maps. The functor $\Sigma^\infty_T$ preserves smash products as well as $\mathbb{A}^1$-weak equivalences.

Give a $T$-Spectrum one can obtain a cohomology theory on $\text{Spc}_*$. Since there are two natural spheres in $\text{Spc}$, the cohomology theories will be bigraded. If $E$ is a $T$-spectrum, let $E(q)[p]$ denote $S^{p,q} \wedge E$. Now, the bigraded cohomology theory associated to $E$ is given by

$$E^{p,q}(X) = \text{Hom}_{\text{SHot}_T}(\Sigma^\infty_T(X_+), E(q)[p]).$$

We can also define the reduced cohomology theory $\tilde{E}^{p,q}$ in the usual way via the short exact sequence

$$0 \to \tilde{E}^{p,q}(X) \to E^{p,q}(X) \to E_{*,*} = E^{p,q}(pt) \to 0$$

We will mostly be concerned with the Eilenberg-MacLane spectra defined in the next section and related spectra.

### 3.2 Motivic Cohomology

One way to construct the Eilenberg-MacLane spectrum in topology is to start with a Moore space, such as a sphere, and use the Dold-Thom theorem which says that the group completion of the infinite symmetric product of a space $X$ is a product of Eilenberg-MacLane spaces:

$$\text{Sym}^\infty(X)^+ \simeq \prod_{n \geq 0} K(H_n(X), n)$$

In particular, the infinite symmetric product of a Moore space gives the associated Eilenberg-MacLane space.

Voevodsky’s approach to constructing the Eilenberg-MacLane spectrum consisted of finding an analogue of the Dold-Thom theorem in the motivic case. It turns out that the taking the “free presheaf with transfers” is the right approach [SV96]. For a smooth schemes $X, Y$, and for a ring $R$, let $L(X, R)(Y)$ be the free abelian group generated by cycles $Z \subset Y \times X$ which are finite and surjective over $Y$. We will denote $L(X, \mathbb{Z})$ simply by $L(X)$. $L$ extends uniquely to a functor

$$L : \text{Spc}_* \to \text{AbPre}(\text{Sm}/\mathbb{S})$$

from the category of spaces to the category of presheaves of abelian groups which preserves colimits (cf. [MVW] for different approaches to $L$).
Theorem 3.2.1. [SV96] Let $X, U$ be a smooth schemes. Then the $L(X)(U)$ is the group completion of the abelian monoid $\text{Hom}(U, \sqcup_{n \geq 0} S^n X)$ where $S^n X = X^n / \Sigma_n$, the $n$-th symmetric product of $X$.

In fact [MVW], $L(X, R)$ is an $A^1$-invariant sheaf in the Nisnevich topology, and therefore gives an element of $\text{Spc}$.

We apply this construction to the sheaves $T^n$ to get the Eilenberg-MacLane spaces

$$K(R(n), 2n) = L(T^n, R) = L(A^n, R) / L(A^n - \{0\}, R).$$

The external product of cycles gives the maps

$$K(R(m), 2m) \wedge K(R(n), 2n) \to K(R(m + n), 2(m + n)).$$

Composing these with the natural map

$$T = A^1 / (A^1 - \{0\}) \to L(A^1) / L(A^1 - \{0\})$$

we get bounding maps

$$T \wedge K(R(n), 2n) \to K(R(n + 1), 2n + 2).$$

Thus the spaces $K(R(n), 2n)$ form a ring spectrum $HR$.

Definition 3.2.2. The motivic cohomology of $X$ with coefficients in a ring $R$ is the $HR$ cohomology of $X$, i.e.

$$H^{p,q}(X; R) = \text{Hom}_{\text{SHot}^r}(\Sigma^\infty_+ T, HR(q)[p]).$$

There is another, equivalent approach to motivic cohomology using hypercohomology. By the Dold-Kan correspondence, $HR(q)[p]$ corresponds to a chain complex of Nisnevich sheaves of $R$-modules $R(q)[p] = Z(q)[p] \otimes R$. We then have [Bro74],

$$\text{Hom}_H(X, HR(q)[p]) = \mathbb{H}^p(X; R(q))$$

the hypercohomology with coefficients in $R(q)$.

In what follows, the following properties of the complexes $Z(q)$ and of motivic cohomology, which follow from the above discussion, that will be very useful to us:

- $Z(0)$ is the constant sheaf $\mathbb{Z}$.
- $Z(1) = O^*$, i.e. it is represented by $G_m = A^1 - \{0\} = \Sigma^{-1}_s T$.
- For a smooth scheme $X$ over a field $k$,

$$H^{2n,n}(X) = A^n(X)$$

where $A^n(X)$ is the Chow group of cycles of codimension $n$ on $X$ modulo rational equivalence. In fact, the rings $\oplus_n H^{2n,n}(X)$ and $A^*(X)$ are isomorphic.
• Let $X \xrightarrow{f} Y \xrightarrow{i} C_f$ be a cofiber sequence in $\text{Spc}$. Then we have a long exact sequence in motivic cohomology:

$$\ldots \to \tilde{H}^{*,*}(C_f) \xrightarrow{i_*} H^{*,*}(Y) \xrightarrow{f_*} H^{*,*}(X) \to \tilde{H}^{*+1,*}(C_f) \to \ldots$$

We will be concerned exclusively with motivic cohomology with $\mathbb{Z}$ and $\mathbb{Z}/2$ coefficients. Since not very much is known about the motivic cohomology of a point it is convenient to calculate the motivic cohomology of spaces as a module over $H^{*,*}(\text{pt})$.

