Minimal Taylor Algebras

Zarathustra Brady
Taylor algebras

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Definition
An idempotent algebra is *Taylor* if the variety it generates does not contain a two element set.

All algebras in this talk will be idempotent, so I won’t mention idempotence further.
Useful facts about Taylor algebras

▶ Theorem (Bulatov and Jeavons)

A finite algebra $\mathbb{A}$ is Taylor iff there is no set in $HS(\mathbb{A})$.

▶ Theorem (Barto and Kozik)

A finite algebra $\mathbb{A}$ is Taylor iff for every number $n$ such that every prime factor of $n$ is greater than $|\mathbb{A}|$, there is an $n$-ary cyclic term $c$, i.e. $c(x_1, x_2, \ldots, x_n) \approx c(x_2, \ldots, x_n, x_1)$.

▶ Corollary

A finite algebra is Taylor iff it has a 4-ary term $t$ satisfying the identity $t(x, x, y, z) \approx t(y, z, z, x)$. 

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A finite algebra is Taylor iff it has a 4-ary term $t$ satisfying the identity

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- Larger CSPs $\iff$ smaller clones.

Definition
An algebra is a minimal Taylor algebra if it is Taylor, and has no proper reduct which is Taylor.

Proposition
Every finite Taylor algebra has a reduct which is a minimal Taylor algebra.

Proof.
There are only finitely many 4-ary terms $t(x, x, y, z) \approx t(y, z, z, x)$. 

$\Box$
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- So it makes sense to study Taylor algebras whose clones are as small as possible.

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  An algebra is a *minimal Taylor algebra* if it is Taylor, and has no proper reduct which is Taylor.

- Proposition
  *Every finite Taylor algebra has a reduct which is a minimal Taylor algebra.*

- Proof.
  There are only finitely many 4-ary terms $t$ which satisfy $t(x, x, y, z) \approx t(y, z, z, x)$.
First hints of a nice theory

Theorem

If $\mathbf{A}$ is a minimal Taylor algebra, $\mathbf{B} \in HSP(\mathbf{A})$, $S \subseteq \mathbf{B}$, and $t$ a term of $\mathbf{A}$ satisfy

1. $S$ is closed under $t$,
2. $(S, t)$ is a Taylor algebra,

then $S$ is a subalgebra of $\mathbf{B}$, and is also a minimal Taylor algebra.
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First hints of a nice theory

- **Theorem**
  
  If \( A \) is a minimal Taylor algebra, \( B \in HSP(A) \), \( S \subseteq B \), and \( t \) a term of \( A \) satisfy
  - \( S \) is closed under \( t \),
  - \((S, t)\) is a Taylor algebra,

  then \( S \) is a subalgebra of \( B \), and is also a minimal Taylor algebra.

- Choose \( p \) a prime bigger than \(|A|\) and \(|S|\).
First hints of a nice theory

Theorem

If \( \mathbb{A} \) is a minimal Taylor algebra, \( \mathbb{B} \in HSP(\mathbb{A}) \), \( S \subseteq \mathbb{B} \), and \( t \) a term of \( \mathbb{A} \) satisfy

\( S \) is closed under \( t \),
\( (S, t) \) is a Taylor algebra,

then \( S \) is a subalgebra of \( \mathbb{B} \), and is also a minimal Taylor algebra.

Choose \( p \) a prime bigger than \( |\mathbb{A}| \) and \( |S| \).
Choose \( c \) a \( p \)-ary cyclic term of \( \mathbb{A} \), \( u \) a \( p \)-ary cyclic term of \( (S, t) \).
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Theorem

If $\mathbb{A}$ is a minimal Taylor algebra, $\mathbb{B} \in HSP(\mathbb{A})$, $S \subseteq \mathbb{B}$, and $t$ a term of $\mathbb{A}$ satisfy

- $S$ is closed under $t$,
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then $S$ is a subalgebra of $\mathbb{B}$, and is also a minimal Taylor algebra.

