Solutions to Problem Set 3

3-9 Given two extreme points \(a\) and \(b\) of a polyhedron \(P\), we say that they are adjacent if the line segment between them forms an edge (i.e. a face of dimension 1) of the polyhedron \(P\). This can be rephrased by saying that \(a\) and \(b\) are adjacent on \(P\) if and only if there exists a cost function \(c\) such that \(a\) and \(b\) are the only two extreme points of \(P\) minimizing \(c^T x\) over \(P\).

Consider the polyhedron (polytope) \(P\) defined as the convex hull of all perfect matchings in a (not necessarily bipartite) graph \(G\). Give a necessary and sufficient condition for two matchings \(M_1\) and \(M_2\) to be adjacent on this polyhedron (hint: think about \(M_1 \triangle M_2 = (M_1 \setminus M_2) \cup (M_2 \setminus M_1)\)) and prove that your condition is necessary and sufficient.)

First consider the situation in which \(M_1\) and \(M_2\) are such that \(M_1 \triangle M_2\) have more than one connected component. Consider one of these connected components, say \(S \subseteq V\), and partition \(M_1\) and \(M_2\) into \(M_1 = M_{1s} \cup M_{1t}\) and \(M_2 = M_{2s} \cup M_{2t}\) where \(M_{1s}\) and \(M_{2s}\) correspond to the edges within \(S\). By definition \(M_{1s} \cup M_{2s} \neq \emptyset\). Now define two other matchings by \(M_3 = M_{1s} \cup M_{2t}\) and \(M_4 = M_{2s} \cup M_{1t}\). Observe that
\[
\chi(M_1) + \chi(M_2) = \chi(M_3) + \chi(M_4)
\]
which implies that any face that contains \(M_1\) and \(M_2\) will also contain \(M_3\) and \(M_4\), and thus cannot be an edge.

Conversely, suppose that \(M_1 \triangle M_2\) has only one connected component, and say that this component has \(k_1\) edges from \(M_1\) and \(k_2\) edges from \(M_2\). We must have that \(|k_1 - k_2| \leq 1\). Now consider the following cost function:
\[
c_e = \begin{cases} 
1 & e \in M_1 \cap M_2 \\
-1 & e \notin (M_1 \cup M_2) \\
k_2 & e \in M_1 \setminus M_2 \\
k_1 & e \in M_2 \setminus M_1.
\end{cases}
\]

Notice that \(c(M_1) = c(M_2) = b\) where \(b := |M_1 \cap M_2| + 2k_1k_2\) and for any other matching \(M\) we have that \(c(M) < b\). Thus the valid inequality \(c^T x \leq b\) induces a face with only the incidence vectors of \(M_1\) and \(M_2\) has vertices. Thus the line segment between \(M_1\) and \(M_2\) defines an edge.

3-10 Show that two vertices \(u\) and \(v\) of a polytope \(P\) are adjacent if and only there is a unique way to express their midpoint \((\frac{1}{2}(u + v))\) as a convex combination of vertices of \(P\).

First suppose \(u, v\) are adjacent, and assume for contradiction that there exist vertices \(w_1, \ldots, w_n\) (at least one of which is not \(u\) or \(v\)) and weights \(\lambda_1, \ldots, \lambda_n > 0\), \(\sum \lambda_i = 1\), such that
\[
\frac{u + v}{2} = \lambda_1 w_1 + \cdots + \lambda_n w_n.
\]
Since $u, v$ are adjacent, there is a cost vector $c$ such that the line segment connecting $u, v$ is exactly the set of points $x$ of $P$ which maximize $c^T x$. But then

$$c^T u = \frac{c^T u + c^T v}{2} = \lambda_1 c^T w_1 + \cdots + \lambda_n c^T w_n < \lambda_1 c^T u + \cdots + \lambda_n c^T u = c^T u,$$

a contradiction.