Let $M^{*,*}$ and $M_2^{*,*}$ denote $H^{*,*}(\text{pt})$ and $H^{*,*}(\text{pt}; \mathbb{Z}/2)$ respectively.

We clearly have, $M_{p,q} = 0$ if $q < 0$, $p > q \geq 0$ or if $q = 0$ and $p < 0$. Also, $M^{1,1} = H^0(\text{Spec } k, \mathbb{G}_m) = k^*$ and $M^{0,1} = 0$.

To relate these two cohomology theories, consider the short exact sequence

$$0 \to \mathbb{Z} \times 2 \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

This gives a long exact sequence in cohomology,

$$\ldots H^{*,*}(X; \mathbb{Z}) \to H^{*,*}(X, \mathbb{Z}) \to H^{*,*}(X, \mathbb{Z}/2) \to H^{*+1,*}(X; \mathbb{Z}) \to \ldots$$

Composing the boundary map and the restriction, we get the Bockstein

$$\beta : H^{*,*}(X; \mathbb{Z}/2) \to H^{*+1,*}(X; \mathbb{Z}/2).$$

It has the same properties as the Bockstein in topology, i.e. $\beta^2 = 0$ and $\beta(uv) = \beta(u)v + u\beta(v)$.

Since $M^{1,1} = k^*$ and $M^{0,1} = M^{2,1} = 0$, we have an exact sequence

$$0 \to M^{0,1} \to k^* \to k^* \to M^{2,1} \to 0$$

where the map $k^* \to k^*$ maps $x$ to $x^2$. Now, let $\tau \in M^{0,1}$ be the class which maps to $-1$, and let $\rho \in M^{2,1}$ be the image of $-1$. By definition, we have that $\beta(\tau) = \rho$.

### 3.3 Thom Isomorphism Theorem

One of the key tools in calculating motivic cohomology is the Thom isomorphism theorem, which is the analogue of the Thom isomorphism in topology.

**Theorem 3.3.1.** [Voe03] Let $E$ be a vector bundle of rank $n$ over $B$. Then there is an isomorphism

$$H^{*,*}(B_+) \to \tilde{H}^{*+2n,*+n}(\text{Th}(E))$$

given by the cup product $a \mapsto a t_E$ where $t_E \in \tilde{H}^{2n,n}(\text{Th}(E))$ is the Thom class.
For a vector bundle $E$ of rank $n$ over $B$, we can define the *Euler class* $e(E) \in H^{2n,n}(B)$ as the restriction of $t_E$ with respect to the zero section map $X_+ \rightarrow Th(E)$. For a line bundle $L$, we have that $e(L)$ coincides with the canonical class of $L$ in $H^{2,1}(B)$ [Voe03].

An important consequence of the Thom isomorphism theorem is the Gysin sequence. Let $Z \hookrightarrow X$ be a closed embedding of codimension $c$. By the homotopy purity theorem, we have a homotopy cofiber sequence

$$X - Z \xrightarrow{i} X \rightarrow Th(N_{X,Z})$$

Applying the Thom isomorphism theorem,

$$H^*(Th(N_{X,Z})) \cong H^{*-2c,*-c}(Z)$$

we get the long exact sequence

$$\ldots \rightarrow H^{*-2c,*-c}(Z) \rightarrow H^{*,*}(X) \rightarrow H^{*,*}(X - Z) \rightarrow H^{*-2c+1,*-c}(Z) \rightarrow \ldots$$

We can use this result to calculate the cohomology of $\mathbb{P}^n$. First, let’s compute the additive structure. From the inclusion $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ with $\mathbb{P}^n - \mathbb{P}^{n-1} = \mathbb{A}^n$, we have

$$\hat{H}^{*-1,*}(\mathbb{A}^n) = 0 \rightarrow \hat{H}^{*-2,*-1}(\mathbb{P}^{n-1}) \rightarrow \hat{H}^{*,*}(\mathbb{P}^n) \rightarrow \hat{H}^{*,*}(\mathbb{A}^n) = 0$$

In particular, we have

$$\hat{H}^{*-2,*-1}(\mathbb{P}^{n-1}) = H^{*,*}(\mathbb{P}^n)$$

as $\mathbb{M}$-modules. Since $\mathbb{P}^0 = \text{Spec } k$, we have $H^{*,*}(\mathbb{P}^0) = \mathbb{M}$. By induction, we get

$$H^{*,*}(\mathbb{P}^n) = \mathbb{M} \oplus t_1 \mathbb{M} \oplus \cdots \oplus t_n \mathbb{M}$$

as an $\mathbb{M}$-module, where $\deg(t_i) = (2i, i)$. Since all the generators of $H^{*,*}(\mathbb{P}^n)$ are in degree $(2k, k)$, the ring structure coincides with the ring structure for the Chow ring. For $\mathbb{P}^n$, we have $A^*(\mathbb{P}^n) = \mathbb{Z}[t]/t^{n+1}$ which is generated in degree $n - k$ by $[L^k]$ where $L^k$ is a $k$-dimensional linear subspace of $\mathbb{P}^n$. It follows that

$$H^{*,*}(\mathbb{P}^n) = \mathbb{M}[t]/t^{n+1}.$$ 

### 3.4 Cellular Spaces and the Künneth Theorem

Another very useful tool for computing cohomology is the Künneth theorem. Unfortunately, there isn’t a general Künneth theorem for motivic cohomology. Nevertheless, such a theorem does exist for a relatively broad class of objects, namely cellular spaces [DI05], and this will suffice for our purposes.