Choose $p$ a prime bigger than $|\mathbb{A}|$ and $|S|$.

Choose $c$ a $p$-ary cyclic term of $\mathbb{A}$, $u$ a $p$-ary cyclic term of $(S, t)$.

Then

$$f = c(u(x_1, x_2, \ldots, x_p), u(x_2, x_3, \ldots, x_1), \ldots, u(x_p, x_1, \ldots, x_{p-1}))$$

is a cyclic term of $\mathbb{A}$. 

First hints of a nice theory

Theorem
If $\mathbb{A}$ is a minimal Taylor algebra, $\mathbb{B} \in HSP(\mathbb{A})$, $S \subseteq \mathbb{B}$, and $t$ a term of $\mathbb{A}$ satisfy
- $S$ is closed under $t$,
- $(S, t)$ is a Taylor algebra,
then $S$ is a subalgebra of $\mathbb{B}$, and is also a minimal Taylor algebra.

Choose $p$ a prime bigger than $|\mathbb{A}|$ and $|S|$.
Choose $c$ a $p$-ary cyclic term of $\mathbb{A}$, $u$ a $p$-ary cyclic term of $(S, t)$.
Then
\[ f = c(u(x_1, x_2, ..., x_p), u(x_2, x_3, ..., x_1), ..., u(x_p, x_1, ..., x_{p-1})) \]
is a cyclic term of $\mathbb{A}$.
Have $f|_S = u|_S$ by idempotence.
A few consequences

- **Proposition**

For $\mathbb{A}$ minimal Taylor, $a, b \in \mathbb{A}$, then $\{a, b\}$ is a semilattice subalgebra of $\mathbb{A}$ with absorbing element $b$ iff

$$\begin{bmatrix} b \\ b \end{bmatrix} \in Sg_{\mathbb{A}^2} \left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix} \right\}.$$
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- Proposition

  For $\mathbb{A}$ minimal Taylor, $a, b \in \mathbb{A}$, then $\{a, b\}$ is a majority subalgebra of $\mathbb{A}$ iff

  $$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \in Sg_{\mathbb{A}^{3\times2}} \left\{ \begin{bmatrix} a & b \\ a & b \end{bmatrix}, \begin{bmatrix} b & a \\ a & b \end{bmatrix}, \begin{bmatrix} a & b \\ b & a \end{bmatrix} \right\}.$$
A few consequences, ctd.

- **Proposition**

  For a minimal Taylor, \( a, b \in A \), then \( \{a, b\} \) is a \( \mathbb{Z}/2^{\text{aff}} \) subalgebra of \( A \) iff

  \[
  \begin{bmatrix}
  b & a \\
  b & a \\
  b & a \\
  \end{bmatrix}
  \in \text{Sg}_{A^{3 \times 2}} \left\{ \begin{bmatrix} a & b \\ a & b \\ b & a \end{bmatrix}, \begin{bmatrix} a & b \\ b & a \\ a & b \end{bmatrix}, \begin{bmatrix} b & a \\ b & a \\ a & b \end{bmatrix} \right\}.
  \]
A few consequences, ctd.

**Proposition**

*For a minimal Taylor, $a, b \in A$, then $\{a, b\}$ is a $\mathbb{Z}/2^{\text{aff}}$ subalgebra of $A$ iff*

$$
\begin{bmatrix}
  b & a \\
  b & a \\
  b & a
\end{bmatrix} \in Sg_{A^{3 \times 2}} \left\{ \begin{bmatrix}
  a & b \\
  a & b \\
  b & a
\end{bmatrix}, \begin{bmatrix}
  a & b \\
  b & a \\
  a & b
\end{bmatrix}, \begin{bmatrix}
  b & a \\
  b & a \\
  a & b
\end{bmatrix} \right\}.
$$

**If there is an automorphism of $A$ which interchanges $a, b$, then we only have to consider**

$$
Sg_{A^3} \left\{ \begin{bmatrix}
  a \\
  a \\
  b
\end{bmatrix}, \begin{bmatrix}
  a \\
  b \\
  a
\end{bmatrix}, \begin{bmatrix}
  b \\
  a \\
  a
\end{bmatrix} \right\}.
$$
Daisy Chain Terms

- It's difficult to write down explicit examples without nice terms.
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- Choose a $p$-ary cyclic term $c$. 