Now suppose that $u, v$ are not adjacent, and let $F$ be the minimal face of $P$ containing $u$ and $v$ ($F$ is defined by the set of all inequalities of $P$ that have equality at both $u$ and $v$). Since $F$ is a polytope, $F$ is the convex hull of its vertices, so $F$ must have at least one vertex $w$ which is not $u$ or $v$. Let $L$ be the intersection of the line connecting $w$ to $\frac{u+v}{2}$ with $F$ (note $w \neq \frac{u+v}{2}$ since $w$ is a vertex). Since $L$ is a polytope defined by some system of equations describing a line together with the inequalities describing the facets of $F$, the vertices of $L$ come from setting some inequalities corresponding to facets of $F$ to equalities. Suppose $p$ is the second vertex of $L$ (the first is $w$), and suppose the corresponding facet of $F$ comes from the inequality $a^T x \leq b$, with equality $a^T p = b$ at $p$. By the minimality of $F$, at least one of $a^T u, a^T v$ is strictly less than $b$, so $p \neq \frac{u+v}{2}$. Thus $\frac{u+v}{2}$ can be written as a convex combination of $w$ and $p$ with a nonzero weight on $w$. Since $p$ can be written as a convex combination of vertices of $P$, we see that $\frac{u+v}{2}$ can be written as a convex combination of vertices of $P$ with a nonzero weight on $w$.

3-12 A stable set $S$ (sometimes, it is called also an independent set) in a graph $G = (V, E)$ is a set of vertices such that there are no edges between any two vertices in $S$. If we let $P$ denote the convex hull of all (incidence vectors of) stable sets of $G = (V, E)$, it is clear that $x_i + x_j \leq 1$ for any edge $(i, j) \in E$ is a valid inequality for $P$.

(a) Give a graph $G$ for which $P$ is not equal to

$$\{x \in \mathbb{R}^{|V|} : x_i + x_j \leq 1 \text{ for all } (i, j) \in E, x_i \geq 0 \text{ for all } i \in V\}$$

(b) Show that if the graph $G$ is bipartite then $P$ equals

$$\{x \in \mathbb{R}^{|V|} : x_i + x_j \leq 1 \text{ for all } (i, j) \in E, x_i \geq 0 \text{ for all } i \in V\}.$$

(a) Take $G$ to be the triangle, with vertex set $V = \{1, 2, 3\}$ and edge set $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$. The vector $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$ satisfies the given inequalities, but the sum of its coordinates is $\frac{3}{2}$, which is larger than the sum of the coordinates of any vertex of $P$, since every stable subset of the triangle has size at most 1.

(b) Note: strictly speaking, the problem statement is incorrect (considering the case where $G$ has just one vertex) - to fix it, we must assume that $G$ has no isolated vertices. So from here on we make this assumption.
The easy direction is checking that each indicator vector $x$ of a stable set $S$ satisfies the given inequalities, which follows immediately from the definition of a stable set. For the other direction - showing that each vector satisfying our system of inequalities is contained in the convex hull $P$ - we give two different proofs.

**Vertex Proof.** Let $A \in \mathbb{R}^{E \times V}$ be the matrix given by

$$A_{ev} = \begin{cases} 
1 & v \in e, \\
0 & v \notin e,
\end{cases}$$

and let $b \in \mathbb{R}^E$ be the vector of all 1s, so our system of inequalities can be written in the form

$$\{x \in \mathbb{R}^V : Ax \leq b, x \geq 0\}.$$

Note that $A^T$ is the matrix coming from the bipartite matching polytope, which we have already shown is totally unimodular. Since the transpose of a T.U. matrix is T.U., every vertex of the polyhedron defined by the system $\{Ax \leq b, x \geq 0\}$ is integral, and since this polyhedron is bounded (each $x_v$ is bounded below by 0 and above by 1 as long as $v$ is incident to at least one edge) it is the convex hull of its vertices. Let $x$ be a vertex of the polyhedron, we will show it is the indicator vector of a stable set. Since $x$ is integral and each coordinate of $x$ is bounded between 0 and 1, $x$ is certainly the indicator vector of some set $S$ - explicitly, $S = \{v \in V \mid x_v = 1\}$. If there was any edge $e$ between two vertices $v, w$ of $S$, we would have $x_v + x_w = 2$, contradicting the inequality $x_v + x_w \leq 1$ corresponding to the edge $e$, so in fact $S$ must be a stable set.

**Facet Proof.** First we check that we are not missing any equalities, by showing that $\dim(P) = |V|$. To see this, note that every set $S$ with $|S| \leq 1$ is stable, so $P$ contains the $|V| + 1$ affinely independent points $(0, 0, ..., 0)^T, (1, 0, ..., 0)^T, (0, 1, ..., 0)^T, ..., (0, 0, ..., 1)^T$.