**Definition 3.4.1.** The class of finite cell complexes in $\mathbf{Spc}$ is the smallest class of spaces such that
1. the class contains the spheres $S^{p,q}$;

2. if $X$ is weakly equivalent to a finite cell complex then $X$ is a finite cell complex;

3. if $X \to Y \to Z$ is a homotopy cofiber sequence and two of the objects are in the class, then so is the third.

**Theorem 3.4.2.** [DI05] Let $A, B \in \text{Spc}$, $E \in \text{T-Spectra}$ such that $A$ is a finite cell complex and $E^{*,*}(A)$ is free as a $E_{*,*}$-module. Then there is a Künneth isomorphism

$$E^{*,*}(A) \otimes_{E_{*,*}} E^{*,*}(B) \cong E^{*,*}(A \wedge B).$$

Cellular spaces play a very important role in topology; in particular, every space is weakly equivalent to a cellular one. A natural question, then, is whether a similar statement is true in motivic homotopy. It is folklore in algebraic geometry that most spaces cannot be cellular. One should be able to see this using a suitable theory of weights. Since spheres only have even weights in their cohomology, it should be impossible to construct schemes with odd weights from spheres. However, no suitable theory of weights exists for motivic cohomology at present.
Chapter 4

The Motivic Steenrod Algebra and Its Dual

Cohomology operations play an important role in any cohomology theory as they describe a finer structure of the theory. The (motivic) Steenrod algebra $A^{\ast\ast}$ is the algebra of stable (motivic) cohomology operations with coefficients in $\mathbb{Z}/2$. These operations also play an important role in motivic cohomology. For instance, Voevodsky used the structure of the motivic Steenrod algebra in his proof of the Milnor conjecture.

The Yoneda lemma states that $A^{\ast\ast} = H^*H$. Thus the problem of determining the Steenrod algebra is reduced to computing the cohomology of the Eilenberg-MacLane spectrum. We approach the problem in this way, rather then as [Voe03] which explicitly constructs the cohomology operations.

In spirit, our argument follows Milnor’s classic paper [Mil58] in that we consider the action on the cohomology of $B\mu_2$. We also make use of the more modern idea relating cohomology theories and formal groups laws. Incorporating this machinery into motivic homotopy theory, we obtain an algebra closely related to the dual of the motivic Steenrod algebra and work out its structure.

4.1 Cohomology of $B\mu_2$

Recall that the group scheme $\mu_2$ is defined as the kernel

$$0 \to \mu_2 \to \mathbb{G}_m \times \mathbb{G}_m \to 0$$

The approach that we take to describing the motivic Steenrod algebra is by considering its action on the cohomology of $B_{gm}\mu_2$. We will only consider $B_{gm}\mu_2$ in this chapter so since there is no ambiguity we will refer to it simply as $B\mu_2$. We define, as before, $E\mathbb{G}_m$ to be the simplicial sheaf of sets such that for $U \in \text{Spc}$, the simplicial set $E\mathbb{G}_m(U) = E(\mathbb{G}_m(U))$. Now, let

$$B\mu_2 = E\mathbb{G}_m \times_{\mathbb{G}_m} \mathbb{G}_m$$
where the action of $G_m$ on $G_m$ is given by $a(b) = a^2 b$, for $a, b \in G_m$. As a simplicial sheaf, we have,

$$(B\mu_2)_n = G_m \times \ldots \times G_m \ (n - \text{times})$$

Now, by Lemma 2.3.1, we have that $EG_m = E_{gm}G_m = \mathbb{A}^\infty - \{0\}$. Therefore, we get another, weakly equivalent, model for $B\mu_2$ given by $X = (\mathbb{A}^\infty - \{0\})/\mu_2$, the $n$-dimensional approximation of $B\mu_2$. For $X_n = \mathbb{A}^n - \{0\}/\mu_2$, the $n$-dimensional approximation of $B\mu_2$. For $X_n$, we have the following fiber bundle

$$\mathbb{A}^1 - 0/\mu_2 \longrightarrow \mathbb{A}^n - 0/\mu_2$$

We clearly have,

$$(\mathbb{A}^1 - 0)/\mu_2 = \text{Spec } k[t, t^{-1}]/\mu_2 = \text{Spec } k[t^2, t^{-2}] \cong \mathbb{A}^1 - \{0\}.$$
We have that \( z^*(t_E) = e(E) \) by definition, and we know \( e(O(-2)) = 2\sigma \) where \( \sigma = c_1(O(-1)) \) is the generator. We thus get a long exact sequence

\[
\ldots \rightarrow H^{*-2,*-1}(\mathbb{P}^{n-1}) \xrightarrow{2\sigma} H^{*,*}(\mathbb{P}^{n-1}) \rightarrow H^{*,*}(X_n) \rightarrow \ldots
\]

Now, passing to \( \mathbb{Z}/2 \) coefficients, \( 2\sigma = 0 \) and so we get the short exact sequence

\[
0 \rightarrow H^{*,*}(\mathbb{P}^{n-1}; \mathbb{Z}/2) \rightarrow H^{*,*}(X_n; \mathbb{Z}/2) \rightarrow H^{*-1,*-1}(\mathbb{P}^{n-1}) \rightarrow 0
\]

Let \( v \in H^{2,1}(X_n) \) be the image of the generator of \( H^{*,*}(\mathbb{P}^{n-1}) \). Since the outer terms in the exact sequence are free \( M_2 \)-modules, the middle term is also free. It follows that

\[
H^{1,1}(X_n; \mathbb{Z}/2) \cong M_2^{1,1} \oplus uM_2^{0,0}
\]

where \( u \in H^{1,1}(X_n; \mathbb{Z}/2) \) is the unique nonzero element whose restriction to a point is zero. Furthermore, from the exact sequence we have that \( H^{*,*}(X_n; \mathbb{Z}/2) \) is generated by \( v_i \) and \( uv^t \) with \( v^n = 0 \) as an \( M_2 \)-module. It remains to determine \( u^2 \).