$\begin{align*}
\forall a < p^2, \quad &\text{can make a ternary term } w(x, y, z) \text{ via:} \\
&\quad w(x, y, z) = c(x, \ldots, x, y, \ldots, y, z, \ldots, z) \\
&\quad \approx \quad w(y, x, x).
\end{align*}$

- Also have $w(x, y, x) = c(x, \ldots, x, y, \ldots, y, x, \ldots, x)$. 


Daisy Chain Terms

- It's difficult to write down explicit examples without nice terms.
- Choose a $p$-ary cyclic term $c$.
- For any $a < \frac{p}{2}$, can make a ternary term $w(x, y, z)$ via:

\[
w(x, y, z) = c(x, \ldots, x, y, \ldots, y, z, \ldots, z).
\]

\[
\begin{array}{ccc}
\underbrace{x, \ldots, x}_{a} & \underbrace{y, \ldots, y}_{p-2a} & \underbrace{z, \ldots, z}_{a}
\end{array}
\]
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- For any \( a < \frac{p}{2} \), can make a ternary term \( w(x, y, z) \) via:

\[
w(x, y, z) = c(x, \ldots, x, y, \ldots, y, z, \ldots, z).
\]

- This satisfies

\[
w(x, x, y) \approx w(y, x, x).
\]
Daisy Chain Terms

- It's difficult to write down explicit examples without nice terms.

- Choose a $p$-ary cyclic term $c$.

- For any $a < \frac{p}{2}$, can make a ternary term $w(x, y, z)$ via:

$$w(x, y, z) = c(x, ..., x, y, ..., y, z, ..., z).$$

- This satisfies

$$w(x, x, y) \approx w(y, x, x).$$

- Also have

$$w(x, y, x) = c(x, ..., x, y, ..., y, x, ..., x).$$
Daisy Chain Terms, ctd.

- From a sequence

\[ a, p - 2a, p - 2(p - 2a), ... \]

we get a sequence of ternary terms:

\[
\begin{align*}
w_0(x, x, y) & \approx w_0(y, x, x) \approx w_1(x, y, x), \\
w_1(x, x, y) & \approx w_1(y, x, x) \approx w_2(x, y, x), \\
\vdots
\end{align*}
\]
Daisy Chain Terms, ctd.

▶ From a sequence

\[ a, p - 2a, p - 2(p - 2a), \ldots \]

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\[ \vdots \]

▶ If \( p \) is large enough and \( a \) is close enough to \( \frac{p}{3} \), then the sequence can become arbitrarily long.
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\[ \vdots \]

- If \( p \) is large enough and \( a \) is close enough to \( \frac{p}{3} \), then the sequence can become arbitrarily long.

- Since there are only finitely many ternary functions in \( \text{Clo}(A) \), we eventually get a cycle.
What do they mean?

- How can daisy chain terms be useful to us?
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- For $a, b \in A$, define a binary relation $D_{ab} \subseteq A^2$ by

$$D_{ab} = \left\{ \begin{bmatrix} c \\ d \\ c \\ d \end{bmatrix} \text{ s.t. } \begin{bmatrix} c \\ d \\ c \\ d \end{bmatrix} \in Sg_{A^3} \left\{ \begin{bmatrix} a \\ a \\ b \\ a \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \\ a \end{bmatrix} \right\} \right\}.$$
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- If $\begin{bmatrix} a \\ a \end{bmatrix} \in D_{ab}$ and there is an automorphism interchanging $a, b$, then $\{a, b\}$ is a majority algebra.
What do they mean?

- How can daisy chain terms be useful to us?

