1. Hochschild homology

Given an associative algebra $A$ over a commutative ring $R$, the Hochschild homology of $A$ with coefficients in an $A$-bimodule is the homotopy of the simplicial abelian group

$$\text{Bar}^R(A) \otimes_{A \otimes A^{op}} M = \text{Bar}^R_{cyc}(A, M)$$

or, equivalently, as the homotopy of its realization. In that expression, $\text{Bar}^R(A)$ stands for the bar resolution of $A$ over $R$. We will denote Hochschild homology by $HH^R(A, M)$.

2. Topological Hochschild homology

Topological Hochschild homology is defined analogously as:

$$\text{THH}^R(A, M) = |\text{Bar}^R_{cyc}(A, M)|$$

for an associative (cofibrant) $R$-algebra $A$ and an $A$-bimodule $M$. $R$ is here a (cofibrant) commutative $S$-algebra. We'll abbreviate $\text{THH}^R(A, A)$ as $\text{THH}^R(A)$ [cofibrant could be simply replaced by units being cofibrations].

This immediately gives us the following relation:

$$\text{THH}^{HR}(HA, HM) = HH^R(A, M)$$

for a commutative ring $R$, a flat $R$-algebra $A$ and an $A$-bimodule $M$. This follows because in this case $HA \wedge_{HR} HN \simeq H(A \otimes_R N)$ for any $N$.

Also, the spectral sequence for a simplicial spectrum gives us the B"okstedt spectral sequence

$$HH^{E_*R}(E_*A, E_*M) \implies E_*(\text{THH}^R(A, M))$$

when $E_*(A)$ is $E_*(R)$-flat.

In what follows, the references to $R$ will often be removed.

3. The case of commutative algebras

For a commutative $R$-algebra $A$, we have

$$\text{THH}(A) \simeq S^1 \otimes A$$

where “$\otimes$” in the right-hand side indicates the enrichment of commutative $R$-algebras over $sSet$. A proof of this statement comes from noticing that the enrichment of commutative $R$-algebras over $sSet$ is given by:

$$X \otimes A = |A^\wedge X|$$

for $X \in sSet$ (the maps in the simplicial object in the right-hand side come from the unit and multiplication of $A$). Taking $X = S^1 = \Delta^1/\partial \Delta^1$ we get the Hochschild complex in the right hand side and the result follows.

4. Another construction of $\text{THH}(A)$

Another construction of $\text{THH}$ can be given which is valid for algebras over the little intervals operad, $D_1$. We set the stage now for giving that construction.

Consider an operad $P$ in Top. Then there is a symmetric monoidal topologically enriched category (over $(\text{FinSet}, II)$) associated to $P$, the so-called category of operators of $P$ or the May-Thomason construction of $P$ (inspired by a similar construction by May and Thomason), which we denote $\mathcal{P}$. The object set of this category is $\mathbb{N}$ and the spaces of morphisms are:

$$\mathcal{P}(k, l) = \coprod_{f \in \text{FinSet}(k, l)} \otimes P(f^{-1}(i))$$

for $k, l \geq 1$. For $k = 0$, we put $\mathcal{P}(0, l) = l$ while for $l = 0$, we put $\mathcal{P}(k, 0) = 0$. The composition in $\mathcal{P}$ is defined using functoriality of $P$. This provides us with a way to construct $\text{THH}$ algebraically.
In particular, $\mathcal{P}(k, 1) = P(k)$. The important property of this category is that a $P$-algebra $A$ in a symmetric monoidal category $C$ is the same as a symmetric monoidal functor $\mathbf{A} : \mathcal{P} \to C$.

Then the category $\mathcal{D}_1$ can be easily described as:

$$\mathcal{D}_1(k, l) = \{ f \in \text{Emb}([0, 1]|_k, [0, 1]|_l) : f \text{ preserves orientation, } f \text{ has locally constant speed}\}$$

and we can give a topological functor

$$\mathcal{D}[S^1] : \mathcal{D}_1^{\text{op}} \to \text{Top}$$

defined on objects by

$$\mathcal{D}[S^1](k) = \{ f \in \text{Emb}([0, 1]|_k, S^1) : f \text{ preserves orientation, } f \text{ has locally constant speed}\}$$

[Note that in the descriptions of $\mathcal{D}_1$ and $\mathcal{D}[S^1]$, the condition on the speed being locally constant does not change the homotopy type of the spaces.]

With this in place, we now have

$$\text{THH}(A) \simeq \mathcal{D}[S^1] \otimes_{\mathcal{D}_1} \mathbf{A}$$

for an associative (cofibrant) $R$-algebra $A$. Note that the above expression makes sense for $A$ a $\mathcal{D}_1$-algebra in the category of $R$-modules. Also, it brings the oriented manifold $S^1$ explicitly into the picture.

5. Morita invariance

If $A, B$ are (cofibrant) associative algebras and $P$ is a $A$-$B$-bimodule (cofibrant as a $R$-module) then

$$(2) \quad \text{THH}(A, P \otimes_B N) = \text{THH}(B, N \otimes_A P)$$

for any $A$-$B$-bimodule $N$. Here, we put

$$M \otimes_A^L N = |\text{Bar}(M, A, N)|$$

Note that the map $M \otimes_A^L N \to M \otimes_A N$ is a weak equivalence if $M$ is a cofibrant right $A$-module. In particular, if $P$ is a cofibrant $A$-$B$-bimodule, we get from (2) that

$$\text{THH}(A, P \otimes_B N) \simeq \text{THH}(B, N \otimes_A P)$$

From this, it follows that given a Morita equivalence between $A$ and $B$, namely:

i. a $A$-$B$-bimodule $P$.

ii. a $B$-$A$-bimodule $Q$.

iii. a weak equivalence of $A$-bimodules $P \otimes_B^L Q \simeq A$.

iv. a weak equivalence of $B$-bimodules $Q \otimes_A^L P \simeq B$.

then

$$(3) \quad \text{THH}(A, M) \simeq \text{THH}(B, Q \otimes_A^L M \otimes_A^L P)$$
The proof is simple:

\[
THH(A, M) \simeq THH(A, A^\mathcal{L} \otimes M)
\]

\[
\simeq THH(A, P^\mathcal{L} \otimes Q^\mathcal{L} \otimes M)
\]

\[
= THH(B, Q^\mathcal{L} \otimes M^\mathcal{L} \otimes P^\mathcal{L})
\]

Actually, one can see from the proof that condition (iv) is not necessary. In view of a remark above, if \(P\) and \(Q\) are cofibrant (left and right, respectively) \(A\)-modules or \(M\) is a cofibrant \(A\)-bimodule then we can drop the superscripts “\(L\)” from \((\mathcal{L})\).

The usual example of a Morita equivalence is between \(A\) and the matrix algebra \(M_n(A) = \text{Hom}_A(A^{\mathcal{L} n}, A^{\mathcal{L} n})\). Then the \(A-M_n(A)\)-bimodule \(\text{Hom}_A(A^{\mathcal{L} n}, A) = A^{\mathcal{L} n}\) and the \(M_n(A)-A\)-bimodule \(\text{Hom}_A(A^{\mathcal{L} n}, A) = A^{\mathcal{L} n} \simeq A^{\mathcal{L} n}\) give the desired Morita equivalence (where the weak equivalences in the conditions above actually become isomorphisms). This Morita equivalence translates to an equivalence

\[
THH(A, M) \simeq THH(M_n(A), M_n(M))
\]

as a particular case of \((\mathcal{L})\).