From the long exact sequence in \( \mathbb{Z} \) coefficients, we have

\[
\mathbb{Z} = H^{0,0}(\mathbb{P}^{n-1}) \xrightarrow{2\sigma} H^{2,1}(\mathbb{P}^{n-1}) = \sigma\mathbb{Z} \rightarrow H^{2,1}(X_n)
\]

Thus for \( \tilde{v} \in H^{2,1}(X_n) \), the image of the generator of \( H^{*,*}(\mathbb{P}^{n-1}) \), we have \( 2\tilde{v} = 0 \).

Now, using the long exact sequence defining the Bockstein map,

\[
H^{1,1}(X_n; \mathbb{Z}) \rightarrow H^{1,1}(X_n; \mathbb{Z}/2) \xrightarrow{\partial} H^{2,1}(X_n; \mathbb{Z}) \xrightarrow{x^2} H^{2,1}(X_n; \mathbb{Z})
\]

we see that there exists \( u' \in H^{1,1}(X_n; \mathbb{Z}/2) \) such that \( \partial(u') = \tilde{v} \). We then have \( u' = au + b \) where \( a \in M_2^{0,0} = \mathbb{Z}/2 \) and \( b \in M_2^{1,1} \). Since the restriction of \( u \) to a point vanishes, the restriction of \( \partial(u') \) to a point is just \( \partial(b) \). But since \( \partial(u') = \tilde{v} \), and the restriction of \( \tilde{v} \) to a point vanishes, we have that \( \partial(u') = 0 \). In particular, since \( \tilde{v} \neq 0 \), \( \partial(u) = \tilde{v} \). Since \( v \in H^{2,1}(X_n; \mathbb{Z}/2) \), is just the mod-2 reduction of \( \tilde{v} \), we have

\[
\beta(u) = v.
\]

Now, consider the constant simplicial sheaf \( B\mathbb{Z}/2 = K(\mathbb{Z}/2, 1) \). We have a map \( B\mathbb{Z}/2 \rightarrow B\mu_2 \) given by the following diagram

\[
\begin{array}{cccc}
\mathbb{Z}/2 \times \mathbb{Z}/2 & \longrightarrow & \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m \\
\downarrow & & \downarrow & \downarrow \\
\mathbb{Z}/2 & \longrightarrow & \mathbb{G}_m \times \mathbb{G}_m \\
\downarrow & (\cdot, 1) & \downarrow & \downarrow \\
\bullet & \longrightarrow & \mathbb{G}_m
\end{array}
\]
Now, let’s compute the map on cohomology. Since the higher cohomology groups vanish, we have that the mod-2 $H^{*,1}$ cohomology of $B\mu_2$ is given by the hypercohomology of

$$H^0(\mathbb{G}_m \times \mathbb{G}_m; \mathbb{G}_m) \xrightarrow{(\cdot)^2} H^0(\mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m; \mathbb{G}_m) \xrightarrow{(\cdot)^2} H^0(\mathbb{G}_m; \mathbb{G}_m)$$

since the chain complex $\mathbb{Z}(1) \otimes \mathbb{Z}/2$ is given by $\mathbb{Z}(1) = \mathbb{G}_m \xrightarrow{\cdot 2} \mathbb{Z}(1) = \mathbb{G}_m$. We have that $\mathbb{G}_m = \text{Spec} k[t, t^{-1}]$ and therefore, $H^0(\mathbb{G}_m; \mathbb{G}_m) = k^* \times \mathbb{Z}$, and is the multiplicative group given by monomials in $t$. Now, since the Bockstein of $u$ is $v$, we see that in $H^0(\mathbb{G}_m \times \mathbb{G}_m; \mathbb{G}_m)$, $u$ comes from the class $(t, 1)$ and therefore it is $t$ in $H^0(\mathbb{G}_m; \mathbb{G}_m)$ because the map

$$H^0(\mathbb{G}_m \times \mathbb{G}_m; \mathbb{G}_m) \to H^0(\mathbb{G}_m \times \mathbb{G}_m; \mathbb{G}_m)$$

is given by squaring, and the map

$$H^0(\mathbb{G}_m; \mathbb{G}_m) \to H^0(\mathbb{G}_m \times \mathbb{G}_m; \mathbb{G}_m)$$

is given by $t \mapsto s/t^2 s$. Now, considering the analogous diagram of $B\mathbb{Z}/2$, we see that the class given by $x \tau$ maps to $u$, where $x \in H^{1,0}(B\mathbb{Z}_2; \mathbb{Z}_2) = H^{1,0}(K(\mathbb{Z}/2, 1); \mathbb{Z}/2)$ is the canonical element. Now, for degree $(1, 0)$ the classifying map for the Bockstein

$$K(\mathbb{Z}/2, 1) \to K(\mathbb{Z}/2, 2)$$

agrees with the classifying map for squaring (because it’s true in topology for the simplicial spaces). Therefore, $\beta(x) = x^2$. Using the same diagram, we see that the map on $H^{2,1}$ is also injective (since $v$ maps to something non-zero).