Now suppose that $F$ is a facet of $P$, defined by maximizing some cost $c^T x$ over vertices of $P$. We will show that the set $\{x \in P \mid c^T x \text{ is maximal}\}$ is contained in some facet of the polyhedron defined by the given system of inequalities. There are two cases.

First case: for some $v \in V$, we have $c_v < 0$. In this case, every $x$ corresponding to a stable set $S$ which maximizes $c^T x$ must have $x_v = 0$, since otherwise the set $S \setminus \{v\}$ is also stable, and if $x'$ is the corresponding vector, then $c^T x' = c^T x - c_v > c^T x$. Thus the face of $P$ corresponding to the cost vector $c$ must be contained in the facet corresponding to the inequality $x_v \geq 0$.

Second case: for some $v \in V$ we have $c_v > 0$. Suppose for contradiction that for each edge $e = \{v, w\}$ containing $v$, there is some stable set $S_w$ which doesn’t contain $v$ or $w$, but such that if $x_w$ is the corresponding indicator vector, then
$c^T x_v$ maximizes $c^T x$ over $x$ in $P$. Let $W$ be any subset of the set of neighbors of $v$, we will show by induction on $|W|$ that there is a stable set $S_W$ which doesn’t contain $v$ or any vertex from $W$, but such that the corresponding indicator vector $x_W$ maximizes $c^T x_W$. Taking $W$ to be the set $N(v)$ of all neighbors of $v$, we will get a stable set $S_{N(v)}$ not containing $v$ or any neighbor of $v$ and maximizing our cost function, but then adding $v$ to this stable set gives us a stable set with a strictly larger cost, giving us our contradiction.

For the inductive step, suppose $W = X \cup Y$, and that we have already constructed stable sets $S_X, S_Y$ maximizing our cost function, not containing $v$, and s.t. $S_X \cap X = S_Y \cap Y = \emptyset$. Let $H$ be the induced subgraph of $G$ with vertex set $S_X \cup S_Y$, and let $C$ be the set of vertices of $H$ which are connected to some element of $X$ in $H$. Let $A, B \subseteq V$ be the two parts of $G$, and suppose $v \in A$. Then by induction on the length of the shortest path (in $H$) connecting a vertex $c$ in $C$ to $X$, we see that $c \in B \iff c \in S_Y$ and $c \in A \iff c \in S_X$. In particular, no vertex of $C$ is in $Y$. Additionally, we see that both $S_X \Delta C$ and $S_Y \Delta C$ are stable sets, and the sum of their costs is equal to the sum of the costs of $S_X$ and $S_Y$, so they both maximize our cost function as well. Thus we can take $S_W = S_Y \Delta C$, which has no elements of $X$ (since neither $S_Y \cap X = C \cap X$ by the definition of $C$) and no elements of $Y$ (since $S_Y \cap Y = \emptyset$ and $C \cap Y = \emptyset$). This completes the inductive step, which as we saw above gives us the required contradiction.

By the above argument, there must be some edge $e = \{v, w\}$ containing $v$ such that every stable set maximizing our cost function contains at least one of the vertices $v, w$. Thus, the face of $P$ corresponding to the cost vector $c$ is contained in the facet corresponding to the inequality $x_v + x_w \leq 1$.

3-13 Let $e_k \in \mathbb{R}^n$ ($k = 0, \ldots, n-1$) be a vector with the first $k$ entries being 1, and the following $n-k$ entries being -1. Let $S = \{e_0, e_1, \ldots, e_{n-1}, -e_0, -e_1, \ldots, -e_{n-1}\}$, i.e. $S$ consists of all vectors consisting of +1 followed by -1 or vice versa. In this problem set, you will study $\text{conv}(S)$.

(a) Consider any vector $a \in \{-1, 0, 1\}^n$ such that (i) $\sum_{i=1}^n a_i = 1$ and (ii) for all $j = 1, \ldots, n-1$, we have $0 \leq \sum_{i=1}^j a_i \leq 1$. (For example, for $n = 5$, the vector $(1, 0, -1, 1, 0)$ satisfies these conditions.) Show that $\sum_{i=1}^n a_i x_i \leq 1$ and $\sum_{i=1}^n a_i x_i \geq -1$ are valid inequalities for $\text{conv}(S)$.