- For $a, b \in \mathbb{A}$, define a binary relation $D_{ab} \leq \mathbb{A}^2$ by

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- If $\begin{bmatrix} a \\ a \end{bmatrix} \in D_{ab}$ and there is an automorphism interchanging $a, b$, then $\{a, b\}$ is a majority algebra.

- Proposition

  If $\mathbb{A}$ has daisy chain terms and $a, b \in \mathbb{A}$, then if we consider $D_{ab}$ as a digraph, it must contain a directed cycle.
Describing a minimal Taylor algebra

- If $p = w_i$, $q = w_{i+1}$ are any pair of adjacent daisy chain terms, then they satisfy the system

\[ p(x, x, y) \approx p(y, x, x) \approx q(x, y, x), \]
\[ q(x, x, y) \approx q(y, x, x). \]
Describing a minimal Taylor algebra

- If \( p = w_i, q = w_{i+1} \) are any pair of adjacent daisy chain terms, then they satisfy the system

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\end{align*}
\]

- Thus \( p, q \) generate a Taylor clone, so \( \text{Clo}(A) = \langle p, q \rangle \) if \( A \) is minimal Taylor.

In particular, the number of minimal Taylor clones on a set of \( n \) elements is at most \( n^2 n^3 \).

Conjecture: Every minimal Taylor clone can be generated by a single ternary function.
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Describing a minimal Taylor algebra

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- Thus $p, q$ generate a Taylor clone, so $\text{Clo}(A) = \langle p, q \rangle$ if $A$ is minimal Taylor.

- In particular, the number of minimal Taylor clones on a set of $n$ elements is at most $n^2 n^3$.

- Conjecture

  Every minimal Taylor clone can be generated by a *single* ternary function.
Daisy chain terms in the basic algebras

- **Proposition**
  
  If $w_i$ are daisy chain terms and $A$ is a semilattice, then each $w_i$ is the symmetric ternary semilattice operation on $A$. 

- **Proposition**
  
  If $w_i$ are daisy chain terms and $A$ is a majority algebra, then each $w_i$ is a majority operation on $A$. 

- **Proposition**
  
  If $w_i$ are daisy chain terms and $A$ is affine, then there is a sequence $a_i$ such that $w_i$ is given by:
  
  $$w_i(x, y, z) = a_i x + (1 - 2a_i) y + a_i z,$$

  with $a_{i+1} = 1 - 2a_i$.

  If $a_0 = 0$, then $w_1$ is the Mal'cev operation $x - y + z$ and $w_{-1}$ is the operation $x + z$. 
Daisy chain terms in the basic algebras

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**Proposition**

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with $a_{i+1} = 1 - 2a_i$.

If $a_0 = 0$, then $w_1$ is the Mal’cev operation $x - y + z$ and $w_{-1}$ is the operation $\frac{x + z}{2}$.
Bulatov’s graph

- Bulatov studies finite Taylor algebras via three types of edges: semilattice, majority, and affine.
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• In minimal Taylor algebras, we can define his edges more simply.
Bulatov’s graph

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- In minimal Taylor algebras, we can define his edges more simply.

**Definition**
If $A$ is minimal Taylor and $a, b \in A$, then $(a, b)$ is an edge if there is a congruence $\theta$ on $Sg\{a, b\}$ s.t.

$$Sg\{a, b\}/\theta$$

is isomorphic to either a two-element semilattice, a two element majority algebra, or an affine algebra.
Connectivity

- Theorem (Bulatov)
  
  *If $\mathbb{A}$ is minimal Taylor, then the associated graph is connected.*
Connectivity

- **Theorem (Bulatov)**
  
  *If \( \mathbb{A} \) is minimal Taylor, then the associated graph is connected.*

  - We can simplify the proof!
Connectivity

- **Theorem (Bulatov)**
  
  *If $\mathcal{A}$ is minimal Taylor, then the associated graph is connected.*

  - We can simplify the proof!
  - If $\mathcal{A}$ is a minimal counterexample:
    - the hypergraph of proper subalgebras must be disconnected,
    - $\mathcal{A}$ is generated by two elements $a, b,$ and
    - $\mathcal{A}$ has no proper congruences.
Connectivity

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  - It’s not hard to show there must be an automorphism interchanging $a, b$. 