Abusing notation, and identifying $u, v$ with their images in $H^{*,1}(B\mathbb{Z}/2; \mathbb{Z}/2)$, we have, $v = \beta(u) = \beta(x \tau) = x^2 \tau + x \rho$. It follows that

$$u^2 = x^2 \tau^2 = v \tau + x \rho \tau = v + \rho u.$$ 

Hence, we have $H^{*,*}(X_n; \mathbb{Z}/2) = M_*^{*,*}[u, v]/(u^2 = \tau v + \rho u, \rho^{n+1})$. Since the maps on cohomology induced by the inclusion maps $X_n \hookrightarrow X_{n+1}$ are injective, we have $H^{*,*}(B\mu_2; \mathbb{Z}/2) = M_*^{*,*}[u, v]/(u^2 = \tau v + \rho u)$.

4.2 Motivic Cohomology and its Formal Group Law

We recall from [Haz78] that an $n$-dimensional formal group law over a ring $A$ is an $n$-tuple of power series $F(X, Y) = (F_1(X, Y), \ldots, F_n(X, Y))$ in $2n$ indeterminates
$X_1, \ldots, X_n, Y_1, \ldots, Y_n$ such that

\[ F_i(X, Y) = X_i + Y_i \mod \text{(degree 2)}, \ i = 1, \ldots, n \]

\[ F_i(F(X, Y), Z) = F_i(X, F(Y, Z)), \ i = 1, \ldots, n \]

Furthermore, if we also have that

\[ F_i(X, Y) = F_i(Y, X), \ i = 1, \ldots, n \]

then we say that the formal group law $F$ is commutative. We clearly have that

\[ F_i(X, Y) = \sum_{j_1, \ldots, j_n, k_1, \ldots, k_n} a_i^{j_1, \ldots, j_n, k_1, \ldots, k_n} \prod X_i^{j_i} \prod Y_m^{k_m} \]

Now, suppose we have a homomorphism $f : A \to B$. We then get a formal group law $f^*(F)$ given by

\[ f^*(F)_i(X, Y) = \sum_{j_1, \ldots, j_n, k_1, \ldots, k_n} f(\alpha_i^{j_1, \ldots, j_n, k_1, \ldots, k_n}) \prod X_i^{j_i} \prod Y_m^{k_m} \]

From the description above, it is clear that formal group laws of dimension $n$ are corepresented by the ring $\Lambda_n = \mathbb{Z}[a_i^{j_1, \ldots, j_n, k_1, \ldots, k_n}]/I$ where $I$ is the ideal generated by the relation on coefficients induced by the conditions for $F$ to be a formal group law.

Now, suppose $F, G$ are formal group laws over $A$, a homomorphism over $A$, $\alpha : F(X, Y) \to G(X, Y)$ is an $n$-tuple of power series $\alpha(X)$ in $n$ indeterminates such that

\[ \alpha(X) \equiv 0 \mod \text{(degree 1)} \]

and

\[ \alpha(F(X, Y)) = G(\alpha(X), \alpha(Y)) \]

The homomorphism $\alpha$ is an isomorphism if there exists a homomorphism $\beta(X) : G(X, Y) \to F(X, Y)$ such that $\alpha(\beta(X)) = X$ and $\beta(\alpha(X)) = X$. Furthermore, we have that an isomorphism $\alpha$ is called a strict isomorphism if $\alpha(X) \equiv X \mod \text{(degree 2)}$. As it will turn out, we will mostly be concerned with strict isomorphisms. In fact, for $F, G$ formal groups laws of dimension $n$ over rings $A, B$ respectively, we have a functor

\[ \text{StrictIso}(F, G) : \text{Rings} \to \text{Sets} \]

which is given by

\[ \text{StrictIso}(F, G)(R) = \{ f : A \to R, g : B \to R, \alpha : f^*F \to g^*G \ \text{a strict isomorphism} \} \]

Thus elements of $\text{StrictIso}(F, G)(R)$ consist pullbacks of $F$ and $G$ to $R$ and a strict isomorphism between them.
Now, let’s get back to motivic cohomology. The Segre map (which classifies tensor products of line bundles) gives us the following diagram

\[
\begin{array}{cccc}
(O(-2) \otimes O(-2)) - z(P^{n-1}) & \rightarrow & B_{\mu_2} \times B_{\mu_2} \rightarrow \hat{S} & \rightarrow B_{\mu_2} \\
\rightarrow & & \rightarrow & \\
\mathbb{P}^\infty & \rightarrow & \mathbb{P}^\infty \times \mathbb{P}^\infty & \rightarrow \mathbb{P}^\infty
\end{array}
\]

Now, on cohomology, we have

\[
H^{*,*}(\mathbb{P}^\infty; \mathbb{Z}/2) = M_2[v] \rightarrow M_2[u_1, v_2]
\]

\[
H^{*,*}(B_{\mu_2}; \mathbb{Z}/2) = M_2[u, v]/(u^2 = \tau v + \rho u) \rightarrow M_2[u_1, u_2, v_1, v_2]/(u_1^2 = \tau v_1 + \rho u_1)
\]

It follows that \(\hat{S}^*(v) = v_1 + v_2\) since this is true for Euler classes of line bundles.