(b) How many such inequalities are there?

(c) Show that any such inequality defines a facet of $\text{conv}(S)$.

(This can be done in several ways. Here is one approach, but you are welcome to use any other one as well. First show that either $e_k$ or $-e_k$ satisfies this inequality at equality, for any $k$. Then show that the resulting set of vectors on the hyperplane are affinely independent (or uniquely identifies it).)

(d) Show that the above inequalities define the entire convex hull of $S$. 
(Again this can be done in several ways. One possibility is to consider the 3rd technique described above.)

(a) Fix \( a \in \{-1, 0, 1\}^n \) satisfying \( \sum_{i=1}^{n} a_i = 1 \) and \( 0 \leq \sum_{i=1}^{j} a_i \leq 1 \) for each \( j = 1, \ldots, n - 1 \). It is enough to show that

\[
-1 \leq \sum_{i=1}^{n} a_i(e_k)_i \leq 1
\]

for each \( k = 0, \ldots, n - 1 \) (it is symmetric for \(-e_k\)'s).

Note that \((e_k)_i = 1\) if \( i \leq k \) and \((e_k)_i = -1\) if \( i > k \). We have

\[
\sum_{i=1}^{n} a_i(e_k)_i = \sum_{i=1}^{k} a_i - \sum_{i=k+1}^{n} a_i = 2 \sum_{i=1}^{k} a_i - 1.
\]

Since \( \sum_{i=1}^{k} a_i \) is 0 or 1, it is between \(-1\) and 1.

(b) Fix \( a \in \{-1, 0, 1\}^n \) as in the previous part. Let \( b_j = \sum_{i=1}^{j} a_i \) for \( j = 1, \ldots, n \). Then, \( b_j \in \{0, 1\} \) for any \( j = 1, \ldots, n - 1 \) and \( b_n = 1 \) by definition of \( a \). On the other hand, if we are given \( b \in \{0, 1\}^n \) with \( b_n = 1 \), we can find the corresponding \( a \in \{-1, 0, 1\}^n \) by letting \( a_1 = b_1 \) and \( a_i = b_i - b_{i-1} \) for \( i = 2, \ldots, n \). This is a bijection between \( a \)'s and \( b \)'s. Hence, there are \( 2^{n-1} \) such \( a \)'s and \( 2^n \) inequalities.

(c) First note that \( a^T e_k \) is either \(-1\) or \(1\), since \( a^T e_k = 2 \sum_{i=1}^{k} a_i - 1 \). Let \( b \) as defined in (b). Then, \( a^T e_k = 1 \) if and only if \( b_k = 1 \) (we say \( b_0 = 0 \)). Thus,

\[
\begin{align*}
\{x \in S \mid a^T x = 1\} &= \{e_k \mid b_k = 1\} \cup \{-e_k \mid b_k = 0\} \\
\{x \in S \mid a^T x = -1\} &= \{e_k \mid b_k = 0\} \cup \{-e_k \mid b_k = 1\}.
\end{align*}
\]

So each inequality defines distinct hyperplanes, because they contain different set of extreme points. Moreover, if we choose exactly one vector from each \( \{e_k, -e_k\} \), then they are affinely independent. For, note that it is enough to show that \( \{e_0, \ldots, e_{n-1}\} \) are linearly independent, and they are indeed linearly independent since \( \{\frac{1}{2}(e_1 - e_0), \frac{1}{2}(e_2 - e_1), \ldots, \frac{1}{2}(e_{n-1} - e_{n-2}), -\frac{1}{2}(e_{n-1} + e_0)\} \) is the standard basis of \( \mathbb{R}^n \).

(d) Note that 0 is in the interior of \( \text{conv}(S) \). Hence, no facet can contain \( \{e_k, -e_k\} \) for any \( k = 0, \ldots, n - 1 \) (otherwise it will contain 0). Since \( \text{conv}(S) \) is full-dimensional, any facet should contain at least \( n \) extreme points, i.e., it contains exactly one from each \( \{e_k, -e_k\} \). So there are at most \( 2^n \) facets of \( \text{conv}(S) \).

On the other hand, we showed in (c) that each of \( 2^n \) inequalities defines distinct facet. Hence they define \( \text{conv}(S) \).