Connectivity

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  - $\mathcal{A}$ has no proper congruences.
- It’s not hard to show there must be an automorphism interchanging $a, b$.
- Consider the binary relation $\mathbb{D}_{ab}$!
Recall the definition of $\mathbb{D}_{ab}$:

$$\mathbb{D}_{ab} = \left\{ \begin{bmatrix} c \\ d \\ c \end{bmatrix} \text{ s.t. } \begin{bmatrix} c \\ d \\ c \end{bmatrix} \in Sg_{A^3} \left\{ \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} a \\ a \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \end{bmatrix} \right\} \right\}.$$
Connectivity, ctd.

- Recall the definition of $\mathcal{D}_{ab}$:

$$
\mathcal{D}_{ab} = \left\{ \begin{bmatrix} c \\ d \end{bmatrix} \text{ s.t. } \begin{bmatrix} c \\ d \\ c \end{bmatrix} \in Sg_{A^3} \begin{Bmatrix} \begin{bmatrix} a \\ a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \end{bmatrix} \end{Bmatrix} \right\}.
$$

- Have $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathcal{D}_{ab}$, want to show that either $\begin{bmatrix} a \\ a \end{bmatrix} \in \mathcal{D}_{ab}$ or $A$ is affine.
Connectivity, ctd.

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\mathbb{D}_{ab} = \left\{ \begin{bmatrix} c \\ d \end{bmatrix} \right\} \text{ s.t. } \begin{bmatrix} c \\ d \\ c \end{bmatrix} \in Sg_{\mathbb{A}^3} \left\{ \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \end{bmatrix} \right\}.
$$

- Have $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{D}_{ab}$, want to show that either $\begin{bmatrix} a \\ a \end{bmatrix} \in \mathbb{D}_{ab}$ or $\mathbb{A}$ is affine.

- The daisy chain terms give us $c, d, e \in \mathbb{A}$ such that

$$
\begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} d \\ e \end{bmatrix} \in \mathbb{D}_{ab}.
$$
Connectivity, ctd.

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$$

- Have $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{D}_{ab}$, want to show that either $\begin{bmatrix} a \\ a \end{bmatrix} \in \mathbb{D}_{ab}$ or $A$ is affine.

- The daisy chain terms give us $c, d, e \in A$ such that

$$
\begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} d \\ e \end{bmatrix} \in \mathbb{D}_{ab}.
$$

- If both $Sg\{a, d\}$ and $Sg\{d, b\}$ are proper subalgebras, then the hypergraph of proper subalgebras is connected.
Connectivity, ctd.

- Recall the definition of $\mathbb{D}_{ab}$:

  $$
  \mathbb{D}_{ab} = \left\{ \begin{bmatrix} c \\ d \end{bmatrix} \text{ s.t. } \begin{bmatrix} c \\ d \\ c \end{bmatrix} \in Sg_{A^3} \left\{ \begin{bmatrix} a \\ b \\ a \\ a \end{bmatrix}, \begin{bmatrix} a \\ b \\ a \\ a \end{bmatrix}, \begin{bmatrix} b \\ a \\ a \\ a \end{bmatrix} \right\} \right\}.
  $$

- Have $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{D}_{ab}$, want to show that either $\begin{bmatrix} a \\ a \end{bmatrix} \in \mathbb{D}_{ab}$ or $A$ is affine.

- The daisy chain terms give us $c, d, e \in A$ such that

  $$
  \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} d \\ e \end{bmatrix} \in \mathbb{D}_{ab}.
  $$

- If both $Sg\{a, d\}$ and $Sg\{d, b\}$ are proper subalgebras, then the hypergraph of proper subalgebras is connected.