Now, let’s determine \(\hat{S}^*(u)\). By the same argument that we used to obtain the short exact sequence for \(\mathbb{Z}/2\) cohomology, we have a commutative diagram

\[
\begin{array}{cccc}
0 & \rightarrow & H^{*,*}(\mathbb{P}^\infty; \mathbb{Z}/2) & \rightarrow H^{*,*}(B_{\mu_2}; \mathbb{Z}/2) & \rightarrow H^{*-1,*-1}(\mathbb{P}^\infty; \mathbb{Z}/2) & \rightarrow 0 \\
\rightarrow & & \rightarrow & & \rightarrow & \\
0 & \rightarrow & H^{*,*}(\mathbb{P}^\infty \times \mathbb{P}^\infty; \mathbb{Z}/2) & \rightarrow H^{*,*}((B_{\mu_2})^2; \mathbb{Z}/2) & \rightarrow H^{*-1,*-1}(\mathbb{P}^\infty \times \mathbb{P}^\infty; \mathbb{Z}/2) & \rightarrow 0
\end{array}
\]

Since the image of \(u \in H^{1,1}(B_{\mu_2}; \mathbb{Z}/2)\) is \(1 \in H^{0,0}(\mathbb{P}^\infty; \mathbb{Z}/2)\), we have that \(\hat{S}^*(u) \neq 0\).

Now, consider the following diagram

\[
\begin{array}{cc}
B_{\mu_2} \times \{\ast\} & \rightarrow B_{\mu_2} \times B_{\mu_2} \leftarrow \{\ast\} \times B_{\mu_2} \\
\downarrow S_L & \downarrow S_R \\
B_{\mu_2} & \rightarrow B_{\mu_2}
\end{array}
\]

Since the tensor product of line bundles is commutative, we have that \(S_L^* = S_R^*\). Now, we have that

\[
H^{1,1}(B_{\mu_2} \times B_{\mu_2}) = \bigoplus_{i_1+i_2=1, j_1+j_2=1} H^{i_1,j_1}(B_{\mu_2}; \mathbb{Z}/2) \otimes H^{i_2,j_2}(B_{\mu_2}; \mathbb{Z}/2)
\]

If \(j_1 < 0\) then \(H^{i_1,j_1}(B_{\mu_2}; \mathbb{Z}/2) = 0\) and so \(j_1 = 1, j_2 = 0\) or vice versa. Suppose without loss of generality, \(j_1 = 1, j_2 = 0\); if \(i_2 < 0\) or \(i_2 \geq 1\) then \(H^{i_2,j_2}(B_{\mu_2}; \mathbb{Z}/2) = 0\). Thus, we have

\[
H^{1,1}((B_{\mu_2})^2; \mathbb{Z}/2) = \bigoplus_{i=0,1} (H^{i,i}(B_{\mu_2}; \mathbb{Z}/2) \otimes H^{1-i,1-i}(B_{\mu_2}; \mathbb{Z}/2))
\]

\[
= H^{1,1}(B_{\mu_2}; \mathbb{Z}/2) \oplus H^{1,1}(B_{\mu_2}; \mathbb{Z}/2)
\]
In particular, the map \( H^{1,1}(B\mu_2 \times B\mu_2) \to H^{1,1}(B\mu_2 \vee B\mu_2; \mathbb{Z}/2) \) is an isomorphism. It follows that \( \bar{S}^*(u) = S_L^*(u) \oplus S_R^*(u) \). In particular, since \( \bar{S}^*(u) \neq 0 \), we have that \( S_L^*(u) = S_R^*(u) = u \) and so \( \bar{S}^*(u) = u_1 + u_2 \).

To summarize, we have the map

\[
B\mu_2 \times B\mu_2 \to B\mu_2
\]

classifying tensor product of bundles induces the map on cohomology

\[
M_2[\mathbb{Z}/2] / (u^2 = \tau v + \rho u) \to M_2[X_1, X_2, Y_1, Y_2] / (X_1^2 = \tau Y_1 + \rho X_1)
\]
given by \( u \mapsto X_1 + X_2 \) and \( v \mapsto X_1 + X_2 \). This gives us a 2-dimensional formal group law

\[
F(X_1, X_2, Y_1, Y_2) = (X_1 + X_2, Y_1 + Y_2)
\]

This is the formal group law associated to the spectrum \( \mathbb{Z}/2 \).

Now, we get maps on spectra corresponding to the two factors and the elements \( u, v \in M_2^*(B\mu_2) \):

We thus get two pairs of cohomology classes \( u_1, v_1 \) and \( u_2, v_2 \) in \( (\mathbb{Z}/2 \wedge \mathbb{Z}/2)^{*,*}(B\mu_2) \).

Unlike in topology, it is not true here that every module over \( \mathbb{Z}/2 \) is free. However, it is a theorem of Voevodsky (which unfortunately is not at present published) that

\[
\mathbb{Z}/2 \wedge \mathbb{Z}/2 \cong \bigvee_{\alpha} \Sigma^n \mathbb{Z}/2 \wedge \bigwedge_{\alpha} \Sigma^n \mathbb{Z}/2
\]

We thus have \( (\mathbb{Z}/2 \wedge \mathbb{Z}/2)^{*,*}(X) = (\mathbb{Z}/2 \wedge \mathbb{Z}/2)_{*,*,*} \otimes_{M_2} H^{*,*}(X; \mathbb{Z}/2) \). In particular,

\[
(\mathbb{Z}/2 \wedge \mathbb{Z}/2)^{*,*}(B\mu_2) = (\mathbb{Z}/2 \wedge \mathbb{Z}/2)_{*,*,*}[u_1, v_1] / (u_1^2 = \tau v_1 + \rho u_1)
\]

Thus we can express \( u_2 \) and \( v_2 \) as power series

\[
u_2 = \alpha_1(u_1, v_1); \quad v_2 = \alpha_2(u_1, v_1)
\]