- Then we can show $\mathbb{D}_{ab}$ is subdirect, and the proof flows naturally from here.
Can we do better?

- Can we get rid of congruences in the definition of the edges?
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- Proposition (Bulatov)

  For every semilattice edge from $a$ to $b$, there is a $b'$ in the congruence class of $b$ such that $\{a, b'\}$ is a two element semilattice algebra.
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  - Similar statements fail for majority edges and affine edges.
  - There are minimal Taylor algebras $A, B$ of size 4 which have congruences $\theta$ such that:
    - $A/\theta$ is a two element majority algebra and $B/\theta$ is $\mathbb{Z}/2^{\text{aff}}$,
    - each congruence class of $\theta$ is a copy of $\mathbb{Z}/2^{\text{aff}}$,
    - every proper subalgebra of $A$ or $B$ is contained in a congruence class of $\theta$,
    - $A$ has a 3-edge term and $B$ is Mal’cev,
    - $\theta$ is the center of $A$ or $B$ in the sense of commutator theory.
Evil algebra #1

$A = (\{a, b, c, d\}, g)$, where $g$ is an idempotent ternary symmetric operation.
Evil algebra \#1

\[ \mathbb{A} = (\{a, b, c, d\}, g), \text{ where } g \text{ is an idempotent ternary symmetric operation.} \]

\( g \) commutes with the cyclic permutation \( \sigma = (a \ b \ c \ d) \) and satisfies

\[
\begin{align*}
  g(a, a, b) &= a, \\
  g(a, a, c) &= c, \\
  g(a, a, d) &= c, \\
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- $\theta$ corresponds to the partition $\{a, c\}, \{b, d\}$.
- The algebra $\mathbb{S} = Sg_{\mathbb{A}^2}\{(a, b), (b, a)\}$ has a congruence $\psi$ corresponding to the partition

  \[
  \begin{bmatrix}
  a \\
  b
  \end{bmatrix}, \begin{bmatrix}
  b \\
  c
  \end{bmatrix}, \begin{bmatrix}
  c \\
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  \end{bmatrix}, \begin{bmatrix}
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  b \\
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  c
  \end{bmatrix}
  \]

  such that $\mathbb{S}/\psi$ is isomorphic to $\mathbb{Z}/2^{\text{aff}}$. 
Evil algebra #2

\[ \mathbb{B} = (\{a, b, c, d\}, p), \text{ where } p \text{ is a Mal’cev operation.} \]
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- $\mathbb{B} = (\{a, b, c, d\}, p)$, where $p$ is a Mal’cev operation.
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- The polynomials $+_a = p(\cdot, a, \cdot), +_b = p(\cdot, b, \cdot)$ define abelian groups:

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Zhuk’s four cases

- Theorem (Zhuk)
  
  If $\mathbb{A}$ is minimal Taylor, then at least one of the following holds:
  
  - $\mathbb{A}$ has a proper binary absorbing subalgebra,
  - $\mathbb{A}$ has a proper “center”,
  - $\mathbb{A}$ has a nontrivial affine quotient, or
  - $\mathbb{A}$ has a nontrivial polynomially complete quotient.

- Definition
  
  $\mathbb{C} \leq \mathbb{A}$ is a center of $\mathbb{A}$ if there exist
  
  - a binary-absorption-free Taylor algebra $\mathbb{B}$ and
  - a subdirect relation $R \leq \text{sd } \mathbb{A} \times \mathbb{B}$, such that
  
  $\mathbb{C} = \{c \in \mathbb{A} \text{ s.t. } \forall b \in \mathbb{B}, [c, b] \in R\}$.

- Theorem (Zhuk)
  
  If $\mathbb{C}$ is a center of $\mathbb{A}$, then $\mathbb{C}$ is a ternary absorbing subalgebra of $\mathbb{A}$. 
Zhuk’s four cases

▶ Theorem (Zhuk)

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$C \leq A$ is a center of $A$ if there exist

▶ a binary-absorption-free Taylor algebra $B$ and
▶ a subdirect relation $R \leq_{sd} A \times B$, such that

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If $C$ is a center of $A$, then $C$ is a ternary absorbing subalgebra of $A$. 
Centers and Daisy Chain terms

Theorem

If $\mathbb{A}$ is minimal Taylor and $\mathbb{M} \in HSP(\mathbb{A})$ is the two element majority algebra on the domain $\{0, 1\}$, then the following are equivalent:

- $C$ is a ternary absorbing subalgebra of $\mathbb{A}$,
- there is a $p$-ary cyclic term $c$ of $\mathbb{A}$ such that whenever $\#\{x_i \in C\} > \frac{p}{2}$, we have $c(x_1, \ldots, x_p) \in C$,
- the binary relation $R \subseteq \mathbb{A} \times \mathbb{M}$ given by
  \[ R = (\mathbb{A} \times \{0\}) \cup (C \times \{0, 1\}) \]
  is a subalgebra of $\mathbb{A} \times \mathbb{M}$,
- every daisy chain term $w_i(x, y, z)$ witnesses the fact that $C$ ternary absorbs $\mathbb{A}$.
Centers produce majority quotients

- If $C, D$ are centers, then for any daisy chain terms $w_i$, we must have
  \[ w_i(C, C, D), w_i(C, D, C), w_i(D, C, C) \subseteq C \]
  and
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so \( C \cup D \) is a subalgebra of \( A \).

- If \( C \cap D = \emptyset \), then the equivalence relation \( \theta \) on \( C \cup D \) with parts \( C, D \) is preserved by each daisy chain term \( w_i \), and \( (C \cup D)/\theta \) is a two element majority algebra.
Binary absorption is strong absorption

**Theorem**

If $\mathbb{A}$ is minimal Taylor, then the following are equivalent:

- $\mathbb{B}$ binary absorbs $\mathbb{A}$,
- there exists a cyclic term $c$ such that if any $x_i \in \mathbb{B}$, then $c(x_1, ..., x_p) \in \mathbb{B}$,
- the ternary relation

$$
\mathbb{R} = \{(x, y, z) \text{ s.t. } (x \not\in \mathbb{B}) \implies (y = z)\}
$$

is a subalgebra of $\mathbb{A}^3$,
- every term $f$ of $\mathbb{A}$ which depends on all its inputs is such that if any $x_i \in \mathbb{B}$, then $f(x_1, ..., x_n) \in \mathbb{B}$.
Theorem

If $\mathbb{A}$ is minimal Taylor and $\mathbb{A} = \text{Sg}\{a, b\}$, then the following are equivalent:

- $\mathbb{B}$ binary absorbs $\mathbb{A}$,
- $\mathbb{A} = \mathbb{B} \cup \{a, b\}$ and there is a congruence $\theta$ such that $\mathbb{B}$ is a congruence class of $\theta$, and $\mathbb{A}/\theta$ is a semilattice.

Minimal Taylor algebras generated by two elements are nicer than general minimal Taylor algebras. It's good enough to understand such algebras.
Minimal Taylor algebras generated by two elements

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Big conjecture

- **Conjecture**

  Suppose $\mathcal{A}$ is minimal Taylor, generated by two elements $a, b$, and has no affine or semilattice quotient. Then each of $a, b$ is contained in a proper ternary absorbing subalgebra of $\mathcal{A}$. 

- **Proposition**

  Suppose the conjecture holds. Then any daisy chain term $w_i$ which is nontrivial on every affine algebra in $\text{HS}(\mathcal{A})$ generates $\text{Clo}(\mathcal{A})$. In particular, $\text{Clo}(\mathcal{A})$ is generated by a single ternary term.

- **Theorem (Kearnes, Szendrei)**

  Suppose a minimal Taylor algebra has no semilattice edges and has its clone generated by a single ternary term. Then it has a $3$-edge term.
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Thank you for your attention.