Consider the restriction

\[
(\mathbb{Z}/2 \wedge \mathbb{Z}/2)^{1,1}(B\mu_2) \to (\mathbb{Z}/2 \wedge \mathbb{Z}/2)^{1,1}(\mathbb{A}^1 - 0/\mathbb{Z}/2) = (\mathbb{Z}/2 \wedge \mathbb{Z}/2)^{0,0}(*)
\]
Since both $u_L$ and $u_R$ have to map to the image of 1 in $(\mathbb{Z}/2 \wedge \mathbb{Z}/2)^{0,0}(\ast)$, we see that $u_R \equiv u_L \mod \text{(degree 2)}$. Similarly, considering the restriction

$$(\mathbb{Z}/2 \wedge \mathbb{Z}/2)^{2,1}(B\mu_2) \longrightarrow (\mathbb{Z}/2 \wedge \mathbb{Z}/2)^{2,1}(\mathbb{A}^2 - 0/\mathbb{Z}/2)$$

we see that since $v_L$ and $v_R$ must map to the image of 1, $v_R \equiv v_L \mod \text{(degree 2)}$. Now, applying $(\mathbb{Z}/2 \wedge \mathbb{Z}/2)^{\ast,\ast}$ to the Segre map, we get a homomorphism

$$(\mathbb{Z}/2 \wedge \mathbb{Z}/2)^{\ast,\ast}(B\mu_2) \longrightarrow (\mathbb{Z}/2 \wedge \mathbb{Z}/2)^{\ast,\ast}(B\mu_2 \times B\mu_2)$$

$u_L, v_L \mapsto X_1^L + Y_1^L, X_2^L + Y_2^L$

$u_R, v_R \mapsto X_1^R + Y_1^R = \alpha_1(X_1^L + Y_1^L), X_2^R + Y_2^R = \alpha_2(X_2^L + Y_2^L)$

Thus, $\alpha$ is a strict isomorphism of the formal group law. Therefore, we get a natural transformation of functors

$$\text{Hom}((\mathbb{Z}/2 \wedge \mathbb{Z}/2)^{\ast,\ast}, -) \rightarrow \text{StrictIso}(F, F)$$

**Proposition 4.2.1.** The functor $\text{StrictIso}(F, F)$ is corepresented by $M_2[\tau_i, \xi_i]$, with $\text{deg}(\tau_i) = (2^{i+1} - 1, 2^i - 1)$ and $\text{deg}(\xi_i) = (2^{i+1} - 2, 2^i - 1)$.

**Proof.** Suppose over some ring $R$, we have a strict isomorphism $\phi$. Abusing notation, we will denote by $\tau, \rho$ the images of these elements in $R$. Now, $\phi(u)$ and $\phi(v)$ are given by some power series in $M = R[[u, v]]/(u^2 = \tau v + \rho u)$. Over $R$, $M$ is clearly generated by $\{v^i, uv^j\}_{i \geq 0}$. Thus, we have

$$\phi(u) = \sum_i (a_{2i}v^i + a_{2i+1}uv^i)$$

$$\phi(v) = \sum_i (b_{2i}v^i + b_{2i+1}uv^i)$$

Since $\phi$ is a strict isomorphism, we have

$$a_0 = b_0 = b_1 = 0 \quad \text{and} \quad a_1 = b_2 = 1$$

Now, since $\phi$ preserves the formal group law, we have that

$$\phi(X_1 + X_2) = \phi(X_1) + \phi(X_2), \quad \phi(Y_1 + Y_2) = \phi(Y_1) + \phi(Y_2)$$

over the ring $R[[X_1, X_2, Y_1, Y_2]]/(X_2^2 = \tau Y_1 + \rho X_1)$. In particular, we get that

$$\phi(u) = u + \sum_{i \geq 0} \tau_i v^{2i}$$

and
\[ \phi(v) = v + \sum_{i \geq 1} \xi_i v^{2i} \]

This gives us a map \( \mathbb{M}_2[\tau_i, \xi_{i+1}]_{i \geq 0} \to R \), and in fact it is clear that any such map gives a strict isomorphism. \( \square \)

As a corollary, we get a canonical map \( \mathbb{M}_2[\tau_i, \xi_i] \to (H\mathbb{Z}/2 \wedge H\mathbb{Z}/2)_{*,*} \). Let \( A_{*,*} \) be the image under this map. We will be largely concerned with establishing the structure of \( A_{*,*} \) and its dual.

### 4.3 Cohomology Operations

The spectrum \( H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \) has two structures as an \( H\mathbb{Z}/2 \) module. Let \( i_L, i_R : H\mathbb{Z}/2 \to H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \) be the left and right module structure maps respectively. Now, the map

\[ i_R : H\mathbb{Z}/2 \to H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \]

is a map of commutative ring spectra. Now, applying this to \( B\mu_2 \), we get a ring homomorphism

\[ \lambda : \text{H}^{*,*}(B\mu_2; \mathbb{Z}/2) \to \text{H}^{*,*}(B\mu_2; \mathbb{Z}/2) \otimes_{\mathbb{M}_2} (H\mathbb{Z}/2 \wedge H\mathbb{Z}/2)^{*,*} \]

The image of \( \lambda \) lies in \( \text{H}^{*,*}(B\mu_2; \mathbb{Z}/2) \otimes_{\mathbb{M}_2} A_{*,*} \), and in fact it is given by Proposition 4.2.1 (since we use the left \( H\mathbb{Z}/2 \) module structure of \( A_{*,*} \subset (H\mathbb{Z}/2 \wedge H\mathbb{Z}/2)_{*,*} \) in the tensor product). Thus we have,

\[ \lambda(s) = s \otimes 1 + \sum_{j \geq 0} t^{2j} \otimes \tau_j \]

and

\[ \lambda(t) = t \otimes 1 + \sum_{j \geq 1} t^{2j} \otimes \xi_j \]

**Proposition 4.3.1.** On \( \mathbb{M}_2 \), \( \lambda \) is given by

\[ \lambda(\tau) = \tau \otimes 1 + \rho \otimes \tau_0, \quad \lambda(\rho) = \rho \otimes 1. \]

Furthermore, in \( A_{*,*} \), \( \tau_i^2 = \tau \xi_{i+1} + \rho \xi_{i+1} + \rho \tau_0 \xi_{i+1} \).

**Proof.** As before, we can write \( u = x\tau \). Since \( \tau \) lives in the cohomology of a point, we can work in \( \mathbb{M}_2[x]/(x^2) = \text{H}^{*,*}(B\mu_2; \mathbb{Z}/2)/I \), for some ideal \( I \), to determine \( \lambda(\tau) \). We have then have,

\[ \lambda(u) = u \otimes 1 + \sum_{j \geq 0} v^{2j} \otimes \tau_j = \lambda(x\tau) \cong x\lambda(\tau) \mod I \]
In particular, since $v = x\rho + x^2\tau$, we get that
\[
\lambda(\tau) = \tau \otimes 1 + \rho \otimes \tau_0.
\]
Similarly, for $\rho$, we have
\[
\lambda(v) = v \otimes 1 + \sum_{j \geq 1} v^{2j} \otimes \xi_j = \lambda(x\rho + x^2\tau) \equiv x\lambda(\rho) \mod I
\]
and so $\lambda(\rho) = \rho \otimes 1$.

Now, we have,
\[
\lambda(s^2) = \tau b \otimes 1 + \rho a \otimes 1 + \sum v^{2i+1} \otimes \tau_j^2
\]
\[
\lambda(\tau v) = \sum \tau v^{2j} \otimes \xi_j + \sum \rho v^{2j} \otimes \tau_0 \xi_j
\]
\[
\lambda(\rho a) = \rho a \otimes 1 + \sum \rho b^{2j} \otimes \tau_j
\]

Using the relation $\lambda(u^2) = \lambda(\tau v) + \lambda(\rho u)$ and equating the coefficients of $v^{2j}$, we get
\[
\tau_j^2 = \tau \xi_{j+1} + \rho \tau_{j+1} + \rho \tau_0 \xi_j.
\]

From the proposition, we see that the map $M_2[\tau_i, \xi_i] \to A$ factors through the ring $M_2[\tau_i, \xi_i]/(\tau_j^2 = \tau \xi_{j+1} + \rho \tau_{j+1} + \rho \tau_0 \xi_j)$. In fact, this ring has a coproduct structure given by the composition of the isomorphisms of the formal group law.

**Proposition 4.3.2.** In $M_2[\tau_i, \xi_i]/(\tau_j^2 = \tau \xi_{j+1} + \rho \tau_{j+1} + \rho \tau_0 \xi_j)$, the coproduct is given by
\[
\Delta(\tau_k) = \sum_{i=0}^{k} \xi_{k-i}^{2i} \otimes \tau_i + \tau_k \otimes 1
\]
\[
\Delta(\xi_k) = \sum_{i=0}^{k} \xi_{k-i}^{2i} \otimes \xi_i
\]

**Proof.** Let $\phi$ and $\phi'$ be two isomorphisms of the formal group law given by
\[
\phi(u) = u + \sum_{i \geq 0} \tau_i v^{2i} \quad \phi'(u) = u + \sum_{i \geq 0} \tau'_i v^{2i}
\]
\[
\phi(v) = \sum_{i \geq 0} \xi_i v^{2i} \quad \phi'(v) = \sum_{i \geq 0} \xi'_i v^{2i}
\]
where $\xi_0 = \xi'_0 = 1$. Now consider the composition

$$\phi(\phi'(u)) = \phi \left( u + \sum_i \tau'_i v^{2i} \right) = u + \sum_j \tau_j v^{2j} + \sum_i \tau'_i \sum_j \xi_j^{2i} v^{2i+j} =$$

$$= u + \sum_k \tau_k v^{2k} + \sum_k \sum_{i=0}^k \tau'_i \xi_{k-i}^{2i} v^{2k}.$$

It follows that

$$\Delta(\tau_k) = \sum_{i=0}^k \xi_{k-i}^{2i} \otimes \tau_i + \tau_k \otimes 1$$

Similarly,

$$\phi(\phi'(v)) = \phi \left( \sum_i \xi'_i v^{2i} \right) = \sum_i \xi'_i \sum_j \xi_j^{2i} v^{2i+j} = \sum_k \sum_{i=0}^k \xi'_i \xi_{k-i}^{2i} v^{2k}.$$ 

Thus,

$$\Delta(\xi_k) = \sum_{i=0}^k \xi_{k-i}^{2i} \otimes \xi_i$$

We have thus worked out the structure of $\mathbb{M}_2[\tau_i, \xi_j]/(\tau_j^2 = \tau \xi_{j+1} + \rho \tau_{j+1} + \rho \tau_0 \xi_j)$ which maps into the dual of the motivic Steenrod algebra. In fact, it is known (though unfortunately not published) that this map is an isomorphism.
Bibliography


